Discrete parafermions and quantum-group symmetries

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Outline

1. Introduction

2. The Bernard-Felder construction

3. Mapping to loop models
1. Introduction
Discretely holomorphic functions

- Discrete function: $F(z)$ on midpoints of square lattice $\mathcal{L}$

- Discrete “Cauchy-Riemann” equation:
  
  $$e^{\frac{i\pi}{4}} F(z_1) - e^{-\frac{i\pi}{4}} F(z_2) - e^{\frac{i\pi}{4}} F(z_3) + e^{-\frac{i\pi}{4}} F(z_4) = 0$$

- Short-hand notation: $\sum_{\diamond} F(z)\delta z = 0$
Loop models in Statistical Mechanics
The Temperley-Lieb loop model

- Plaquette configurations:

- Lattice configurations:

- Boltzmann weights:
  \[ W(C) = x^{N_x(C)} y^{N_y(C)} n^{N_{\ell}(C)} \]

- Partition function:
  \[ Z = \sum_{\text{config. } C} W(C) \]
Loop models in Statistical Mechanics

Correlation functions

- Averaging on Boltzmann weights:

\[ \langle f(C) \rangle := \frac{1}{Z} \sum_C W(C) f(C). \]

- Two-leg correlation function:

\[ G(z_1, z_2) := \frac{1}{Z} \sum_{C \mid z_1, z_2 \in \text{same loop}} W(C) \]

- Phases in scaling limit:
  - Non-critical phase: \( G(z_1, z_2) \sim \exp(-|z_1 - z_2|/\xi) \)
  - Critical phase: \( G(z_1, z_2) \sim |z_1 - z_2|^{-2\chi_2} \)

- “Coulomb-gas” studies \(\Rightarrow\) TL model is critical for \(0 < n \leq 2\).
Pick a pair of boundary points \((a, b)\) \(\rightarrow\) define \(BC_{ab}\).

Define correlation function:

\[
F_s(z) := \frac{1}{Z_{ab}} \sum_{\text{\(C\) \(z \in\) open path}} W(C) \, e^{i s \theta_{a \to z}(C)}
\]

[\(\theta_{a \to z} := \text{winding angle of red arc from } a \text{ to } z\)]

**Theorem:** if \(n = 2 \sin \frac{\pi s}{2}\) then \(\forall \diamond \in \Omega, \sum_{\diamond} F_s(z) \delta z = 0.\)
Algebraic structure behind discrete holomorphicity?

- Discretely holomorphic observables like $F_s$ exist in various models: TL, $O(n)$, $\mathbb{Z}_N$ clock models . . .
- Rhombic lattice $\Rightarrow$ additional parameter $\alpha$

\[ e^{-\frac{i\alpha}{2}} F(z_1) + e^{\frac{i\alpha}{2}} F(z_2) - e^{-\frac{i\alpha}{2}} F(z_3) - e^{\frac{i\alpha}{2}} F(z_4) = 0 \quad \text{(CR}_\alpha) \]

- Observations:
  1. $F_s$ satisfies $\text{CR}_\alpha$ when $W \equiv \text{integrable}$ Boltzmann weights
  2. $\alpha \equiv$ spectral parameter
- Q: general relation discrete holomorphicity $\leftrightarrow$ integrability?
Discrete holomorphicity in Physics and Mathematics

- [Dotsenko,Polyakov 88] : Linear relations for fermions in Ising
- [Smirnov 01–06] : Conf. inv. for interfaces in perco+Ising
- [Cardy,Riva,Rajabpour,Yl 06–09] : Discr. holo. in various lattice models, obs. relation to integrability
- [Smirnov,Chelkak,Hongler,Izyurov,Kytölä 09–12] : Scaling limit of interfaces+corr. func. in Ising
- [Duminil-Copin,Smirnov 10] : Proof of connectivity constant for SAW on honeycomb
- [Beaton,de Gier,Guttmann,Jensen 11–12] : Critical boundary parameter for SAW on honeycomb
- [Fendley 12] : Discr. holo. from topological QFT
- [Alam,Batchelor 12] : CR eq ↔ star-triangle in $\mathbb{Z}_N$ models
- [Hongler,Kytölä,Zahabi 12] : Discr. holo. for non-local currents in Ising, transfer-matrix formalism
2. The Bernard-Felder construction
Hopf algebras

