Topological Order and Quantum Computation

Anyons

Michele Burrello

February 13, 2014
1. **Species of anyons (topological charges)** in the toric code:
   - $\mathbb{I}$: vacuum or identity (it is always present).
   - $e$: it is created by $Z$ strings.
   - $m$: it is created by $X$ strings.
   - $\psi$: it is the simultaneous presence of $e$ and $m$.

2. We can write down the **fusion rules**:
   - $e$ and $m$ are their own conjugate particles: if I change twice an $A$ or $B$ stabilizer I go back to the vacuum state:
     \[ e \times e = m \times m = \mathbb{I} \]
   - Definition of $\psi$:
     \[ e \times m = \psi \]
   - It follows:
     \[ e \times \psi = m; \quad m \times \psi = e; \quad \psi \times \psi = \mathbb{I} \]
Braidings in the toric code

- $e$ and $m$ singularly behave as bosons.
- $e$ and $m$ singularly behave as bosons.
- The mutual statistics of $e$ and $m$ is given by $R_{em} = e^{i \frac{\pi}{2}}$.
- $e$ and $m$ singularly behave as bosons.
- The mutual statistics of $e$ and $m$ is given by $R_{em} = e^{i \frac{\pi}{2}}$.
- $\psi$ is a fermion.
Consider a generic topologically ordered system on a torus. For each kind of anyon \( a \) in the system, we can define two string symmetries, \( T_1 \) and \( T_2 \), that correspond to:

1. Create a pair of anyons.
2. Wind them around one non-trivial loop.
3. We reannihilate them.
Braiding and degeneracy of the ground states

The commutation relation between $T_1$ and $T_2$ is related to the braiding statistics $R_{aa}$ of the anyon $a$:

$$T_1 T_2 T_1^{-1} T_2^{-1} :$$

If $R_{aa}^2 = 1$, then $[T_1, T_2] = 0$, so there is no degeneracy (bosons or fermions).

If $R_{aa}^2 \neq 1$, then $[T_1, T_2] \neq 0$, thus there are two non-commuting symmetries and the ground state of the system is degenerate.
Localized and gapped indistinguishable objects whose exchange statistics is described by a generic unitary operator.
Localized and gapped indistinguishable objects whose exchange statistics is described by a generic unitary operator.

- These unitary operators describe the adiabatic evolution of the system and may be represented in terms of world lines.
- The result of the exchanges does not depend on the detail of the path the anyons undergo.
Localized and gapped indistinguishable objects whose exchange statistics is described by a generic unitary operator.

- These unitary operators describe the adiabatic evolution of the system and may be represented in terms of world lines.
- The result of the exchanges does not depend on the detail of the path the anyons undergo.
For fermions or bosons, the wavefunction of a set of indistinguishable particles at fixed position depends only on their permutation.

For anyons, instead, we must keep track of their time evolution, since $R \neq R^{-1}$.

The anyon world lines in $2 + 1$D are self-avoiding strands.

Their exchange statistics is defined by the **braid group**.
Braid Group (Oktoberfest definition)
The braid group is generated by the counterclockwise and clockwise exchanges of neighboring anyons $\sigma_i, \sigma_i^\dagger$.

Disjoint operators commute:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1$$

Neighboring operators obey the Yang-Baxter relation:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

For $\sigma_i^2 = 1$ we recover the permutations.
Yang Baxter Braiding

Algebra relations

For non-adjacent operators:

\[ [\sigma_i, \sigma_k] = 0 \quad \text{if} \quad |i - k| \geq 2 \]

Yang Baxter Relations:

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]
Yang Baxter Braiding

Algebra relations

For non-adjacent operators:

\[ [\sigma_i, \sigma_k] = 0 \quad \text{if} \quad |i - k| \geq 2 \]

Yang Baxter Relations:

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]
Algebra relations

For non-adjacent operators:

\[ [\sigma_i, \sigma_k] = 0 \quad \text{if} \quad |i - k| \geq 2 \]

Yang Baxter Relations:

\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]
Abelian anyons

- The easiest non-trivial representation of the braid group is provided by Abelian anyons:

\[ \sigma \rightarrow R_{aa} = e^{i\theta_a}. \]

- Abelian anyons can be described in terms of charge-flux composite objects where:

\[ \theta_a = q_a \Phi_a \]

- **Spin-Statistics**: The exchange of two Abelian anyons \( a \) gives the same phase as a \( 2\pi \) rotation of \( q_a \) around \( \Phi_a \):

\[
\frac{1}{2} q_a \Phi_a + \frac{1}{2} q_a \Phi_a = q_a \Phi_a
\]
For two Abelian anyons:

\[ a \times b = c \]

