Spontaneous Breaking of U(N) symmetry in invariant Matrix Models & Ergodicity Breaking

Fabio Franchini

Support by:

arXiv:1412.6523
arXiv:1503.03341
Outlook

• Consider a Quantum System

• Localization/extendedness of wavefunctions is a basis-dependent property

• However, eigen-energy statistics (Poisson/Wigner Dyson) characterizes insulating/conducting systems

• Seeking for a basis independent, general structure of (Anderson) insulators
Results

• The U(N) symmetry matrix models are endowed with can be spontaneously broken

• Thermodynamic limit also takes symmetry’s rank to infinity

• Eigenvectors encode non-trivial information!

• Certain models break U(N) in a critical way: similarity with Metal/Insulator Transition

• These models are in the family of CS/ABJM
Outline

1. Intro 1: Disorder & Localization

2. Intro 2: Matrix Models

3. Spontaneous Symmetry Breaking:
   - Geometrical argument
   - Symmetry Breaking term
   - Numerical finite size detection

4. Weakly Confined Matrix Models
   - Spectral Statistics (known)
   - Energy landscape (new)

5. Conclusions & Outlook
Part 1

Introduction on Localization due to Disorder
Disorder & Localization

- **Anderson Model**: \( \mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[ c_l^\dagger c_j + c_j^\dagger c_l \right] \) (Anderson. ‘58)

- Tight-binding model (nearest neighbor hopping)

- Random on-site energies: \( \epsilon_j \in [-W, +W] \)

- 1 (& 2) Dimensions: localized for any \( W \neq 0 \)

- Higher D: Small \( W \): conducting
  
  (weak localization, Random Matrices)

  ➢ \( W > W_c \): insulating

  (localized at low energies)

- **Hard problem** (uncontrolled perturbation expansion)
Metal/Insulator Transition

\[ \mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[ c_l^\dagger c_j + c_j^\dagger c_l \right] \]

\( \epsilon_j \in [-W, +W] \quad W < W_c \quad D \geq 3 \)

- At \( E = E_m \) : Mobility Edge
  separating extended
  from localized states

- Transition as
  Intermediate state
  (multifractal)

Van Tiggelen group (PRL 2009)
Multifractality

• At each height $|\Psi|^2 = \alpha$, the wavefunction's amplitude draws a "curve" with a different fractal dimension $f(\alpha)$.
To characterize localization:

- **Extended:** $\text{IPR}_q \simeq N^{1-q} = L^{-d(q-1)}$

- **Localized:** $\text{IPR}_q \simeq \text{const}$

- **Critical state:**

  $$\text{IPR}_q \simeq L^{-d_q(q-1)}$$

  $$= \int N^{-q\alpha + f(\alpha)} d\alpha$$

  $0 < d_q < d$ : fractal dimensions

  $f(\alpha)$ : multi-fractal spectrum

Van Tiggelen group (PRL 2009)
Multifractality

\[ L^{-d_q(q-1)} = \int N^{-q\alpha + f(\alpha)} d\alpha \]

- Fractal dimensions \( d_q \) /fractal spectrum \( f(\alpha) \) known analytically only in perturbative regimes (\( d_q \simeq 0, d \))
Landau Zener Picture

- Qualitative picture on eigenvalue/eigenvector connection

- 2-level system:

\[
\begin{pmatrix}
\epsilon_1 & V \\
V^* & \epsilon_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
E_1 & 0 \\
0 & E_2
\end{pmatrix}
\]

\[
\delta = E_1 - E_2 = \sqrt{(\epsilon_1 - \epsilon_2)^2 + |V|^2}
\]

<table>
<thead>
<tr>
<th>&quot;Localized&quot;</th>
<th>&quot;Extended&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V \ll \epsilon_1 - \epsilon_2$</td>
<td>$V \gg \epsilon_1 - \epsilon_2$</td>
</tr>
<tr>
<td>$\delta \approx \epsilon_1 - \epsilon_2$</td>
<td>$\delta \approx</td>
</tr>
<tr>
<td>$\Psi_{1,2} \approx \psi_{1,2} + \mathcal{O}\left(\frac{1}{\epsilon_1 - \epsilon_2}\right)\psi_{2,1}$</td>
<td>$\Psi_{1,2} \approx \psi_{1,2} \pm \psi_{2,1}$</td>
</tr>
</tbody>
</table>
Part 2

