Non-Abelian Anyons in the Quantum Hall Effect

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Outline

- Incompressible Hall fluids: bulk & edge excitations
- CFT description
- Non-Abelian statistics
- Partition function
- Signatures of non-Abelian statistics:
  - Coulomb blockade & thermopower
Quantum Hall Effect

- 2 dim electron gas at low temperature $T \sim 10$ mK and high magnetic field $B \sim 10$ Tesla

- Conductance tensor

\[ J_i = \sigma_{ij} E_j, \quad \sigma_{ij} = R_{ij}^{-1}, \quad i, j = x, y \]

- Plateaux: $\sigma_{xx} = 0$, $R_{xx} = 0$  
  
  \[ \sigma_{xy} = R_{xy}^{-1} = \frac{e^2}{h} \nu, \quad \nu = 1(\pm 10^{-8}), 2, 3, \ldots, \frac{1}{3}, \frac{2}{5}, \ldots, \frac{5}{2}, \]

- High precision & universality

- Uniform density ground state:

\[ \rho_o = \frac{eB}{hc} \nu \]

Incompressible fluid
Laughlin's quantum incompressible fluid

Electrons form a droplet of fluid:
- incompressible = gap
- fluid = \( \rho(x, y) = \rho_o = \text{const.} \)

\[ D_A = BA/\Phi_o, \quad \# \text{degenerate orbitals} = \# \text{quantum fluxes}, \quad \Phi_o = \frac{hc}{e} \]

filling fraction:
\[ \nu = \frac{N}{D_A} = 1, 2, \ldots \frac{1}{3}, \frac{1}{5}, \ldots \]

density for quantum mech.

\[ \nu = 1 \quad \nu = \frac{1}{3} \]
Laughlin's wave function

\[ \Psi_{gs}(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^{2k+1} e^{-\sum |z_i|^2/2} \quad \nu = \frac{1}{2k+1} = 1, \frac{1}{3}, \frac{1}{5}, \ldots \]

- \( \nu = 1 \) filled Landau level: obvious gap \( \omega = \frac{eB}{mc} \gg kT \)
- \( \nu = \frac{1}{3} \) non-perturbative gap due to Coulomb interaction

\[ \Psi_{\eta} = \prod_i (\eta - z_i) \Psi_{gs} \]

**effective theories**

- **quasi-hole = elementary vortex**
- **fractional charge** \( Q = \frac{e}{2k+1} \) & **statistics** \( \frac{\theta}{\pi} = \frac{1}{2k+1} \)

\[ \Psi_{\eta_1,\eta_2} = (\eta_1 - \eta_2)^{2k+1} \prod_i (\eta_1 - z_i) (\eta_2 - z_i) \Psi_{gs} \]

**Anyons** vortices with long-range topological correlations
Conformal field theory of edge excitations

The edge of the droplet can fluctuate: edge waves are massless

- edge \sim Fermi surface: linearize energy \quad \varepsilon(k) = \frac{v}{R}(k - k_F), \quad k = 0, 1, \ldots
- relativistic field theory in 1+1 dimensions, chiral \quad (X.G.Wen '89)

chiral compactified c=1 CFT \quad (chiral Luttinger liquid)
CFT descriptions of QHE

- same function by analytic continuation from the circle:
  - both equivalent to Chern-Simons theory in 2+1 dim (Witten '89)

- c=1 Luttinger CFT:
  - wavefunctions: spectrum of anyons and braiding
  - edge correlators: physics of conduction experiments
Non-Abelian fractional statistics

- $\nu = \frac{5}{2}$ described by Moore-Read “Pfaffian state” ~ Ising CFT x boson
- Ising fields: $I$ identity, $\psi$ Majorana = electron, $\sigma$ spin = anyon
- fusion rules:
  - $\psi \cdot \psi = I$  
    2 electrons fuse into a Bosonic bound state
  - $\sigma \cdot \sigma = I + \psi$  
    2 channels of fusion = 2 conformal blocks
  \[
  \langle \sigma(0)\sigma(z)\sigma(1)\sigma(\infty) \rangle = a_1 F_1(z) + a_2 F_2(z)
  \]
  Hypergeometric
  state of 4 anyons is two-fold degenerate  
  (Moore, Read '91)
- statistics of anyons ~ analytic continuation  
  2x2 matrix