Then:

\[ R_{ab}^c = \exp[i(\theta_c - \theta_a - \theta_b)/2] \]
For two Abelian anyons:

\[ a \times b = c \]

Then:

\[ R_{ab}^c = \exp\left[ i \left( \theta_c - \theta_a - \theta_b \right) / 2 \right] \]
For two Abelian anyons:

\[ a \times b = c \]

Then:

\[ R_{ab}^c = \exp\left[i \left( \theta_c - \theta_a - \theta_b \right)/2 \right] \]
For two Abelian anyons:

\[ a \times b = c \]

Then:

\[ R_{ab}^{c} = \exp\left[i \left( \theta_c - \theta_a - \theta_b \right)/2 \right] \]
For two Abelian anyons:

\[ a \times b = c \]

Then:

\[ R_{ab}^c = \exp\left[i\left(\theta_c - \theta_a - \theta_b\right)/2\right] \]
For two Abelian anyons:

\[ a \times b = c \]

Then:

\[ R_{ab}^c = \exp\left[ i \left( \theta_c - \theta_a - \theta_b \right) / 2 \right] \]

We could also write:

\[ \phi_a(z_1) \phi_b(z_2) = \frac{1}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_c}} \phi_c(z_1) \]
Non-Abelian Anyons correspond to higher dimensional representations of the Braid group.
Non-Abelian Anyons correspond to higher dimensional representations of the Braid group.

To obtain these higher dimensions we need to introduce a new degeneracy.

A pair of non-Abelian anyons may assume different states, characterized by different topological charges:

\[ a \times b = c \oplus d \oplus e \oplus \ldots \]

Each pair define a Hilbert space, and the braidings are unitary operators on these spaces.

Braidings of neighboring pairs of non-Abelian anyons, in general, do not commute.
Fusion Rules

Let’s consider the simple case of spin $\frac{1}{2}$ (Qubit):

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1 \quad \rightarrow \quad 2 \otimes 2 = 1 \oplus 3$$

- A particle with spin $1/2$ is described by a two-dimensional Hilbert space.
- When two of them fuse, they give rise to a singlet or to a triplet. This is a **Fusion Rule**
Fusion Rules

Let’s consider the simple case of spin $\frac{1}{2}$ (Qubit):

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1 \quad \rightarrow \quad 2 \otimes 2 = 1 \oplus 3$$

- A particle with spin $1/2$ is described by a two-dimensional Hilbert space
- When two of them fuse, they give rise to a singlet or to a triplet. This is a Fusion Rule
Non-Abelian anyons are characterized by non trivial fusion rules.

Ising Anyons / Majorana modes:

$$\sigma \times \sigma = \mathbb{I} + \varepsilon$$

$$\gamma \times \gamma = \mathbb{I} + \psi$$

Fibonacci Anyons:

$$\tau \times \tau = \mathbb{I} + \tau$$

In general one writes:

$$a \times b = \sum_c N_{ab}^c c$$
To describe a non-Abelian anyon model we need a theory characterized by the following elements:

- **Fusion Rules**: $N^c_{ab}$
- **Associativity Rules**: $(F^d_{abc})_{xy}$
- **Braiding Rules**: $\sigma \rightarrow R^c_{ab}$

These rules must have a coherent structure and must obey several constraints.
A non-Abelian anyonic model is defined starting from a finite set of particles (Topological charges). These particles are linked by the fusion rules:

\[ a \times b = \sum_c N^c_{ab}c \quad \rightarrow \quad V_a \otimes V_b = \bigoplus_c N^c_{ab}V^c_{ab} \quad \rightarrow \quad d_a d_b = \sum_c N^c_{ab}d_c \]

where \( N^c_{ab} = 0, 1; \ V^c_{ab} = V_c \) are Hilbert spaces and \( d_i \) are their quantum dimension.
A non-Abelian anyonic model is defined starting from a finite set of particles (Topological charges). These particles are linked by the fusion rules:

\[ a \times b = \sum_c N_{ab}^c \quad \rightarrow \quad V_a \otimes V_b = \bigoplus_c N_{ab}^c V_{ab} \quad \rightarrow \quad d_a d_b = \sum_c N_{ab}^c d_c \]

where \( N_{ab}^c = 0, 1 \); \( V_{ab}^c = V_c \) are Hilbert spaces and \( d_i \) are their quantum dimension.

\( a \) is a non-Abelian anyon if \( \sum_c N_{aa}^c \geq 2. \)

This means that a pair of \( a \) anyons may be found in at least two degenerate states.
- $N_{ab}^c$ can be understood as a (transfer) matrix: $(N_a)^{b_{i+1}}_{b_i}$.