Introduction on Matrix Models
Matrix Models

\[ Z = \int \mathcal{D} M e^{-W(M)} \]

- Several applications: nuclear theory, mesoscopic conduction, 2-D quantum gravity, string theory, statistical physics, econophysics, neuroscience, chaos theory, number theory, integrability…

- Reflects a large universality

- Matrices can be link between points; fields in adjoint or bi-fundamental, representation of operators in many-body theory (Hamiltonians, Scattering…)...
Random Matrices

\[ \mathcal{Z} = \int DMe^{-W(M)} \quad \mathcal{F} = \ln \mathcal{Z} \]

• If \( W(M) \) real: statistical model

• Consider \( M \) as a Hamiltonian:
  - Interaction between every degree of freedom
  - Matrix entries randomly from a distribution

• Describe quantum “chaotic” systems

• Universality determined by symmetry:
  - Orthogonal, Unitary, Symplectic, ... ensembles
Invariant Ensembles

• Action invariant under rotations: \( W(M) = \text{Tr} V(M) \)

• Switch to eigenvalues/eigenvectors: \( M = U^\dagger \Lambda U \)

\[
Z = \int DMe^{-\text{Tr}V(M)} = \int DU \int d^N \lambda \Delta^\beta (\{\lambda\}) e^{-\sum_j V(\lambda_j)}
\]

Eigenvectors uniformly distributed over the \( N \)-dimensional sphere (Hilbert space): independent from \( V(\lambda) \)

Van der Monde Determinant: 
\[
\Delta (\{\lambda\}) = \prod_{j>l}^{N} (\lambda_j - \lambda_l)
\]
(from Jacobian)
Jacobian introduces interaction between eigenvalues

Effective Coulomb gas:

$$\mathcal{L} = -\beta \sum_{j > l} \ln |\lambda_j - \lambda_l| + \sum_j V(\lambda_j)$$

- Eigenvalues as 1-D particles with
  - logarithmic interaction
  - external confining potential $V(\lambda)$

- Eigenvalue distribution from equilibrium configuration
Wigner-Dyson Universality

\[ \mathcal{L} = -\beta \sum_{j > l} \ln |\lambda_j - \lambda_l| + \sum_j V(\lambda_j) \]

- Distribution of the distance between n.n. eigenvalues (level spacing) universal:

\[ P(s) \propto s^\beta e^{-A(\beta)s^2} \]

- Universality captured by Gaussian ensemble: \( V(\lambda) = \frac{\lambda^2}{2} \)

- Valid for any polynomial \( V(\lambda) \)
Entries of Unitary matrix follow the Porter-Thomas Distribution:

\[ P \left( \left| \tilde{U}_{ij} \right|^2 \right) = N \exp \left[ -N \left| \tilde{U}_{ij} \right|^2 \right] \]
Invariant Ensembles

\[ Z = \int \mathcal{D}U \int d^N \lambda \Delta^\beta (\{\lambda\}) e^{-\sum_j V(\lambda_j)} \]

- Wigner Dyson distribution & level repulsion:
  Jacobian introduces interaction between eigenvalues
- Extended states/conducting phases:
  uniform distribution means eigenvectors typically have all non-vanishing entries
- Eigenvalues interact through their eigenvectors:
  \( WD \Leftrightarrow \text{extended states} \)
To study localization problems, introduce non-invariant random matrix ensembles (Random Banded Matrices)

\[ Z = \int DMe^{-\sum_{j,l} A_{jl} |M_{jl}|^2} \Rightarrow \langle M_{nm}^2 \rangle = A_{nn}^{-1} \]

\[ A_{nm} = e^{\frac{|n-m|}{B}} \]

\[ A_{nm} = 1 + \frac{(n-m)^2}{B^2} \]

\[ \rightarrow \] Localized states (Poisson statistics) (Mirlin et al. ‘96)

\[ \rightarrow \] Multi-Fractal states (Critical Statistics) (Evers & Mirlin, ‘00)
Invariant vs. non-Invariant Ensembles