\[
\left( \begin{array}{c}
F_1 \\
F_2
\end{array} \right) \left( ze^{i2\pi} \right) = \left( \begin{array}{cc}
1 & 0 \\
0 & -1
\end{array} \right) \left( \begin{array}{c}
F_1 \\
F_2
\end{array} \right) (z)
\]

\[
\left( \begin{array}{c}
F_1 \\
F_2
\end{array} \right) \left( (z - 1)e^{i2\pi} \right) = \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right) \left( \begin{array}{c}
F_1 \\
F_2
\end{array} \right) (z)
\]

(all CFT redone for Q. Computation: M. Freedman, Kitaev, Nayak, Slingerland,....., 00'-10' )
Topological quantum computation

- qubit = two-state system \( |\chi\rangle = \alpha |0\rangle + \beta |1\rangle \)
- QC: perform \( U(2^n) \) unitary transformations in \( n \) qubit Hilbert space
- Proposal: (Kitaev; M. Freedman; Nayak; Simon; Das Sarma '06)

\[
2^n - 1
\]

use non-Abelian anyons for qubits and operate by braiding

- 4-spin system \( \alpha |F_1\rangle + \beta |F_2\rangle \) is 1 qubit (2n-spin has dim \( 2^{n-1} \))
- anyons topologically protected from decoherence (local perturbations)
- more stable but more difficult to create and manipulate

\[ \text{great opportunity} \]

\[ \text{new experiments and model building} \]
Models of non-Abelian statistics

- Study Rational CFTs with non-Abelian excitations:
  - best candidate: Pfaffian & its generalization, the Read-Rezayi states
    \[ \nu = 2 + \frac{k}{k+2}, \quad \{ \frac{k}{M} = 2, 3, \ldots \} \quad \frac{SU(2)_k}{U(1)_{2k}} \]
  - alternatives: other (cosets of) non-Abelian affine groups \( U(1) \times \frac{G}{H} \)

- Identify their N sectors of fractional charge and statistics
  - Abelian (electron) & non-Abelian (quasi-particles)

- Compute physical quantities that could be signatures of non-Abelian statistics:
  - Coulomb blockade conductance peaks
  - thermopower & entropy
use partition function

- quantity defining Rational CFT \((\text{Cardy '86; many people})\)
- complete inventory of states (bulk & edge)
- **modular invariance** as building principle:
  - \(S\) matrix and fusion rules
  - further modular conditions for charge spectrum
  - straightforward solution for any non-Abelian state
  - useful to compute physical quantities
- Inputs:
  - non-Abelian RCFT (i.e. \(\frac{G}{H}\))
  - Abelian field representing the electron \(\text{"simple current"}\)
- Output is unique

\[ U(1) \times \frac{G}{H} \]
**Annulus partition function**

\[ i2\pi \tau = -\beta \frac{v}{R} + it, \quad \beta = \frac{1}{k_B T} \]

\[ i2\pi \zeta = \beta (-V_0 + i\mu) \]

\[ Z_{\text{annulus}} = \sum_{\lambda=1}^{p} |\theta_\lambda(\tau, \zeta)|^2, \quad \theta_\lambda(\tau, \zeta) = \text{Tr}_{\mathcal{H}(\lambda)} \left[ e^{i2\pi \tau (L_0 - c/24) + i2\pi \zeta Q} \right] \]

**modular invariance conditions**

**geometrical properties & physical interpretation**  
(A. C., Zemba, '97)

\[ T^2 : Z(\tau + 2, \zeta) = Z(\tau, \zeta), \quad L_0 - \overline{L}_0 = \frac{n}{2} \quad \text{half-integer spin excitations globally} \]

\[ S : Z \left( \frac{-1}{\tau}, \frac{-\zeta}{\tau} \right) = Z(\tau, \zeta), \quad \text{completeness} \quad \theta_\lambda \left( \frac{-1}{\tau} \right) = \sum_{\lambda'} S_{\lambda\lambda'} \theta_{\lambda'}(\tau) \quad \text{S matrix} \]

\[ U : Z(\tau, \zeta + 1) = Z(\tau, \zeta), \quad Q - \overline{Q} = n \quad \text{integer charge excitations globally} \]

\[ V : Z(\tau, \zeta + \tau) = Z(\tau, \zeta), \quad \Delta Q = \nu \quad \text{add one flux: spectral flow} \]

\[ \theta_\lambda(\zeta + \tau) \sim \theta_{\lambda+1}(\tau) \]
Disk partition function

