Feynman Rules for Any Spin. III*

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Free fields, which transform according to general irreducible representations of the homogeneous Lorentz group, are constructed for any spin. The most general interactions may be built up from these fields, without explicit derivatives. The propagators for these fields are caluclated in terms of previously calculated propagators for $(2j+1)$-component fields and Racah coefficients. A general formula is given which describes the decomposition of these propagators into various "daughter" spin exchanges.

I. INTRODUCTION AND SUMMARY

SOME years ago I started a series[1,2] of papers on the relativistic quantum dynamics of particles with arbitrary spin. The aim of this series was to develop a formalism for calculating the S matrix in perturbation theory, in which the spins of the particles involved would appear as adjustable parameters. However, this work was left incomplete, in that I did not state the rules for constructing the most general Lorentz-invariant interactions, and I did not calculate the propagators that would appear in such general theories.

Since then, theoretical physicists have given a good deal of attention to the problems of higher spin as they affect S-matrix theory, particularly with regard to the analysis of kinematic singularities and zeros, the phenomena of conspiracy and evasion, daughter Regge trajectories, etc. Often the ideas developed in this work have been tested in or even inspired by the "theoretical laboratory" of Feynman diagrams.[3] It therefore seemed likely to me that a paper completing the series on Feynman rules for any spin might be of some use.

Section II shows how to construct the most general irreducible free fields $\psi_{ab}^{(AB)}$ which transform according to the representation (A,B) of the homogeneous Lorentz group. The material of this section is largely contained in previous work,[4] and is presented here for the sake of clarity and completeness. In Sec. III it is shown that the most general possible interactions can be formed out of these irreducible fields *without explicit derivatives*, by using the usual rules of angular momentum addition to couple all A,a indices and, *separately*, all B,b indices, to form rotational scalars.

The only problem here is to calculate the propagators of those general fields. The residue of the poles in any such propagator can be determined in a perfectly straightforward way; for a particle of spin j, momentum $\mathbf{p}$, and mass m created by a field of type (A_2,B_2) and destroyed by a field of type (A_1,B_1) this residue is

$$\pi_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}\left(\frac{\mathbf{p}}{m};j\right)=(-)^{2B_2}\sum_{\lambda a_1' a_2' b_1' b_2'}(-)^{j+\lambda}\times\left(\frac{|\mathbf{p}|+(\mathbf{p}^2+m^2)^{1/2}}{m}\right)^{b_1'-a_1'+b_2'-a_2'}C_{A_1B_1}(j\lambda;a_1'b_1')\times C_{A_2B_2}(j-\lambda;a_2'b_2')D_{a_1a_1'}{}^{(A_1)}[R(\hat{p})]D_{b_1b_1'}{}^{(B_1)}[R(\hat{p})]\times D_{a_2a_2'}{}^{(A_2)}[R(\hat{p})]D_{b_2b_2'}{}^{(B_2)}[R(\hat{p})],\quad(1.1)$$

where C denotes a Clebsch-Gordan coefficient,[5] and $D[R(\hat{p})]$ is the usual unitary matrix representation of the rotation $R(\hat{p})$ which carries the z axis into the direction of $\mathbf{p}$. (The π matrix is the same as appears when we sum over helicities in unitarity relations.) In order to calculate the propagator $\Delta(q;j,m)$ for four-momenta q^μ which are *not* on the mass shell, it is necessary to construct a polynomial $N(q;j,m)$ which reduces to π on the mass shell:

$$N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(\mathbf{p},(\mathbf{p}^2+m^2)^{1/2};j,m)=\pi_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(\mathbf{p}/m;j)\quad(1.2)$$

and which has the correct Lorentz transformation properties for any q. This task is performed in Secs. IV and V; the answer is

$$N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q;j,m)=(2j+1)\times\sum_{n,\nu,\mu}W(A_1B_1A_2B_2;jn)\Pi_{\nu-\mu}{}^{(n)}(q)m^{-2n}(-)^{A_1-B_2-j+\mu}\times C_{A_1A_2}(n\nu;a_1a_2)C_{B_1B_2}(n,-\mu;b_1b_2).\quad(1.3)$$

Here W is the usual Racah coefficient[5]; the sums over n, ν, and μ run over all integers or half-integers for which W and the Clebsch-Gordan coefficients C do not vanish; and the matrix $\Pi^{(n)}$ is a homogeneous polynomial of order $2n$ in the four-vector q^μ, calculated in an earlier paper[1] and given here by Eqs. (5.4) and (5.5). The propagator is given by

$$\Delta_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q;j,m)=\frac{N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q;j,m)}{q^2+m^2-i\epsilon}.\quad(1.4)$$

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[1] S. Weinberg, Phys. Rev. **133**, B1318 (1964).

[2] S. Weinberg, Phys. Rev. **134**, B882 (1964).

[3] In particular, see L. Van Hove, Phys. Letters **24B**, 183 (1967); L. Durand, III, Phys. Rev. **154**, 1537 (1967); R. Blankenbecler and R. L. Sugar, *ibid.* **168**, 1597 (1968); R. Blankenbecler, R. L. Sugar, and J. D. Sullivan, *ibid.* **172**, 1451 (1968).

[4] Reference 1, Sec. VIII. For a more explicit formulation, see S. Weinberg, in *Lectures on Particles and Field Theory* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964), Vol. II, p. 439. The results of Sec. II are also implicit in E. Wichmann (unpublished).

[5] All Clebsch-Gordan and Racah coefficients and rotation matrices are in the notation of M. E. Rose, *Elementary Theory of Angular Momentum* (Wiley-Interscience, Inc., New York, 1957).