• Invariant: basis independent
  → Wigner-Dyson eigenvalue statistics
  ⇒ de-Haar measure for eigenvector
  ⇒ delocalized systems
  analytical techniques

• Non-Invariant: basis dependent
  → Poisson/critical eigenvalue statistics
  eigenvector connected with eigenvalue
  ⇒ localized/critical systems
  mostly numerical approaches
Loophole: Spontaneous Breaking of Rotational Invariance

- Invariant models are endowed with superior (non-perturbative) analytical techniques
- Spontaneous breaking of rotational invariance:
  \[ \Rightarrow \text{Eigenvectors contain non-trivial information} \]
  \[ \Rightarrow \text{Invariant machinery for localization problems!} \]
- Recall a ferromagnet:
  \[ \Rightarrow \text{From partition function, rotational invariance} \]
  \[ \Rightarrow \text{no spontaneous magnetization} \]
  \[ \Rightarrow \text{Need symmetry breaking term} \]
PART 3

Spontaneous Breaking of Rotational Symmetry in Invariant Multi-Cuts Matrix Model
**Multi-Cut Solutions**

\[ Z = \int \mathcal{D}U \int d^N \lambda \ e^{-\sum_j V(\lambda_j) + 2 \sum_{j > l} \ln|\lambda_j - \lambda_l|} \]

- \( V(x) \) with several, well separated, minima
  \( \Rightarrow \) disconnected support for eigenvalues (multi-cuts)
- For example: double well potential
  \[ V_{2W}(x) = \frac{1}{4} x^4 - \frac{t}{2} x^2 \]
  (2-cuts for \( t > 2 \))

**Level Density:**
\[ \rho(x) = \sum_{j}^{N} \delta(x - \lambda_j) \]
Understanding the matrix SSB

- Geometrical argument: line element
  \[ ds^2 = \text{Tr} \left( dM \right)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2 \sum_{j>l} (\lambda_j - \lambda_l)^2 |dA_{jl}|^2 \]

- Angular degrees of freedom live on spheres of radii
  \[ r_{jl} = |\lambda_j - \lambda_l| \]

- Two lengths scales:
  - Eigenvalues spacing: \( O \left( \frac{1}{N} \right) \)
  - Support of distribution: \( O \left( 1 \right) \)
  - Small arc lengths:
    \[ r_{jl} \sim O \left( \frac{1}{N} \right) \rightarrow dA_{jl} \sim O \left( 1 \right) \]
    \[ r_{jl} \sim O \left( 1 \right) \rightarrow dA_{jl} \sim O \left( \frac{1}{N} \right) \]

- \( \beta = 2 \), Unitary
  \[ dA \equiv U^\dagger dU \]
  \[ M = U^\dagger \Lambda U \]
Multi-Cuts SSB

• Level repulsion resolves degeneracy:

  \[ \Rightarrow \text{each of the } n \text{ cuts contains } m_j \text{ eigenvalues} \]

• Gap between cuts breaks rotational invariance:

  \[ U(N) \xrightarrow{N \to \infty} \prod_{j=1}^{n} U(m_j) \]

• Three Arguments:

  ★ Brownian motion;
  ★ Numerical finite size analysis;
  ★ Symmetry Breaking Term

Double well

\[ U(N) \xrightarrow{N \to \infty} U(N/2) \times U(N/2) \]
(assume N even)

F.F. arXiv:1412.6523
Symmetry Breaking Term

\[ W(J) = \ln \int dM e^{-N \text{Tr} V(M) + JN \text{Tr}(\Lambda T - M S)} \]

- Calculate (dis-)order parameter:
  \[ \frac{d}{dJ} \lim_{N \to \infty} W(J) \bigg|_{J=0} = 0 \]
  Symmetry is Broken!