Bi-algebra structure

- **Product** $m : \begin{cases} A \otimes A \to A \\ a \otimes b \mapsto a \cdot b \end{cases}$

- **Coproduct** $\Delta : \begin{cases} A \to A \otimes A \\ a \mapsto \sum_i a_i' \otimes a_i'' \end{cases}$

  - $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b), \quad \Delta(a + \lambda b) = \Delta(a) + \lambda \Delta(b)$
  - $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$

- **Example:** enveloping algebra of a Lie algebra $\mathfrak{g}$
  - $\mathfrak{g}$ Lie algebra, with bracket $[X_a, X_b] = i f_{abc} X_c$
  - $A := U(\mathfrak{g}) = \text{span}(\text{words on alphabet } \{X_a\})$
  - bracket $\equiv$ commutator $([a, b] = ab - ba)$
  - Trivial coproduct $\Delta(X_a) = X_a \otimes 1 + 1 \otimes X_a$
Hopf algebras

Tensor-product representations

- $V$ finite-dimensional vector space
  Map $\pi : \mathcal{A} \to \text{End}(V)$ is a representation of $\mathcal{A}$ iff:
    - $\pi$ is linear and surjective,
    - $\pi$ is a morphism: $\pi(ab) = \pi(a)\pi(b)$.
- Coproduct = tool to construct higher-dim. representations:
  
  \[
  \Delta(a) = \sum_i a'_i \otimes a''_i \quad \longrightarrow \quad \pi_{12}(a) := \sum_i \pi_1(a'_i) \otimes \pi_2(a''_i)
  \]

- Iterate:
  
  \[
  \sum_i a^{(1)}_i \otimes \cdots \otimes a^{(L)}_i
  \]

- Example: $\mathcal{A} = U(g)$, for a Lie algebra $g$
  
  \[
  \pi^{(L)}(X_a) = \sum_{m=1}^{L} 1 \otimes \cdots \otimes 1 \otimes \pi(X_a) \otimes 1 \otimes \cdots \otimes 1
  \]
Hopf algebras
The $R$-matrix

- The two representations $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are isomorphic.
- Intertwiner $R_{12} : V_1 \otimes V_2 \to V_2 \otimes V_1$ such that:
  \[
  \forall a \in \mathcal{A}, \quad R_{12} \pi_{12}(a) = \pi_{21}(a) R_{12}
  \]
- Expand coproduct $[\pi_{12}(a) = \sum_i \pi_1(a'_i) \otimes \pi_2(a''_i)]$:

\[
\sum_i \begin{array}{c}
\text{V}_2 \\
\text{V}_1
\end{array}
\begin{array}{c}
a'_i \\
a''_i
\end{array}
\xrightarrow{R_{12}}
\begin{array}{c}
\text{V}_2 \\
\text{V}_1
\end{array}
\begin{array}{c}
a'_i \\
a''_i
\end{array}
\]

- Consistency condition = Yang-Baxter equation:
  \[
  (R_{23} \otimes 1).(1 \otimes R_{13}).(R_{12} \otimes 1) = (1 \otimes R_{12}).(R_{13} \otimes 1).(1 \otimes R_{23})
  \]
Generators of \( \mathcal{A} \): \( \{ J_1, J_2 \ldots \} \) and \( \{ \mu_1, \mu_2 \ldots \} \).