The most interesting consequence of this result is that, although the propagator $\Delta(q;\,j,m)$ is *not* directly given by the residue matrix π for mass m and spin j, it may nevertheless be expressed as a linear combination of these residue matrices for "mass" $\sqrt{(-q^2)}$ and a range of spins j'. It is shown in Sec. V that

$$N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q;\,j,m)=\sum_{j'} F_{jj'}(-q^2/m^2;\,A_1B_1A_2B_2)\times\pi_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(\mathbf{q}/\sqrt{-q^2};\,j')\,,\quad(1.5)$$

where $F_{jj'}$ is a polynomial in $\sqrt{(-q^2/m^2)}$:

$$F_{jj'}(x;\,A_1B_1A_2B_2)=(-)^{j'-j}(2j+1)\sum_n(2n+1)\times x^nW(A_1B_1A_2B_2;\,jn)W(A_1B_1A_2B_2;\,j'n)\,.\quad(1.6)$$

For instance, the $(\frac{1}{2},\frac{1}{2})\rightarrow(\frac{1}{2},\frac{1}{2})$ propagator of the vector field for a $j=1$ particle has the numerator (written with vector indices instead of a's and b's):

$$N_{\mu,\nu}{}^{(\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2})}(q;\,1,m)=g_{\mu\nu}+q_\mu q_\nu/m^2=(g_{\mu\nu}-q_\mu q_\nu/q^2)+(1+q^2/m^2)(q_\mu q_\nu/q^2)\,,$$

the two terms in the last line containing the residue matrices $\pi_{\mu,\nu}(\mathbf{q}/\sqrt{-q^2};\,j')$ for $j'=1$ and $j'=0$. It is precisely this decomposition that gives rise to the daughter trajectories when direct-channel poles are summed to give Regge poles.[6] Thus the general decomposition formula (1.5) offers to us the opportunity of exploiting the known properties of the Racah coefficients to explore relations among Regge trajectories.

II. GENERAL FIELDS—A REVIEW

A free field is a linear combination of creation and annihilation operators which transforms under the inhomogeneous Lorentz group according to the rule

$$U[\Lambda,a]\psi_n(x)U^{-1}[\Lambda,a]=\sum_m D_{nm}[\Lambda^{-1}]\psi_m(\Lambda x+a)\,,\quad(2.1)$$

where $U[\Lambda,a]$ is the Hilbert-space operator representing the Lorentz transformation $x^\mu\rightarrow\Lambda_\nu{}^\mu x^\nu+a^\mu$, and $D_{nm}[\Lambda]$ is a finite matrix representation of the homogeneous Lorentz group. The annihilation operators used to construct ψ transform according to the rule[7]

$$U[\Lambda,a]a(\mathbf{p},\lambda)U^{-1}[\Lambda,a]=e^{ip\cdot a}\left(\frac{\omega(\Lambda\mathbf{p})}{\omega(\mathbf{p})}\right)^{1/2}\times\sum_{\lambda'}D_{\lambda\lambda'}{}^{(j)}[\mathcal{W}^{-1}(\mathbf{p},\Lambda)]a(\Lambda\mathbf{p},\lambda')\,.\quad(2.2)$$

Here $a(\mathbf{p},\lambda)$ is the annihilation operator for a particle of momentum $\mathbf{p}$, energy $p^0\equiv\omega(\mathbf{p})\equiv(\mathbf{p}^2+m^2)^{1/2}$, helicity λ, and spin j; $D_{\lambda\lambda'}{}^{(j)}$ is the usual unitary matrix representation of the three-dimensional rotation group; and $\mathcal{W}(\mathbf{p},\lambda)$ is the Wigner rotation appropriate to helicity states[8]:

$$\mathcal{W}(\mathbf{p},\lambda)\equiv B^{-1}(|\Lambda\mathbf{p}|/m)R^{-1}(\Lambda\hat{p})\Lambda R(\hat{p})B(|\mathbf{p}|/m)\,,\quad(2.3)$$

where $B(u)$ is the boost along the z direction which carries the vector (0,0,0,1) into $(0,\,0,\,u,\,(1-u^2)^{1/2})$ and $R(\hat{p})$ is the rotation which carries the z axis into the direction of $\mathbf{p}$. Comparison of (2.1) with (2.2) shows that any linear combination of the operators $a(\mathbf{p},\lambda)$ which transforms according to (2.1) must be of the form

$$\psi_n{}^{(+)}(x)=(2\pi)^{-3/2}\int d^3p[2\omega(\mathbf{p})]^{-1/2}\times e^{ip\cdot x}D_{nm}[R(\hat{p})B(|\mathbf{p}|/m)]\sum_\lambda u_m(\lambda)a(\mathbf{p},\lambda)\,,\quad(2.4)$$

with $u_m(\lambda)$ constants, defined so that for any rotation $\mathcal{R}$,

$$\sum_m D_{nm}[\mathcal{R}]u_m(\lambda)=\sum_{\lambda'}u_m(\lambda')D_{\lambda'\lambda}{}^{(j)}[\mathcal{R}]\,.\quad(2.5)$$

Also, $(-)^{j-\lambda}a^\dagger(\mathbf{p},\,-\lambda)$ transforms like $a(\mathbf{p},\lambda)$ under homogeneous Lorentz transformations $\Lambda^\mu{}_\nu$, so we can form another field out of the antiparticle creation operators

$$\psi_n{}^{(-)}(x)=(2\pi)^{-3/2}\int d^3p[2\omega(\mathbf{p})]^{-1/2}\times e^{-ip\cdot x}D_{nm}[R(\hat{p})B(|\mathbf{p}|/m)]\times\sum_\lambda v_m(\lambda)(-)^{j-\lambda}b^\dagger(\mathbf{p},\,-\lambda)\,,\quad(2.6)$$

with $v_m(\lambda)$ another "wave function" which also satisfies (2.5), and which we will see below to be just proportional to $u_m(\lambda)$.