Annulus -> Disk (w. bulk q-hole $\overline{Q} = \frac{\lambda}{p}$)

$Z_{\text{annulus}} \rightarrow Z_{\text{disk}}, \lambda = \theta_{\lambda}(\tau, \zeta)$

$$\theta_{\lambda}(\tau, \zeta) = K_{\lambda}(\tau, \zeta; p) = \frac{1}{\eta} \sum_n e^{i2\pi \left[ \frac{\tau(n^p+\lambda)^2}{2p} + \zeta \frac{np+\lambda}{p} \right]}, \quad \nu = \frac{1}{p}, \quad c = 1$$

- $U : Q - \overline{Q} = n$ sectors with charge $Q = \frac{\lambda}{p} + n$
  basic quasiparticle + n electrons

- $T^2$ : electrons have half-integer dimension (=J),
  and integer relative statistics with all excitations

- $\#$ sectors $p = \text{dim} (S_{\lambda\lambda'}) = \text{Wen's topological order}$

we recover phenomenological conditions on the spectrum
Pfaffian & Read-Rezayi states

\[ Z_{\text{annulus}}^{RR} = \sum_{\ell=0}^{k} \sum_{a=0}^{\hat{p}-1} | \theta^\ell_a(\tau, \zeta) |^2, \quad Z_{\text{disk}}^{RR} = \theta^\ell_a = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)} \chi_{a+2\beta} \]

**Ex: Pfaffian (k=2)**

- Ground state + electrons

\[ Z_{\text{annulus}}^{\text{Pfaffian}} = |K_0I + K_4\psi|^2 + |K_0\psi + K_4I|^2 + |(K_1 + K_{-3}) \sigma|^2 \]

\[ + |K_2I + K_{-2}\psi|^2 + |K_2\psi + K_{-2}I|^2 + |(K_3 + K_{-1}) \sigma|^2 \]

- \( K_\lambda \quad \text{charge parts} \quad Q = \frac{\lambda}{4} + 2n \)
- \( I, \psi, \sigma \quad \text{Ising parts (Majorana fermion)} \)
  - 6 sectors
  - Also \( Q = 0, \pm \frac{1}{2} \quad \text{Abelian excitations} \)
• charge and neutral q. #'s are coupled by “parity rule”
• but S-matrix of $\theta^l_a$ is factorized $S_{a\ell,a'\ell'} \sim e^{i2\pi aa'/N/M} s_{\ell\ell'}$

• generalization to other N-A models: (A.C, G. Viola, ’10)
  – Non-Abelian Fluids $U(1) \times SU(2)_k$
  – Anti-Read-Rezayi $U(1) \times SU(2)_k$
  – Bonderson-Slingerland $U(1) \times \text{Ising} \times SU(n)_1$
  – N-A Spin Singlet state $U(1)_q \times U(1)_s \times \frac{SU(3)_k}{U(1)^2}$

unique result once RCFT and electron field have been chosen
Experiments on non-Abelian statistics

- (a) interference of edge waves  
  → Aharonov-Bohm phase, checks fractional statistics  
  - experiment is hard

- (b) electron tunneling into the droplet
  → Coulomb blockade conductance peaks
  - check quasi-particle sectors

- Thermopower

(Chamon et al. '97; Kitaev et al. '06)  
(Goldman et al. '05; Willett et al. '09)  
(Stern, Halperin '06)  
(Ilan, Grosfeld, Schoutens, Stern '08)  
(Stern et al.; A.C. et al. '09 - '10)  
(Cooper, Stern; Yang, Halperin '09; Chickering et al. '10)
Thermopower

- **fusion of** $n$ non-Abelian quasiparticles:
  - multiplicity $\sim (d_\ell)^n$, $n \to \infty$
- **Entropy:** $S(T = 0) \sim n \log(d_\ell)$,  
  
  $d_\ell = \frac{s_{\ell 0}}{s_{00}} > 1$  
  
  (quantum dimension)

from $Z$:  
  