- Starting from the anyon $b_i$, $N_a$ defines the possible states $b_{i+1}$ that can be obtained adding $a$. 
Fusion Rules

- $N^c_{ab}$ can be understood as a (transfer) matrix: $(N_a)_b^{i+1}$.
- Starting from the anyon $b_i$, $N_a$ defines the possible states $b_{i+1}$ that can be obtained adding $a$.
- Consider a chain of $a$ anyons:

\[
\begin{array}{cccccccc}
  & a & a & a & a & a & a & a \\
  a & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \ldots
\end{array}
\]

- A state in this chain is defined by the string $\{b_i\}$ and lives in the space:

\[
V^b_{a_1\ldots a_n} = \bigoplus_{b_1,\ldots,b_{n-1}} V^b_{a_1a_2} \otimes V^b_{b_1a_3} \otimes V^b_{b_2a_4} \otimes \cdots \otimes V^c_{b_{n-2}a_n}.
\]
Consider a chain of \(a\) anyons:

\[
\begin{array}{cccccccc}
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \ldots \\
\text{b}_1 & \text{b}_2 & \text{b}_3 & \text{b}_4 & \text{b}_5 & \text{b}_6 & & \\
\end{array}
\]

A state in this chain is defined by the string \(\{b_i\}\) and lives in the space:

\[
V_{b_1 \ldots b_n}^{a_1 \ldots a_n} = \bigoplus_{b_1, \ldots, b_{n-1}} V_{a_1 a_2}^{b_1} \otimes V_{b_1 a_3}^{b_2} \otimes V_{b_2 a_4}^{b_3} \otimes \ldots \otimes V_{b_{n-2} a_n}^{b_n}.
\]

The number of total orthogonal states (strings) is:

\[
\dim (V_{a_1 \ldots a_n}^{b_a}) = (N_{a_2} N_{a_3} \ldots N_{a_n})_{a_1}^{b_n} = \left[ (N_a)^{n-1} \right]_{a}^{b_n} \approx d_a^{n-1}
\]

\(d_a\) is the highest eigenvalue of \(N_a\), it is called quantum dimension of \(a\).
The model is characterized by two sectors: the Vacuum $\mathbb{I}$ and the Fibonacci anyon $\tau$.

**Fusion Rules:**

$$\tau \times \tau = \mathbb{I} + \tau$$

$$\mathbb{I} \times \tau = \tau$$

These fusion rules correspond to:

$$N_{\tau} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \implies d_{\tau}^2 - d_{\tau} - 1 = 0 \implies d_{\tau} = \frac{1 + \sqrt{5}}{2} \equiv \phi$$
\[ \tau \times \tau = \mathbb{I} + \tau, \quad N_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \]

Constraint: there cannot be two consecutive vacua 1.

The number of states grows like the Fibonacci numbers.

\[ d_\tau = \frac{1 + \sqrt{5}}{2} \] is the golden ratio!
For an anyonic theory to be consistent the fusion rules $N$ must be associative:

$$\sum_x N_{ab}^x N_{xc}^d = \sum_y N_{ay}^d N_{bc}^y$$

These relations characterize the fusion process $abc \rightarrow d$ in the fusion space $V_{abcd}^{d} = V_{(ab)c}^{d} = V_{a(bc)}^{d}$. The two descriptions of the space $V_{abcd}^{d}$ correspond to different orthogonal bases.

There must be a unitary operator that relates these bases:

$$(F_{d}^{abc})_{xy}$$ is this transformation.
Topologically equivalent to:
Topologically equivalent to:
F Matrices

Topologically equivalent to:

Topologically equivalent to:

Michele Burrello  Topological Order and Quantum Computation  Anyons
A couple of anyons $a \times b$ can be in a superposition of states $V_{ab}^k$ defined by the fusion rules:

$$
\phi_a(z_1)\phi_b(z_2) = \frac{\phi_c(z_2)}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_c}} + \frac{\phi_d(z_2)}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_d}} + \ldots
$$

The clockwise exchange $R_{ab}$ does not affect their total charge:

$$
R_{ab} = \begin{pmatrix}
R_{ab}^a & 0 & 0 & 0 \\
0 & R_{ab}^b & 0 & 0 \\
0 & 0 & R_{ab}^c & 0 \\
0 & 0 & 0 & \ddots
\end{pmatrix}
$$

where:

$$
(R_{ab}^c)^2 = e^{-2\pi i(\Delta_a + \Delta_b - \Delta_c)}$$

The representations of the braid generators $\sigma_i$ are given by combinations of $F$ and $R$. 

Michele Burrello  Topological Order and Quantum Computation  Anyons
Differently from Ising anyons and Majorana modes, Fibonacci anyons allow for universal quantum computation with braidings only.