Any such fields $\psi^{(\pm)}$ may be expressed as direct sums of irreducible fields $\psi^{(A,B,\pm)}$ which transform according to the $(2A+1)(2B+1)$-dimensional representation $D_{ab,a'b'}{}^{(A,B)}[\Lambda]$ of the homogeneous Lorentz group. (Here A and B are integers and/or half-integers characterizing the representation, and a and b are indices running by unit steps from $-A$ to A and from $-B$ to B, respectively.) In these irreducible representations the boost $B(u)$ is represented by real diagonal matrices

$$D_{ab,a'b'}{}^{(AB)}[B(u)]=[u+(1-u^2)^{1/2}]^{b-a}\delta_{bb'}\delta_{aa'}\,,\quad(2.7)$$

while rotations $\mathcal{R}$ are represented by direct products of ordinary unitary rotation matrices[5]

$$D_{ab\text{-}a'b'}{}^{(AB)}[\mathcal{R}]=D_{a,a'}{}^{(A)}[\mathcal{R}]D_{bb'}{}^{(B)}[\mathcal{R}]\,.\quad(2.8)$$

It follows then from (2.5) that $u_{ab}(\lambda)$ is just proportional to a Clebsch-Gordan coefficient:

$$u_{ab}(\lambda)\propto C_{AB}(j\lambda;\,ab)\,,\quad(2.9)$$

[6] Blankenbecler and Sugar (Ref. 3).

[7] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[8] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

and the same is true of $v_{ab}(\lambda)$. Using (2.7)–(2.9) in (2.4) and (2.5) we find for our irreducible fields

$$\psi_{ab}{}^{(AB,+)}(x)=(2\pi)^{-3/2}\int d^3p[2\omega(\mathbf{p})]^{-1/2}\sum_{a'b'\lambda} D_{a,a'}{}^{(A)}[R(\mathbf{p})]D_{b,b'}{}^{(B)}[R(\hat{p})]\times\left(\frac{|\mathbf{p}|+\omega(\mathbf{p})}{m}\right)^{b'-a'}C_{AB}(j\lambda;a'b')a(\mathbf{p},\lambda)e^{ip\cdot x}, \quad (2.10)$$

$$\psi_{ab}{}^{(AB,-)}(x)=(2\pi)^{-3/2}\int d^3p[2\omega(\mathbf{p})]^{-1/2}\sum_{a'b'\lambda} D_{a,a'}{}^{(A)}[R(\hat{p})]D_{b,b'}{}^{(B)}[R(\hat{p})]\times\left(\frac{|\mathbf{p}|+\omega(\mathbf{p})}{m}\right)^{b'-a'}C_{AB}(j\lambda;a'b')(-)^{j-\lambda}b^\dagger(\mathbf{p},-\lambda)e^{-ip\cdot x}. \quad (2.11)$$

It remains to be shown how $\psi^{(+)}$ and $\psi^{(-)}$ are to be put together to form a causal free field ψ. The requirement here is physical; we are only allowed to construct the interaction Lagrangian from fields with Lorentz-covariant propagators. It will be shown in Sec. IV that the correct causal combination is

$$\psi_{ab}{}^{(AB)}(x)=\psi_{ab}{}^{(AB,+)}(x)+(-)^{2B}\psi_{ab}{}^{(AB,-)}(x). \quad (2.12)$$

In general any interaction Lagrangian must involve not only the fields $\psi^{(AB)}$ but also their Hermitian adjoints $\psi^{(AB)\dagger}$. However, these adjoints do not have the same Lorentz-transformation properties as the fields $\psi^{(AB)}$, and it therefore proves convenient to replace all field adjoints with "antiparticle fields," defined by interchanging the particle and antiparticle operators in $\psi^{(AB)}$:

$$\tilde{\psi}_{a,b}{}^{(AB)}(x)\equiv(2\pi)^{-3/2}\int d^3p[2\omega(\mathbf{p})]^{-1/2}\times\sum_{a'b'\lambda}D_{aa'}{}^{(A)}[R(\hat{p})]D_{bb'}{}^{(B)}[R(\hat{p})]\times\left(\frac{|\mathbf{p}|+\omega(\mathbf{p})}{m}\right)^{b'-a'}C_{AB}(j\lambda;a'b')\times[b(\mathbf{p},\lambda)e^{ip\cdot x}+(-)^{2B+j-\lambda}a^\dagger(\mathbf{p},-\lambda)e^{-ip\cdot x}]. \quad (2.13)$$

Evidently $\tilde{\psi}^{(AB)}$ has the same Lorentz-transformation properties as $\psi^{(AB)}$, while direct calculation shows that

$$\psi_{a,b}{}^{(AB)\dagger}(x)=(-)^{2B-j+a+b}\tilde{\psi}_{-b,-a}{}^{(BA)}(x). \quad (2.14)$$

In particular, a self-charge-conjugate particle will have $\tilde{\psi}\equiv\psi$, and (2.14) then plays the role of a Hermiticity condition on the field.

The transformation properties of the particle and antiparticle annihilation operators under space inversion (P), charge conjugation (C), and time reversal (T) are well known:

$$\begin{aligned}
Pa(\mathbf{p},\lambda)P^{-1}&=(-)^{j+\lambda}\exp[i\lambda\phi(\hat{p})]\eta_P a(-\mathbf{p},-\lambda),\\
Pb(\mathbf{p},\lambda)P^{-1}&=(-)^{j+\lambda}\exp[i\lambda\phi(\hat{p})]\bar{\eta}_P b(-\mathbf{p},-\lambda),\\
Ca(\mathbf{p},\lambda)C^{-1}&=\eta_C b(\mathbf{p},\lambda),\\
Cb(\mathbf{p},\lambda)C^{-1}&=\bar{\eta}_C a(\mathbf{p},\lambda),\\
Ta(\mathbf{p},\lambda)T^{-1}&=\eta_T\exp[-i\lambda\phi(\hat{p})](-)^{2j}a(-\mathbf{p},\lambda),\\
Tb(\mathbf{p},\lambda)T^{-1}&=\bar{\eta}_T\exp[-i\lambda\phi(\hat{p})](-)^{2j}b(-\mathbf{p},\lambda),
\end{aligned} \quad (2.15)$$

where $\phi(\hat{p})$ is a phase defined by the relation[2]

$$D_{a,-a'}{}^{(A)}[R(\hat{p})]=(-)^{A-a'}\times\exp[ia'\phi(\hat{p})]D_{aa'}{}^{(A)}[R(-\hat{p})], \quad (2.16)$$

and the η's and $\bar{\eta}$'s are intrinsic phases depending on the particle type. In order that the inversions do not change the *relative* phase of the particle and antiparticle parts of $\psi(A,B)$, it is necessary that the inversion phases of particles and antiparticles be related by

$$\bar{\eta}_P=(-)^{2j}\eta_P^*,\quad \bar{\eta}_C=\eta_C^*,\quad \bar{\eta}_T=\eta_T^*. \quad (2.17)$$