- Finite N:
  \[ \frac{dW(J)}{dJ} \bigg|_{J=0} = \langle |\text{Tr}(\Lambda T - M S)| \rangle \neq 0 \]
  Eigenvectors misaligned

\[ M = U^\dagger \Lambda U \]
\[ S = V^\dagger TV \]
Symmetry Breaking Term

\[ W(J) = \ln \int dM e^{-N \text{Tr} V(M) + J N \text{Tr} (\Lambda T - M S)} \]

- **Calculate (dis-)order parameter:**
  - \( \frac{d}{dJ} \lim_{N \to \infty} W(J) \bigg|_{J=0} = 0 \)
  - **Finite N:** \( \frac{dW(J)}{dJ} \bigg|_{J=0} \neq 0 \)

\[ \int dM e^{-N \text{Tr} V(M) + J N \text{Tr} (\Lambda T - M S)} \propto Z_0 + Z_1 \left( e^{-2JN|\lambda_j - \lambda'_i|} \right) + \ldots \]

- **Instantons:**
  - Pairs of eigenvalues tunneling between cuts
  - Restore broken symmetries

\[ M = U^\dagger \Lambda U \]
\[ S = V^\dagger TV \]
Multi-Cuts SSB: Conclusions

- Gap in the eigenvalue distribution
  - Deviation from WD universality
  - Spontaneous breaking of rotational symmetry
  - Eigenvectors localized in patch of Hilbert space spanned by the other eigenvectors in the same cut

- Broken symmetries restored by instantons

- Abstract characterization of localization without reference to basis: IPR not sufficient, need response under perturbation (application to MBL?)
PART 4

Weakly Confined Matrix Models
&
The Metal/Insulator Transition
**Weakly Confined Invariant Models**

\[ Z = \int DMe^{-\text{Tr}V(M)}, \quad V(\lambda) \overset{|\lambda| \to \infty}{\approx} \frac{1}{2\kappa} \ln^2 |\lambda| \]

- Soft confinement *sets* them apart from usual polynomial potentials
  - WD universality does not apply
  - Indeterminate moment problem

- Arise in *localization limit* of Chern-Simons/ABJM:
  - (Marino ’02; Kapustin et al. ’10; …)

- Solvable through *orthogonal polynomials*:
  - *q*-deformed Hermite/Laguerre Polynomials
  - (Muttalib et al. ’93; Tierz’04)
Weakly Confined Matrix Models

\[ V(\lambda) \xrightarrow{|\lambda| \to \infty} \frac{1}{2\kappa} \ln^2 |\lambda| \]

- Intermediat level spacing statistics
- Same eigenvalue correlations as power law (critical) Random Banded Matrices

(Muttalib et al. ‘93)

- Critical level statistics signals fractal eigenstates?
- Critical Spontaneous Breaking of U(N) Invariance?

(Canali, Kravtsov, ‘95)
Weakly Confined Matrix Models & their applications

- 2+1-Dimensional Chern-Simons
- 2-Dimensional q-deformed Yang-Mills
- Six-Vertex Model, Painlevé equation, Integrability
- Topological String Theories
- Vicious Random Walkers
- ABJM Models
- Glassy Behavior & Replica Symmetry Breaking
- Anderson Metal/Insulator Transition

Matrix Model SSB and Localization

n. 33

Fabio Franchini
WCMM Energy Landscape

• Take exactly log-normal ensemble (positive eigenvalues)

\[ Z = \int D U \int_{\lambda > 0} d^N \lambda \Delta \{\{\lambda}\} e^{-\frac{1}{2\kappa} \sum_j \ln^2 \lambda_j} \]

• Exponential mapping: \[ \lambda_j = e^{\kappa x_j} \]

\[ Z \propto \int d^N x_j \prod_{n < m} (e^{\kappa x_n} - e^{\kappa x_m})^2 e^{-\frac{\kappa}{2} \sum_{l=1}^N [x_l^2 - 2x_l]} \]