To encode a qubit we use a system of 4 anyons whose total charge is trivial:

Each pair annihilates.

Each pair gives a single $\tau$
The unitary matrix $F^{\tau \tau \tau}_{\tau}$ can be calculate from a particular constraint called pentagon equation:

\[ F_{11} = F_{1\tau} F_{\tau 1} \]
\[ F_{11} + F^{2}_{\tau \tau} = 1 \]

The resulting matrix is:

\[ F = \begin{pmatrix} \varphi & \sqrt{\varphi} \\ \sqrt{\varphi} & -\varphi \end{pmatrix} \quad \text{with} \quad \varphi = d^{-1}_\tau = \frac{1 - \sqrt{5}}{2} \]
To process a single qubit we must find the operators $\sigma$ that defines the braidings.

From the Yang-Baxter eq. (or the hexagon equation) one finds out the $R$ matrix:

$$R = \begin{pmatrix} e^{\frac{4}{5}\pi i} & 0 \\ 0 & -e^{\frac{2}{5}\pi i} \end{pmatrix}$$

In a Fibonacci chain, to find the representations of $\sigma$’s, we need to make a basis transformation in order to apply the $R$ - matrix:
\[ \sigma_3 = \sigma_1 = R^{-1} = \begin{pmatrix} e^{-\frac{4}{5}\pi i} & 0 \\ 0 & -e^{-\frac{2}{5}\pi i} \end{pmatrix} \]

\[ \sigma_2 = F \sigma_1 F = \begin{pmatrix} -\varphi e^{-i\frac{\pi}{5}} & -\sqrt{\varphi} e^{i\frac{2\pi}{5}} \\ -\sqrt{\varphi} e^{i\frac{2\pi}{5}} & -\varphi \end{pmatrix} \]

\[ \sigma_1 = \sigma_3 = R^{-1} = \begin{pmatrix} e^{-\frac{4}{5}\pi i} & 0 \\ 0 & -e^{-\frac{2}{5}\pi i} \end{pmatrix} \]
To the purpose of Universal Quantum Computation we want to approximate, at any give accuracy, any single-qubit gate using as generators the braidings $\sigma_1$ and $\sigma_2$.

For Fibonacci anyons the elementary braidings generate an **infinite group**, dense in $SU(2)$.

$$\sigma_3^{-2} \sigma_2^2 \sigma_3^{-4} \sigma_2^2 \sigma_3^{-4} \sigma_2^2 \sigma_3^{-4} \sigma_2^2 \sigma_3^{-2} \cong -iX \pm 0.0031$$
Brute Force search
Bonesteel et al.

Total weaves:

\[ B_N \approx 3^N \]

Expected error:

\[ \epsilon_N \approx \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

\[ N = 1 \]

Total weaves:
\[ B_N \approx 3^N \]

Expected error:
\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
\[ N = 2 \]

Total weaves:
\[ B_N \approx 3^N \]

Expected error:
\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

\[ N = 3 \]

Total weaves:
\[ B_N \approx 3^N \]

Expected error:
\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

Total weaves:
\[ B_N \approx 3^N \]

Expected error:
\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

\[ N = 5 \]

Total weaves:
\[ B_N \approx 3^N \]

Expected error:
\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

Total weaves:
\[ B_N \cong 3^N \]

Expected error:
\[ \varepsilon_N \cong \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

$N = 7$

Total weaves:

$B_N \approx 3^N$

Expected error:

$\varepsilon_N \approx \frac{1}{3^{N/3}}$
Total weaves:

$$B_N \approx 3^N$$

Expected error:

$$\varepsilon_N \approx \frac{1}{3^{N/3}}$$
Brute Force search
Bonesteel et al.

\( N = 9 \)

Total weaves:
\[ B_N \approx 3^N \]

Expected error:
\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

\( N = 10 \)

Total weaves:

\[ B_N \approx 3^N \]

Expected error:

\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
Brute Force search
Bonesteel et al.

Total weaves:
\[ B_N \approx 3^N \]

Expected error:
\[ \varepsilon_N \approx \frac{1}{3^{N/3}} \]
Two-qubit operators
Hormozi, Bonesteel and Simon, PRL 103 (2009)

\[ a = 0 \]
\[ b = 0 \]

\[ \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \]

\[ \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix} \]
Introductions to anyons:
J. Preskill, Lecture 9,
http://www.theory.caltech.edu/~preskill/ph219/topological.pdf

G. K. Brennen and J. K. Pachos,
*Why should anyone care about computing with anyons?*,

Introduction to Fibonacci anyons:
Trebst, Troyer, Wang and Ludwig,
*A short introduction to Fibonacci anyon models*,