A straightforward calculation then gives

$$\begin{aligned}
P\psi_{ab}{}^{(AB)}(\mathbf{x},t)P^{-1}&=\eta_P(-)^{A+B-j}\psi_{ba}{}^{BA}(-\mathbf{x},t),\\
P\tilde{\psi}_{ab}{}^{(AB)}(\mathbf{x},t)P^{-1}&=\eta_P^*(-)^{A+B+j}\tilde{\psi}_{ba}{}^{BA}(-\mathbf{x},t),\\
C\psi_{ab}{}^{(AB)}(x,t)C^{-1}&=\eta_C\tilde{\psi}_{ab}{}^{(AB)}(\mathbf{x},t),\\
C\tilde{\psi}_{ab}{}^{(AB)}(x,t)C^{-1}&=\eta_C^*\psi_{ab}{}^{(AB)}(\mathbf{x},t),\\
T\psi_{ab}{}^{(AB)}(\mathbf{x},t)T^{-1}&=\eta_T(-)^{A+B-a-b}\psi_{-a,-b}{}^{(AB)}(\mathbf{x},-t),\\
T\tilde{\psi}_{ab}{}^{(AB)}(\mathbf{x},t)T^{-1}&=\eta_T^*(-)^{A+B-a-b}\tilde{\psi}_{-a,-b}{}^{(AB)}(\mathbf{x},-t).
\end{aligned} \quad (2.18)$$

The advantage in constructing the interaction Lagrangian out of these general irreducible fields will, I hope, be made clear in Sec. III. However, the reader should not suppose that we will thereby be able to write down interactions that are in any respect more general than those which could be built up from the simplest $(2j+1)$-component fields $\psi^{(j,0)}$ and their derivatives. Indeed it is easy to see that *any irreducible field* $\psi^{(AB)}$ *for a particle of spin* j *may be constructed by applying a suitable differential operator of order 2B to the field* $\psi^{(j,0)}$, provided, of course, that A, B, and j satisfy the triangle inequality

$$|A-B|\leq j\leq A+B, \quad (2.19)$$

which is required for the nonvanishing of the Clebsch-Gordan coefficients in (2.10) and (2.11). Consider the object

$$\psi_{\sigma,\mu_1\cdots\mu_{2B}}{}^{(j)}(x)\equiv\left\{\frac{\partial}{\partial x^{\mu_1}}\cdots\frac{\partial}{\partial x^{\mu_{2B}}}\right\}\psi_{\sigma,0}{}^{(j,0)}(x), \quad (2.20)$$

where { } denotes the traceless part, e.g.,

$$\left\{\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right\}\equiv\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}-\tfrac{1}{4}g_{\mu\nu}\Box^2,\text{ etc.}$$

A symmetric traceless tensor of rank $2B$ constitutes the irreducible representation (B,B) of the homogeneous Lorentz group, so the object (2.20) belongs to the reducible representation

$$(B,B)\otimes(j,0)=(B+j,\,B)\oplus(B+j-1,\,B)\oplus\cdots\oplus(|B-j|,\,B). \quad (2.21)$$

That is, by taking linear combinations of the derivatives (2.20) we may project out a field with any one of the transformation types in (2.21):

$$\tau_{Aa,Bb}{}^{\sigma\mu_1\cdots\mu_{2B}}\psi_{\sigma,\mu_1\cdots\mu_{2B}}{}^{(j)}\propto\psi_{a,b}{}^{(AB)}, \quad (2.22)$$

where

$$|B-j|\leq A\leq B+j. \quad (2.23)$$

Further, none of these linear combinations can vanish, because then $\psi_{\sigma,0}{}^{(j,0)}$ would satisfy a nontrivial wave equation, while we know that a field with $2j+1$ components, which is constructed as a linear combination of $2j+1$ independent creation and/or annihilation operators, can satisfy only the trivial wave equations $(\Box^2-m^2)^N\psi=0$. The inequality (2.23) is equivalent to the triangle inequality (2.19), so (2.22) provides a prescription for constructing any field $\psi^{(AB)}$ with A, B, and j satisfying (2.19). [For example, from the self-dual antisymmetric tensor $F^{\mu\nu}$, which has Lorentz transformation type (1,0), we may construct the $(\frac{1}{2},\frac{1}{2})$ field $\partial_\mu F^{\mu\nu}$, the (0,1) field $\epsilon_{\mu\nu\lambda\rho}\partial^\lambda\partial_\sigma F^{\rho\sigma}$, etc. From the $(\frac{1}{2},0)$ field $(1+\gamma_5)\psi$ we may construct the $(0,\frac{1}{2})$ field $\gamma\cdot\partial(1+\gamma_5)\psi=(1-\gamma_5)\gamma\cdot\partial\psi$, etc.]