Assume the coproduct of \( \mathcal{A} \) has the following form:

\[
\Delta(J_k) = J_k \otimes 1 + \mu_k \otimes J_k
\]

\[
\Delta(\mu_k) = \mu_k \otimes \mu_k
\]

Iteration of coproduct \( \Rightarrow \) “conserved charges”:

\[
Q_k := \Delta^{L-1}(J_k) = \sum_{m=1}^{L} \mu_k \otimes \cdots \otimes \mu_k \otimes J_k \otimes 1 \otimes \cdots \otimes 1
\]

Non-local currents:

\[
\psi_k(m) := \mu_k \otimes \cdots \otimes \mu_k \otimes J_k \otimes 1 \otimes \cdots \otimes 1
\]

\[
\psi_k(m) = \psi_k(1) \otimes \cdots \otimes \psi_k(m) \otimes \cdots \otimes \psi_k(L)
\]
Commutation relations

- From intertwining relations \([R_{12} \pi_{12}(a) = \pi_{21}(a) R_{12}]\):
  - For \(a = J_k\):
    - For \(a = \mu_k\):
  - Transfer matrix:
    - Conservation laws:
      \[\forall a \in A, \quad T.\pi^{(L)}(a) = \pi^{(L)}(a).T\]
The affine quantum group $\mathcal{A} = U_q(\widehat{s\ell}_2)$

- Generators: $E_0, E_1, F_0, F_1, T_0, T_1$
  \{E_0, E_1, F_0, F_1\} = \text{raising/lowering ops}, \quad \{T_0, T_1\} = \text{diag. ops}.$

- Product rules:
  \[
  [T_0, T_1] = 0 \\
  T_i E_j T_i^{-1} = q^{2(-1)^\delta_{ij}} E_j \\
  (\text{+ higher order rules} \ldots)
  \]

- Coproduct rules:
  \[
  \Delta(E_i) = E_i \otimes 1 + T_i \otimes E_i \\
  \Delta(T_i) = T_i \otimes T_i \\
  \]

- Introduce $\bar{E}_i := qT_i F_i \Rightarrow \Delta(\bar{E}_i) = \bar{E}_i \otimes 1 + T_i \otimes \bar{E}_i$

- BF structure: \{\mu_k\} = \{T_0, T_1\}. 
Evaluation representations of $A = U_q(\hat{sl}_2)$

- Representations are labelled by a complex number $u$
  
  Explicit form:

  $$
  \begin{align*}
  \pi_u : \\
  E_0 &\mapsto \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} & \tilde{E}_0 &\mapsto \begin{pmatrix} 0 & u^{-1} \\ 0 & 0 \end{pmatrix} & T_0 &\mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \\
  E_1 &\mapsto \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} & \tilde{E}_1 &\mapsto \begin{pmatrix} 0 & 0 \\ u^{-1} & 0 \end{pmatrix} & T_1 &\mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}
  \end{align*}
  $$

- Intertwiner: $R(u/v)\pi_{u,v} = \pi_v,u R(u/v)$

  $$
  R(u/v) = \begin{pmatrix}
  [qu/v] & 0 & 0 & 0 \\
  0 & [u/v] & 1 & 0 \\
  0 & 1 & [u/v] & 0 \\
  0 & 0 & 0 & [qu/v]
  \end{pmatrix}, \quad [z] = \frac{z - z^{-1}}{q - q^{-1}}
  $$
Application to the six-vertex model

- Use basis for $V_u$: $\{\uparrow, \downarrow\}$.

Plaquette configurations:

- Boltzmann weights:

$$R_{6V} = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_5 & \omega_4 & 0 \\ 0 & \omega_3 & \omega_6 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}$$

- When $R_{6V} \equiv R_{U_q(\widehat{sl}_2)}$, the 6V model is integrable.
3. Mapping to loop models
From the TL model to the 6V model

[Baxter, Kelland, Wu 73]

▶ Orient each loop independently:

\[ n = 2 \cos 2\pi \lambda \]

\[ = e^{2i\pi \lambda} + e^{-2i\pi \lambda} \]

▶ Partition function:

\[ Z = \sum_C x^{N_x(C)} y^{N_y(C)} e^{2i\pi \lambda [N_+^\ell(C) - N_-^\ell(C)]} \]

▶ Distribute phase factors locally:
From the TL model to the 6V model (2)

▶ Vertex configurations:

▶ Six-vertex weights arising from loop model:

\[
\begin{align*}
\omega_1 &= \omega_2 = x, \quad \omega_3 = \omega_4 = y, \\
\omega_5 &= e^{+2i\lambda \alpha} x + e^{-2i\lambda (\pi - \alpha)} y \\
\omega_6 &= e^{-2i\lambda \alpha} x + e^{+2i\lambda (\pi - \alpha)} y
\end{align*}
\]

▶ Set \( q = -e^{2i\lambda \pi} \), \( w = e^{-2i\lambda \alpha} \):

\[
\begin{align*}
\omega_1 &= \omega_2 = [qw], \quad \omega_3 = \omega_4 = [w] \quad \Rightarrow \quad \omega_5 = \omega_6 = 1.
\end{align*}
\]
Conserved currents in the 6V model

\[
\begin{cases}
\Delta(E_0) = E_0 \otimes 1 + T_0 \otimes E_0 \\
\Delta(T_0) = T_0 \otimes T_0
\end{cases}
\Rightarrow \text{BF current } \psi_0
\]

\[
\psi_0(m) = T_0 \otimes T_0 \otimes \cdots \otimes T_0 \otimes E_0 \otimes 1 \otimes \cdots \otimes 1
\]

\[\uparrow \text{m–th}\]

\[\psi_0(z_1) - \psi_0(z_2) - \psi_0(z_3) + \psi_0(z_4) = 0.\]

\[\begin{array}{c}
V \\
Z_1 \\
Z_2 \\
Z_3 \\
Z_4 \\
V'
\end{array}\]

\[\Rightarrow \text{Commutation with } \text{R-matrix } \Rightarrow \text{linear relation:}\]

\[\psi_1, \tilde{\psi}_0, \tilde{\psi}_1.\]
Mapping of conserved currents

What is the meaning of $\langle \psi_0(z) \rangle$ in terms of loops?

$\psi_0(z)$ cannot sit alone on a closed loop

$\psi_0 = u \times \begin{array}{c} \uparrow \end{array}$

$\Rightarrow \langle \psi_0(z) \rangle = \frac{u}{Z} \sum_{C \mid z \in \gamma} W(C) \times (\text{phase factor})$
Mapping of conserved currents (2)

Identification of phase factors

\[ \theta_{b \to z} = \theta_{a \to z} + \pi, \quad q = e^{i\pi(2\lambda-1)} \]

phase factor:

\[
e^{i\lambda(\theta_{a \to z} + \theta_{b \to z})} \times q^{\frac{\theta_{a \to z} + \theta_{b \to z} - \pi}{2\pi}} = A e^{i(4\lambda-1)\theta_{a \to z}}
\]

\[ \Rightarrow \langle \psi_0(z) \rangle = \frac{uA}{Z} \sum_{C | z \in \gamma} W(C) e^{i(4\lambda-1)\theta_{a \to z}} = uA \times F_s(z) \]

spin: \( s = 4\lambda - 1 \) (remember Theorem in Intro)
Mapping of conserved currents (3)

Cauchy-Riemann relation

- Set \( u = 1/u' = w^{1/2} \Rightarrow u/u' = w = e^{-2i\lambda\alpha} \)

- Conservation relation:

\[
\langle \psi_0(z_1) - \psi_0(z_2) - \psi_0(z_3) + \psi_0(z_4) \rangle = 0
\]
\[
\Leftrightarrow \quad v F_s(z_1) - u F_s(z_2) - v F_s(z_3) + u F_s(z_4) = 0
\]
\[
\Leftrightarrow \quad \sum_{\diamond} F_s(z) \delta z = 0
\]

- Conservation of BF current \( \Rightarrow \) \( CR_{\alpha} \) relation
Conclusions

▶ What we have **also** obtained:
  ▶ Boundary CR equation $\leftrightarrow$ integrable $K$-matrix
  ▶ Discrete parafermions in other models: dilute $O(n)$, chiral Potts (cf R. Weston’s talk)
  ▶ Massive regime of chiral Potts: $\bar{\partial}F = m\chi$

▶ For future work:
  ▶ Observables from $E_0^2$, $E_0 \otimes E_0$, etc?
  ▶ Find “other half” of CR equations?
  ▶ More relations at roots of unity?
Thank you for your attention!