• Each term of the Van der Monde shifts the equilibrium of the parabolic potential: different effective potential felt by each eigenvalue for each term for the VdM

F.F. arXiv:1503.03341
• Full partition function known (orthogonal polynomials)

\[ Z \propto e^{\frac{\kappa}{\kappa}} N (4N^2 - 1) (2\pi \kappa)^{N/2} \frac{N!}{N} \prod_{n=1}^{N-1} \left(1 - q^n\right)^{N-n} \]

• Each term of the expansion of the product

- Corresponds to a different saddle (equilibrium conf.)
- Has the same leading energy (differ for the powers of $q$)
- $q^j$ fugacity of the instantons

• Instantons restore broken symmetries: from the $U(1)^N$ configuration, to the full $U(N)$ when all instantons act

F.F. arXiv:1503.03341
WCMM Outlook

\[ Z = \int \mathcal{D}U \int_{\lambda > 0} d^N \lambda \Delta (\{ \lambda \}) e^{-\frac{1}{2\kappa} \sum_j \ln^2 \lambda_j} \]

- Critical eigenvalue statistics from complex landscape
- Each saddle can be interpreted as endowed with a reduced symmetry with respect to \( U(N) \)
- **To do:** Employ replica approach for WCMM: Anderson transition as a full-RSB
- **Conjecture:** calculate IPR and multi-fractal spectrum from contributions of the different saddles
SSB Structure

- Each saddle point corresponds to a different SSB

- Unitary matrix from Hermitian matrix: \( U = e^{iA} \)

\[
ds^2 = \text{Tr} \left( dM \right)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2 \sum_{j > l} (\lambda_j - \lambda_l)^2 |dA_{jl}|^2 \\
\text{subject to } dA \equiv U^\dagger dU
\]

- \( U(1)^N \) saddle has all \( dA_{ij} = 0 \)

- Conjecture:
  Each instanton \(-q^n\) “turns on” one element: \( dA_{i,i+n} \neq 0 \)
Multi-fractal Spectrum

- Numerical check of conjecture
- Unitary matrix from Hermitian matrix: $U = e^{iA}$
- Generate each element $A_{jl}$ with probability

$$
q^{j-l} \quad 1 - q^{j-l}
$$

sample $A_{jl}$ uniformly

$A_{jl} = 0$

$\Rightarrow$ MULTIFRACTALITY!
Multi-fractal Spectrum

- Inverse Participation Ratios of $U = e^{iA}$ scale with fractional power of $N$

$\rightarrow$ Multi-fractal spectrum from invariant matrix model!

- Matrix $M = U^\dagger \Lambda U$ has power-law Gaussian elements!
Conclusions

• Invariant Matrix Models usually applied only to extended/conducting states: eigenvectors discarded

• Deviation of eigenvalue statistics from Wigner-Dyson signals loos of ergodicity: gap between eigenvalues mutually localize their eigenvectors: U(N) broken

• Invariant Models techniques for localization problems!

• WCMM has complex energy landscape → critical SSB
- Matching WCMM critical SSB with Metal/Insulator Transition multi-fractal spectrum?
- Critical exponents of SSB
- Direct characterization of eigenvector behavior
- Connection between SSB & Replica Symmetry Breaking
- WCMM & Matrix models arise in string theory: meaning of the $N \to \infty$ $U(N)$ symmetry breaking?
- ...
Brownian Motion Picture

- Level repulsion resolves degeneracy:
  \[ \Rightarrow \text{each of the } n \text{ cuts contains } m_j \text{ eigenvalues} \]

- Gap between cuts breaks rotational invariance: \[ U(N) \xrightarrow{N \to \infty} \prod_{j=1}^{n} U(m_j) \]