III. INTERACTIONS

The remarks at the end of Sec. II indicate that we could, if we liked, construct interactions solely out of the $(2j+1)$-component fields $\psi^{(j,0)}$ and their derivatives. However, it is very much more convenient to take the opposite approach, and write the interaction Lagrangian in terms of general irreducible fields $\psi^{(AB)}$, $\tilde{\psi}^{(AB)}$, *with no explicit derivatives*. The Nth derivative of a field $\psi^{(AB)}$ has the Lorentz transformation type $(\frac{1}{2}N,\frac{1}{2}N)\otimes(A,B)$, and is therefore nothing but a direct sum of irreducible fields of type (A',B'), with $A+\frac{1}{2}N\geq A'\geq|A-\frac{1}{2}N|$ and $B+\frac{1}{2}N\geq B'\geq|B-\frac{1}{2}N|$. (Some of these fields will actually vanish, whenever $\psi^{(AB)}$ satisfies a nontrivial Nth-order wave equation.) Hence there is no loss of generality in writing the interaction in terms of irreducible fields without derivatives.

The great advantage of this approach is that it reduces to a triviality the problem of constructing general Lorentz-invariant interactions. An infinitesimal Lorentz transformation $\Lambda_\nu{}^\mu=\delta_\nu{}^\mu+\omega_\nu{}^\mu$ (with $\omega_{\mu\nu}=-\omega_{\nu\mu}$) will induce on $\psi^{(AB)}$ the infinitesimal transformation

$$U[1+\omega,\,0]\psi_{ab}{}^{(AB)}(x)U^{-1}[1+\omega,\,0]=\psi_{ab}{}^{(AB)}(x+\omega x)-i\sum_{a'}\boldsymbol{\theta}\cdot\mathbf{J}_{aa'}{}^{(A)}\psi_{a'b}{}^{(AB)}(x)-i\sum_{b'}\boldsymbol{\theta}^*\cdot\mathbf{J}_{bb'}{}^{(B)}\psi_{ab'}{}^{(AB)}(x), \quad (3.1)$$

where

$$\theta_1=\omega_{23}-i\omega_{10},\quad \theta_2=\omega_{31}-i\omega_{20},\quad \theta_3=\omega_{12}-i\omega_{30}.$$

Hence the most general Lorentz-invariant interaction is constructed by using the usual rules of angular momentum addition to couple together all a indices and, *separately*, all b indices, to form rotational scalars. For instance, the most general Lorentz-invariant interaction which can destroy three particles with arbitrary spins j_1, j_2, j_3 is

$$\mathfrak{L}_{123}'(x)=\sum_{A_1A_2A_3}\sum_{B_1B_2B_3}\sum_{a_1a_2a_3}\sum_{b_1b_2b_3}g_{123}(A_1A_2A_3;\,B_1B_2B_3)\times\begin{pmatrix}A_1 & A_2 & A_3\\ a_1 & a_2 & a_3\end{pmatrix}\begin{pmatrix}B_1 & B_2 & B_3\\ b_1 & b_2 & b_3\end{pmatrix}\psi_{1,a_1b_1}{}^{(A_1B_1)}(x)\times\psi_{2,a_2b_2}{}^{(A_2B_2)}(x)\psi_{3,a_3b_3}{}^{(A_3B_3)}(x)+\text{Hermitian adjoint}, \quad (3.2)$$

where ( ) denotes the usual Wigner $3j$ symbols,[5] and the sums over A_1, A_2, A_3, B_1, B_2, and B_3 run over values for which the triplets $A_1B_1j_1$, $A_2B_2j_2$, and $A_3B_3j_3$ satisfy triangle inequalities like (2.19). [Indeed, (3.2) is *too* general, because it contains gradient terms which do not contribute to the action $\int d^4x\ \mathfrak{L}'(x)$.] We may write the Hermitian-adjoint term in (3.2) explicitly in terms of the antiparticle fields $\tilde{\psi}$ given by (2.19); using the well-known properties of the $3j$ symbols

$$\begin{pmatrix}A_1 & A_2 & A_3\\ a_1 & a_2 & a_3\end{pmatrix}=(-)^{A_1+A_2+A_3}\begin{pmatrix}A_1 & A_2 & A_3\\ -a_1 & -a_2 & -a_3\end{pmatrix},$$

$$\begin{pmatrix}A_1 & A_2 & A_3\\ a_1 & a_2 & a_3\end{pmatrix}=0\quad\text{if}\quad a_1+a_2+a_3\neq 0,$$

we find that

$$\mathfrak{L}_{123}'(x)=\sum_{A_1A_2A_3}\sum_{B_1B_2B_3}\sum_{a_1a_2a_3}\sum_{b_1b_2b_3}\begin{pmatrix}A_1 & A_2 & A_3\\ a_1 & a_2 & a_3\end{pmatrix}\begin{pmatrix}B_1 & B_2 & B_3\\ b_1 & b_2 & b_3\end{pmatrix}\times[g_{123}(A_1A_2A_3;\,B_1B_2B_3)\psi_{1,a_1b_1}{}^{(A_1B_1)}(x)\psi_{2,a_2b_2}{}^{(A_2B_2)}(x)\psi_{3,a_3b_3}{}^{(A_3B_3)}(x)+\tilde{g}_{123}(A_1A_2A_3;\,B_1B_2B_3)\tilde{\psi}_{1,a_1b_1}{}^{(A_1B_1)}(x)\tilde{\psi}_{2,a_2b_2}{}^{(A_2B_2)}(x)\tilde{\psi}_{3,a_3b_3}{}^{(A_3B_3)}(x)], \quad (3.3)$$

where

$$\tilde{g}_{123}(A_1A_2A_2;\, B_1B_2B_3) = g_{123}^*(B_1B_2B_3;\, A_1A_2A_3) \prod_{n=1}^{3} (-)^{j_n-A_n-B_n}. \quad (3.4)$$

The general rule is that in constructing the total interactions we must use the antiparticle fields $\tilde{\psi}$ as well as the particle fields ψ, with antiparticle coupling constants determined by Hermiticity conditions like (3.4).