  $S = \left(1 - \tau \frac{d}{d\tau}\right) \log \left[\theta^\ell_a(\tau, \zeta)\theta^0_0(\tau, \zeta)\right] \sim \log \frac{s_{\ell 0}}{s_{00}} + S_{\text{Edge}}, \quad \tau \sim \frac{\beta}{R} \to 0$

- put temperature $\nabla T$ and potential $\nabla V_o$ gradients between the edges
- at equilibrium:  
  
  $J = -\sigma \ \nabla V_o - \alpha \ \nabla T = 0$

- **thermopower**

  $S_{\text{Seebeck}} = \frac{\alpha}{\sigma} = -\frac{\nabla V_o}{\nabla T} = \frac{S}{eN_e}$  
  
  (Cooper, Stern; Yang, Halperin '09)

- it could be observable by varying $B$ off the plateau center

  $S_{\text{Seebeck}} = \left|\frac{B - B_o}{e^* B_0}\right| \log(d_1)$  
  
  (Chickering et al '10)
Coulomb blockade

- Droplet capacity stops the electron
- Bias & T \sim 0: needs energy matching
  \[ E(n + 1, S) = E(n, S) \]
  current peak
- energy deformation by \( \Delta S \sim \Delta Q_{\text{bkg}} \)

\[
E(n, S) = \frac{v}{R} \frac{(\lambda + pn - \sigma)^2}{2p} \propto (Q - Q_{\text{bkg}})^2
\]

\[
\Delta \sigma = \frac{B \Delta S}{\Phi_0} = \frac{1}{\nu}, \quad \Delta S = \frac{e}{n_o}
\]

\[
U(1) \quad \text{equidistant peaks}
\]
\[
U(1) \times \frac{G}{H} \quad \text{modulated pattern}
\]

\[
\Delta \sigma_m^\ell = \frac{1}{\nu} + \frac{v_n}{v} (h_m^{\ell+2} - 2h_m^{\ell} + h_m^{\ell-2})
\]

\[
\frac{v_n}{v} \sim \frac{1}{10}
\]
\( \theta^\ell_a = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)} \chi^\ell_{a+2\beta} \)

- compares states in the same sector
- spectroscopy of lowest CFT states
- \( T = 0 \): cannot distinguish NA state from “parent” Abelian state

\( T > 0 \) corrections

\[ \langle Q \rangle_T \sim \frac{\partial}{\partial V_0} \log \theta^\ell_a \]

- two scales:
  \[ 0 < T_n < T_{ch}, \quad T_n = \frac{v_n}{R}, \quad T_{ch} = \frac{v}{R} \sim 10 \ T_n \]

\( T < T_n : \quad \Delta \sigma^\ell_m = \cdots + \frac{T}{T_{ch}} \log \left( \frac{(d^\ell_m)^2}{d^\ell_{m+2} d^\ell_{m-2}} \right) \), \( d^\ell_m \) multiplicity of neutral states in (331) & Anti-Pfaff, not in Pfaff

\( T_n < T < T_{ch} : \quad \cdots + \propto \frac{T}{T_{ch}} e^{-\frac{h^1T}{T_n}} \frac{s_{\ell 1}}{s_{\ell 0}} \), \( S \) matrix of non-Abelian part

\[ \text{test non-Abelian part of disk partition function} \]

(Bonderson et al. '10)

(Stern et al., Georgiev, AC et al. '09, '10)
$T > 0$ \textbf{off-equilibrium}

- energy offset $\Delta E_\sigma > 0$ and bias $\Delta V_o > 0$

  relevant regime: $T < \Delta V_o < \Delta E_\sigma$

\textbf{thermal-activated Coulomb-Blockade conduction}

$$\Gamma \sim d_a^\ell (\Delta E_\sigma^2 - \Delta V_o^2) e^{-\beta(\Delta E_\sigma - \Delta V_o)}$$

- real-time experiment of peak counting
- sensible to level multiplicity $d_a^\ell$ (but qualitative)
Conclusions

- non-Abelian anyons could be seen
- **partition function:**
  - it is simple enough
  - it defines the CFT, its sectors, fusion rules etc.
  - it is useful to compute observables
  - it can be the basis for further model building