- Dyson Brownian Motion for equilibrium distribution shows scale separation:
  \[
  d\lambda_j = -\frac{dV(\lambda_j)}{d\lambda_j} \ dt + \frac{1}{N} \sum_{l \neq j} \frac{dt}{\lambda_j - \lambda_l} + \frac{1}{\sqrt{N}} dB_j(t) \\
  d\tilde{U}_j(t) = -\frac{1}{2N} \sum_{l \neq j} \frac{dt}{(\lambda_j - \lambda_l)^2} \tilde{U}_j + \frac{1}{\sqrt{N}} \sum_{l \neq j} \frac{dW_{jl}(t)}{\lambda_j - \lambda_l} \tilde{U}_l
  \]

Matrix Model SSB and Localization  n. 42  Fabio Franchini
Generating a Random Matrix

- **Gaussian Models:** \( Z = \int D M e^{-\text{Tr} M^2} = \int \prod dM_{jl} e^{-\sum_j M_{jl}^2} \)
  
  → each matrix entries sampled independently

- **One-Cut Models:** \( Z = \int D M e^{-\text{Tr} V(M)} = \int D M e^{-\text{Tr} \sum_k g_k M^k} \)
  
  → entries correlated: generated as perturbation of Gaussian case in a Metropolis scheme

- **Multi-Cut Solutions:** Gaussian case unstable
  
  → start from initial seed and evolve it to equilibrium

  → **SSB:** final configuration has memory of eigenvectors of initial seed
Symmetry Breaking Term

- To detect SSB introduce symmetry breaking term
- Most natural one is \( \text{Tr} \left( [M, S] \right)^2 \), but too hard to handle

\[ W(J) = \ln \int dM e^{-N \text{Tr} V(M) + JN |\text{Tr} (\Delta M - MS)|} \]

\( J \): source strength

\( M = U^\dagger \Lambda U \)
\( S = V^\dagger TV \)

\( S \): given Hermitian Matrix
Favors alignment of eigenvectors

Absolute value can be removed by sorting eigenvalues in increasing order
Symmetry Breaking: Double Well

\[ W(J) = \ln \int dM e^{-N \text{Tr} V(M) + J N |\text{Tr} (\Lambda T - M S)|} \]

- **Double well:** \( U(N) \xrightarrow{N \to \infty} U(N/2) \times U(N/2) \) (assume \( N \) even)

- **Take** \( S \) with 2 sets of \( N/2 \)-degenerate eigenvalues: \( t = \pm 1 \) to induce correct symmetry breaking

- **Calculate (dis-)order parameter:**

\[
\left. \frac{dW(J)}{dJ} \right|_{J=0} = \langle |\text{Tr} (\Lambda T - M S)| \rangle = 0 \quad \text{Symmetry Broken}
\]

\[
\neq 0 \quad \text{Eigenvectors misaligned}
\]
Symmetry Breaking Term

\[ W(J) = \ln \int dM e^{-N \text{Tr} V(M) + JN | \text{Tr}(\Lambda T - M S)|} \]

- **Use Itzykson-Zuber formula:** (Itzykson & Zuber, ‘80)
  \[
  \int dU e^{\text{Tr} A U B U^\dagger} \propto \frac{\det [e^{a_j b_l}]}{\Delta (\{a\}) \Delta (\{b\})} \]

- **After regularization for degenerate eigenvalues:**
  \[
  \int dU e^{JN \text{Tr} M S} \propto \frac{1}{\Delta (\{\lambda\})} \sum_{\{\alpha\} \cup \{\alpha'\} = \{\lambda\}} e^{-JN} \sum_j (\alpha_j - \alpha'_j) \Delta (\{\alpha\}) \Delta (\{\alpha'\})
  \]