As a bonus, our use of general irreducible fields without derivatives to construct the interaction Lagrangian greatly simplifies the conditions imposed by conservation of parity, charge conjugation, and time reversal. Applying to (3.2) or (3.3) the transformations (2.18), we see that $\int d^4x\, \mathcal{L}'(x)$ will conserve P, C, and T if

$$P:\quad g(B_1B_2B_3;\, A_1A_2A_3) = g(A_1A_2A_3;\, B_1B_2B_3) \prod_{n=1}^{3} \eta_{Pn}(-)^{A_n+B_n-j_n},$$

$$C:\quad g^*(B_1B_2B_3;\, A_1A_2A_3) = g(A_1A_2A_3;\, B_1B_2B_3) \prod_{n=1}^{3} \eta_{Cn}(-)^{A_n+B_n-j_n},$$

$$T:\quad g(A_1A_2A_3;\, B_1B_2B_3) = g^*(A_1A_2A_3;\, B_1B_2B_3) \prod_{n=1}^{3} \eta_{Tn}.$$

The TCP theorem appears here as the statement that any interaction will automatically conserve TCP provided that the T, C, and P phases are chosen (as they always can be[9]), so that for each particle type

$$\eta_T\eta_C\eta_P = 1.$$

IV. PROPAGATORS

Suppose we do construct the interaction out of various irreducible fields $\psi^{(AB)}$ and $\tilde{\psi}^{(AB)}$, without explicit derivatives. The ingredients that will appear in the Feynman rules for calculating the S matrix will be coupling constants like $g(A_1A_2A_3B_1B_2B_3)$ in Eq. (3.2), external-line wave functions given by the coefficients of the creation and annihilation operators in the fields, and internal-line propagators. We now turn to the problem of calculating these propagators.

The first step is to compute the vacuum expectation value of a time-ordered product

$$\langle T\{\psi_{a_1b_1}^{(A_1B_1)}(x)\tilde{\psi}_{a_2b_2}^{(A_2B_2)}(y)\}\rangle_0 \equiv \theta(x-y)\langle\psi_{a_1b_1}^{(A_1B_1)}(x)\tilde{\psi}_{a_2b_2}^{(A_2B_2)}(x)\rangle_0 + (-)^{2j}\theta(y-x)\langle\tilde{\psi}_{a_2b_2}^{(A_2B_2)}(y)\psi_{a_1b_1}^{(A_1B_1)}(x)\rangle_0, \quad (4.1)$$

representing a particle of spin j which is created at y and destroyed at x (or vice versa for the antiparticle.) Incidentally, there is nothing at all exotic about the possibility that a particle may be created and destroyed by fields of different types; for instance, in the propagator $(-ip_\mu\gamma^\mu+m)/(p^2+m^2)$ of the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ Dirac field, the term $-ip_\mu\gamma^\mu/(p^2+m^2)$ comes from terms like (4.1) with A_1,B_1 and A_2,B_2 equal to $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ or $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$, while the term $m/(p^2+m^2)$ comes from terms with A_1,B_1 and A_2,B_2 equal to $(\frac{1}{2},0)$ and $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ and $(0,\frac{1}{2})$.

Referring back to the definitions (2.10)–(2.13) of the fields ψ and $\tilde{\psi}$, we find for the Wightman functions in (4.1) the values

$$\langle\psi_{a_1b_1}^{(A_1B_1)}(x)\tilde{\psi}_{a_2b_2}^{(A_2B_2)}(y)\rangle_0 = \int \frac{d^3p\; e^{ip\cdot(x-y)}}{(2\pi)^3 2\omega(\mathbf{p})} \pi_{a_1b_1,a_2b_2}^{(A_1B_1,A_2B_2)}\left(\frac{\mathbf{p}}{m};\, j\right), \quad (4.2)$$

$$\langle\tilde{\psi}_{a_2b_2}^{(A_2B_2)}(y)\psi_{a_1b_1}^{(A_1B_1)}(x)\rangle_0 = (-)^{2B_1+2B_2+2j} \int \frac{d^3p\; e^{ip\cdot(y-x)}}{(2\pi)^3 2\omega(\mathbf{p})} \pi_{a_1b_1,a_2b_2}^{(A_1B_1,A_2B_2)}\left(\frac{\mathbf{p}}{m};\, j\right), \quad (4.3)$$

$$\pi_{a_1b_1,a_2b_2}^{(A_1B_1,A_2B_2)}(\mathbf{p}/m;\, j) \equiv (-)^{2B_2} \sum_{\lambda a_1' a_2' b_1' b_2'} (-)^{j+\lambda} \left(\frac{|\mathbf{p}|+\omega(\mathbf{p})}{m}\right)^{b_1'-a_1'+b_2'-a_2'} D_{a_1a_1'}^{(A_1)}[R(\hat{p})] D_{b_1b_1'}^{(B_1)}[R(\hat{p})] \times D_{a_2a_2'}^{(A_2)}[R(\hat{p})] D_{b_2b_2'}^{(B_2)}[R(\hat{p})] C_{A_1B_1}(j\lambda;\, a_1'b_1') C_{A_2B_2}(j-\lambda;\, a_2'b_2'). \quad (4.4)$$

The Green's function (4.1) is then

$$\langle T\{\psi_{a_1b_1}^{(A_1B_1)}(x),\tilde{\psi}_{a_2b_2}^{(A_2B_2)}(y)\}\rangle_0 = \int \frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})} \pi_{a_1b_1,a_2b_2}^{(A_1B_1,A_2B_2)}(\mathbf{p}/m;\, j) \times [\theta(x-y)e^{ip\cdot(x-y)} + (-)^{2B_1+2B_2}\theta(y-x)e^{ip\cdot(y-x)}]. \quad (4.5)$$

We will show in Sec. V that π is the value on the mass shell of a *polynomial* N, i.e.,

$$\pi_{a_1b_1,a_2b_2}^{(A_1B_1,A_2B_2)}(\mathbf{p}/m;\, j) = N_{a_1b_1,a_2b_2}^{(A_1B_1,A_2B_2)}(\mathbf{p},\omega(\mathbf{p});\, j,m), \quad (4.6)$$

where $N(p;\, j,m)$ is a polynomial in the general four-vector p^μ, with

$$N_{a_1b_1,a_2b_2}^{(A_1B_1,A_1B_2)}(-p;\, j,m) = (-)^{2B_1+2B_2} N_{a_1b_1,a_2b_2}^{(A_1B_1,A_2B_2)}(p;\, j,m). \quad (4.7)$$

[9] G. Feinberg and S. Weinberg, Nuovo Cimento 14, 571 (1959).

Hence (4.5) may be written as

$$\langle T\{\psi_{a_1b_1}{}^{(A_1B_1)}(x),\tilde{\psi}_{a_2b_2}{}^{(A_2B_2)}(y)\}\rangle_0=\int\frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})}[\theta(x-y)N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(-i\partial/\partial x;\,j,m)e^{ip\cdot(x-y)}$$
$$+\theta(y-x)N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(-i\partial/\partial x;\,j,m)e^{-ip\cdot(x-y)}].\quad(4.8)$$

This propagator would be Lorentz-covariant were it not for the θ functions. Let us therefore define a new Green's function by moving the differential operators N to the left of the θ functions:

$$\langle T^*\{\psi_{a_1b_1}{}^{(A_1B_1)}(x),\tilde{\psi}_{a_2b_2}{}^{(A_2B_2)}(y)\}\rangle_0\equiv N_{a_1b,{}_1a_2b_2}{}^{(A_1B_1,A_2B_2)}(-i\partial/\partial x;\,j,m)$$
$$\times\int\frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})}[\theta(x-y)e^{ip\cdot(x-y)}+\theta(y-x)e^{-ip\cdot(x-y)}].\quad(4.9)$$

The integral here is a scalar, so that (4.9) has the correct Lorentz-transformation properties to give an invariant S matrix. But (4.9) is not equal to (4.8), the difference involving noncovariant derivatives of the θ functions in (4.9). Thus (4.8) consists of a covariant part (4.9), plus noncovariant terms involving $\delta(x^0-y^0)$ and its derivatives. Since these noncovariant terms are temporally local, they can be cancelled by adding noncovariant terms to the interaction Hamiltonian density (which has to be temporally local by its definition in the interaction representation). When this is done, the effective propagator is the covariant (4.9).

Note that if we had defined the field $\psi^{(AB)}$ as

$$\psi_{ab}{}^{(AB)}(x)=\psi_{ab}{}^{(AB,+)}(x)+\xi(A,B)\psi_{ab}{}^{(AB,-)}(x)\,,$$

then the negative-frequency terms in (4.8), and hence (4.9), would appear multiplied with an extra factor

$$\xi(A_1B_1)\xi^*(A_2B_2)(-)^{2B_1+2B_2}.$$

But the propagator (4.9) would not be covariant unless this factor were unity, i.e., unless

$$\xi(AB)=\xi(-)^{2B},\quad|\xi|=1.$$

The phase factor ξ can be absorbed into the definition of the antiparticle operator $b^\dagger$; thus Lorentz invariance forces us to define the fields $\psi^{(AB)}$ as in Eq. (2.11), with $\xi(A,B)=(-)^{2B}$.

Returning now to our calculation, we find for the covariant propagator in momentum space

$$\Delta_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q)$$
$$\equiv i\int d^4x\,e^{-iq\cdot(x-y)}\langle T^*\{\psi_{a_1b_1}{}^{(A_1B_1)}(x),\tilde{\psi}_{a_2b_2}{}^{(A_2B_2)}(y)\}\rangle_0$$
$$=N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q;\,j,m)/(q^2+m^2-i\epsilon)\,.$$

It remains to calculate the polynomial N.

V. CALCULATION OF THE NUMERATOR POLYNOMIAL

In order to display the quantity π given by (4.4) as the value on the mass shell of a polynomial N, we recouple the A's and B's; instead of coupling (A_1,a_1') and (B_1,b_1') to form (j,λ) and (A_2,a_2') and (B_2,b_2') to form $(j,-\lambda)$, we will couple (A_1,a_1') and $(A_2,a_2')$to form (n,ν) and couple (B_1,b_1') and $(B_2,b_2{}^{1\prime})$ to form $(n,-\nu)$. To do this we use the sum rules[5]

$$\sum_\lambda C_{A_1B_1}(j\lambda;\,a_1'b_1')C_{A_2B_2}(j-\lambda;\,a_2'b_2')(-)^{j+\lambda}$$
$$=(2j+1)\sum_{n\nu}(-)^{A_1+B_2+j-\nu}W(A_1B_1A_2B_2;\,jn)C_{A_1A_2}(n\nu;\,a_1'a_2')C_{B_1B_2}(n-\nu\;b_1'b_2')\,,$$

$$\sum_{a_1'a_2'}C_{A_1A_2}(n\nu;\,a_1'a_2')D_{a_1a_1'}{}^{(A_1)}[R(\hat{p})]D_{a_2a_2'}{}^{(A_2)}[R(\hat{p})]=\sum_{\nu'}C_{A_1A_2}(n\nu';\,a_1a_2)D_{\nu'\nu}{}^{(n)}[R(\hat{p})],$$

where W is the usual Racah coefficient. Equation (4.4) then becomes

$$\pi_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(\mathbf{p}/m;\,j)=(2j+1)\sum_{n\nu'\nu''}W(A_1B_1A_2B_2;\,jn)$$
$$\times(-)^{A_1-B_2-j+\nu''}m^{-2n}C_{A_1A_2}(n\nu';\,a_1a_2)C_{B_1B_2}(n\nu'';\,b_1b_2)\Pi_{\nu',-\nu''}{}^{(n)}(\mathbf{p},\omega(\mathbf{p}))\,,\quad(5.1)$$

with $\Pi^{(n)}$ defined for a general four-vector q^μ by

$$\Pi_{\nu',\nu''}{}^{(n)}(q)=\sum_\nu(-)^{\nu''-\nu}(-q^2)^{n+\nu}(q^0+|\mathbf{q}|)^{-2\nu}D_{\nu',\nu}{}^{(n)}[R(\hat{q})]D_{-\nu'',-\nu}{}^{(n)}[R(\hat{q})].\quad(5.2)$$