  **Sum over ways to partition eigenvalues of M according to degeneracies of S**

\[ M = U^\dagger \Lambda U \]
\[ S = V^\dagger T V \]
Symmetry Breaking Term

\[\int dM e^{-N\text{Tr}V(M) + JN|\text{Tr}(\Lambda T - M\Sigma)|} \propto\]

\[\int_{\lambda>0, \lambda'<0} d\frac{N}{2} \lambda d\frac{N}{2} \lambda' e^{-N \sum_j V(\lambda_j) - N \sum_l V(\lambda'_l)} \times \Delta^2 (\{\lambda\}) \Delta^2 (\{\lambda'\}) \prod_{j,l} (\lambda_j - \lambda_l) \times\]

\[\times \left[ 1 + \sum_{j,l=1}^{N/2} e^{-2JN(\lambda_j - \lambda'_l)} \prod_{p=1}^{N/2} \prod_{q=1}^{N/2} \frac{(\lambda_l - \lambda'_p)(\lambda_j - \lambda'_q)}{(\lambda'_j - \lambda_p)(\lambda'_l - \lambda_q)} + \ldots \right]\]

• Hence: \[\frac{d}{dJ} \lim_{N \to \infty} W(J) \Bigg|_{J=0} = 0\]

• At finite N: \[\langle |\text{Tr}(\Lambda T - M\Sigma)| \rangle \neq 0\]

\[\int dM e^{-N\text{Tr}V(M) + JN|\text{Tr}(\Lambda T - M\Sigma)|} \propto Z_0 + Z_1 \left( e^{-2JN|\lambda_j - \lambda'_l|} \right) + \ldots\]

Instantons:
- Pairs of eigenvalues tunneling between wells
- Restore broken symmetries

Matrix Model SSB and Localization
Finite Size Analysis

- Without preferred, reference basis; localization means rigidity of eigenvectors under perturbations

- Take double well matrix model:
  \[ Z = \int \mathcal{D}M e^{-N \text{Tr} \left[ \frac{1}{4} M^4 - \frac{t}{2} M^2 \right]} \]

- Generate a representative matrix:
  \[ M = U^\dagger \Lambda U \]

- Apply perturbation \( \Delta M \) (sparse Gaussian Matrix)

- Find eigenvectors of perturbed matrix:
  \[ M + \Delta M = U'^\dagger \Lambda' U' \]

- Consider eigenvectors of perturbed matrix in original eigenvector basis (rotation due to perturbation):
  \[ \tilde{U} = U' U^\dagger \]
Finite Size Analysis

\[ Z = \int \mathcal{D}M e^{-N \text{Tr} \left[ \frac{1}{4} M^4 - \frac{t}{2} M^2 \right]} \]

Equilibrium conf. from Coulomb gas

\[ M = U^\dagger \Lambda U \]

Randomly generated

\[ M + \Delta M = U'^\dagger \Lambda' U' \]

Sparse Gaussian Matrix

Typical Unitary matrix \( \tilde{U} = U' U^\dagger \) connecting the eigenvectors before and after the perturbation (t=4; N=1000; sparse matrix with n=200 non zero elements, drawn from Gaussian with zero mean and variance N)
Finite Size Analysis

\[ Z = \int \mathcal{D}M e^{-N \text{Tr} \left[ \frac{1}{4} M^4 - \frac{t}{2} M^2 \right]} \]

\[ \mathcal{P} \left( \left| \tilde{U}_{ij} \right|^2 \right) = \chi \exp \left[ -\chi \left| \tilde{U}_{ij} \right|^2 \right] \]

\[ \chi_D = \frac{N}{2} \quad \chi_{OD} = \frac{2tN^2}{n} \]

\( t=4, \, N=1000, \) sparse matrix with \( n=200 \) non zero elements, drawn from Gaussian with zero mean and variance \( N \)
Finite Size Analysis

\[ O_{jl} = \sum_{m} \left| \tilde{U}_{mj} \right|^2 \left| \tilde{U}_{ml} \right|^2 \]

Overlaps between eigenstates

\[
\langle |O_{jl}| \rangle_D = \langle |\Delta \tilde{U}_{jl}| \rangle_D = \frac{1}{\chi_D} = \frac{2}{N} \\
\langle |O_{jl}| \rangle_{OD} = 2\langle |\tilde{U}_{jl}| \rangle_{OD} = 2\langle |\Delta \tilde{U}_{jl}| \rangle_{OD} = \frac{2}{\chi_{OD}} = \frac{n}{tN^2}
\]

Off-diagonal blocks suppressed as $1/N$ compared to diagonal ones

Onset of localizations!