The point of the recoupling is that $\Pi^{(n)}(p)$ transforms according to the (n,n) representation of the homogeneous

Lorentz group, and is therefore proportional to the traceless part of the tensor $q_{\mu_1}\cdots q_{\mu_{2n}}$. In fact, Eq. (5.2) show that $\Pi^{(n)}$ is just the same as the homogeneous polynomial calculated in earlier work,[1] of the form

$$\Pi_{\nu',\nu''}{}^{(n)}(q)=(-)^{2n}t_{\nu',\nu''}{}^{\mu_1\cdots\mu_{2n}}q_{\mu_1}\cdots q_{\mu_{2n}}, \tag{5.3}$$

and given for integer n by

$$\Pi^{(n)}(q)=(-q^2)^n+\sum_{m=0}^{n-1}\frac{(-q^2)^{n-1-m}}{(2m+2)!}(2\mathbf{q}\cdot\mathbf{J})[(2\mathbf{q}\cdot\mathbf{J})^2-(2\mathbf{q})^2]$$
$$\times[(2\mathbf{q}\cdot\mathbf{J})^2-(4\mathbf{q})^2]\cdots[(2\mathbf{q}\cdot\mathbf{J})^2-(2m\mathbf{q})^2][2\mathbf{q}\cdot\mathbf{J}-(2m+2)q^0], \tag{5.4}$$

and for half-integer n by

$$\Pi^{(n)}(q)=(-q^2)^{n-1/2}(q^0-2\mathbf{q}\cdot\mathbf{J})+\sum_{m=1}^{n-1/2}\frac{(-q^2)^{n-m-1/2}}{(2m+1)!}$$
$$\times[(2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][(2\mathbf{q}\cdot\mathbf{J})^2-(3\mathbf{q})^2]\cdots[(2\mathbf{q}\cdot\mathbf{J})^2-((2m-1)\mathbf{q})^2][(2m+1)q^0-\mathbf{q}\cdot\mathbf{J}]. \tag{5.5}$$

Thus the polynomial N which reduces to (5.1) on the mass shell is given by

$$N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q;\,j,m)=(2j+1)\sum_{n\nu'\nu''}W(A_1B_1A_2B_2;\,jn)$$
$$\times(-)^{A_1-B_2-j+\nu''}m^{-2n}C_{A_1A_2}(n\nu';\,a_1a_2)C_{B_1B_2}(n\nu'';\,b_1b_2)\Pi_{\nu',-\nu''}{}^{(n)}(q), \tag{5.6}$$

with Π determined by (5.3) or by (5.4) and (5.5). We note that $\Pi^{(n)}(-q)=(-)^{2n}\Pi^{(n)}(q)=(-)^{2B_1+2B_2}\Pi^{(n)}(q)$, so that N has the promised reflection property (4.7).

With Eq. (5.6) our main task is complete, but it is interesting to express N back in terms of the spin sums π with mass $\sqrt{(-q^2)}$. Since (5.1) holds for any mass m, we can set $\mathbf{p}=\mathbf{q}$ and $m=\sqrt{(-q^2)}$, and find

$$\pi_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(\mathbf{q}/\sqrt{-q^2};\,j)=(2j+1)\sum_{n\nu'\nu''}W(A_1B_1A_2B_2;\,jn)$$
$$\times(-)^{A_1-B_2-j+\nu''}(-q^2)^{-n}C_{A_1A_2}(N\nu';\,a_1a_2)C_{B_1B_2}(n\nu'';\,b_1b_2)\Pi_{\nu',-\nu''}{}^{(n)}(q). \tag{5.7}$$

We may now use the orthogonality property of the Racah coefficients

$$(2n+1)\sum_j(2j+1)W(A_1B_1A_2B_2;\,jn)W(A_1B_1A_2B_2;\,jn')=\delta_{nn'}$$

and solve (5.7) for the term of nth order in q:

$$(-q^2)^{-n}\sum_{\nu'\nu''}(-)^{-B_1-B_2+\nu''}C_{A_1A_2}(n\nu',a_1a_2)C_{B_1B_2}(n\nu'';\,b_1b_2)\Pi_{\nu',-\nu''}{}^{(n)}(q)$$
$$=(2n+1)\sum_j(-)^{j+A_1+B_1}W(A_1B_1A_2B_2;\,jn)\pi_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(\mathbf{q}/\sqrt{-q^2};\,j)$$

Putting this back into (5.6), we find that

$$N_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(q;\,j,m)=\sum_{j'}F_{jj'}(-q^2/m^2;\,A_1B_1A_2B_2)\pi_{a_1b_1,a_2b_2}{}^{(A_1B_1,A_2B_2)}(\mathbf{q}/\sqrt{-q^2};\,j'), \tag{5.8}$$

where F is a polynomial in $\sqrt{(-q^2/m^2)}$:

$$F_{jj'}(x;\,A_1B_1A_2B_2)=(-)^{j'-j}(2j+1)\sum_n(2n+1)x^nW(A_1B_1A_2B_2;\,jn)W(A_1B_1A_2B_2;\,j'n). \tag{5.9}$$

Thus the propagator for a particle of spin j behaves off the mass shell as if it described a range of spins j', with relative weights given by the polynomial $F_{jj'}$.

Note added in proof. Similar topics are treated, in a different formalism, by M. D. Scadron, Phys. Rev. **165**, 1640 (1968).