1/\chi(N,n,t)
Consider simplex of eigenvalues in increasing order

\[ Z \propto \int d^N x \prod_{n<m} (e^{\kappa x_n} - e^{\kappa x_m})^2 e^{-\frac{\kappa}{2} \sum_{l=1}^{N} [x_l^2 - 2x_l]} \]

In the \( \kappa \to \infty \) limit, all terms left inside the VdM vanish

Eigenvalue crystallize on a lattice

(Bogomolny et al. ‘97)

\[ \lim_{\kappa \to \infty} Z \propto N! e^{\frac{\kappa}{6} N(4N^2 - 1)} \int d^N x e^{-\frac{\kappa}{2} \sum_{l=1}^{N} (x_l + 1 - 2l)^2} \]
**Eigenvalue Crystallization**

\[ Z \propto N! \int d^N x \prod_{n<m} \left[ 1 - e^{\kappa(x_n - x_m)} \right]^2 e^{-\frac{\kappa}{2} \sum_{l=1}^{N} [x_l^2 - 2(2l-1)x_l]} \]

- Eigenvalue crystallization for \( \kappa \to \infty \)

\[ Z \propto \int d^N x \left[ e^{-\frac{\kappa}{2} \sum_{l=1}^{N} (x_l + 1 - 2l)^2} + \ldots \right] \]

(Bogomolny et al. ‘97)

- Corresponds to SSB: \( U(N) \to U(1)^N \)

(Exponential separation between eigenvalues completely freezes eigenstates dynamics)

(Pato, ‘00)
\[ Z \propto N! \int d^N x \prod_{n<m} \left[ 1 - e^{\kappa(x_n - x_m)} \right]^2 e^{-\frac{\kappa}{2} \sum_{l=1}^{N} \left[ x_l^2 - 2(2l-1)x_l \right]} \]

- Finite \( \kappa \) corrections organized in powers of \( q = e^{-\kappa} \)

\[ q^0 \]

\[ -q^1 \times (N - 1) \]

\[ +q^2 \times \frac{(N - 2)(N - 3)}{2} \]

\[ +q^2 - q^2 \times (N - 2) \ldots \]
Each term in VdM shifts the equilibrium point of 2 eigenvalues (one notch closer to one-another).

Each term of the VdM generates a new equilibrium configuration (saddle point of the partition function).

Saddles connected by instantons with weight $-q^n$. 

$$Z \propto N! \int d^N x \prod_{n<m} \left[ 1 - e^{\kappa (x_n - x_m)} \right]^2 e^{-\frac{\kappa}{2} \sum_{l=1}^{N} [x_l^2 - 2(2l-1)x_l]}$$
• Full partition function known (orthogonal polynomials)

\[ \mathcal{Z} \propto e^{\frac{\kappa}{6} N (4N^2 - 1)} (2\pi \kappa)^{N/2} N! \prod_{n=1}^{N-1} (1 - q^n)^{N-n} \]

• Natural interpretation in terms of instantons
  
  - 1 instanton connecting a pair of eigenvalues N-1 apart
  - 2 instantons connecting a pair N-2 apart
  - ...
  - N-1 instantons connecting a pair of N.N. eigenvalues

• Metal/Insulator Transition as glassy phase?
Exponential number of saddle points

Every equilibrium configuration which preserves the center of mass is realized by the action of the instantons

Each equilibrium configuration has the same leading energy: they differ only for the powers of $q$

Instantons restore broken symmetries: from the $U(1)^N$ configuration at $\kappa \to \infty$, to the full $U(N)$ when all instantons bring each eigenvalue to the same equilibrium point

$$Z \propto e^{\frac{\kappa}{6} N (4N^2 - 1)} (2\pi \kappa)^{N/2} N! \prod_{n=1}^{N-1} (1 - q^n)^{N-n}$$