Multipole Expansion in the Quantum Hall Effect

Andrea Cappelli
(INFN and Physics Dept., Florence)
with E. Randellini (Florence)

Outline

- Chern-Simons effective action: bulk and edge
- Wen-Zee term: shift and Hall viscosity
- Incompressible fluids and W-infinity symmetry
- 1/B expansion, higher-spin fields, coupling to gravity
- Universal and non-universal effects

Chern-Simons effective action

$$S[A] = \frac{\nu}{4\pi}\int AdA = \frac{\nu}{4\pi}\int \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$ Laughlin state $\nu = \frac{1}{p}$

$$\rho = \frac{\delta S}{\delta A_0} = \frac{\nu}{2\pi}\mathcal{B} = \frac{\nu}{2\pi}\left(B + \delta B(x)\right)$$ Density

$$J^i = \frac{\delta S}{\delta A_i} = \frac{\nu}{2\pi}\varepsilon^{ij}\mathcal{E}^j$$ Hall current

Introduce Wen's hydrodynamic matter field a_μ and current $j^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu a_\rho$

$$S[A] = \int \rho_0 A_0 + \int -\frac{\gamma}{2} ada + A \cdot j \qquad \sim \int -ada + Ada \quad \Rightarrow \quad \gamma = \frac{2\pi}{\nu}$$

- Hall current is topological
- Sources of a_μ field are anyons
- Needs boundary action $S_b[\varphi]$, $A|_b = \partial\varphi$ ⇒ massless edge states
- Bulk topological theory is tantamount to conformal field theory on boundary

  ⇒ universal transport coeff. $\sigma_H = \frac{\nu}{2\pi}$

Wen-Zee-Fröhlich action

- Add spatial metric background g_{ij} and coupling to $O(2)$ spin connection ω_μ

$$g_{ij} = e_i^a e_j^a, \quad \omega_\mu^{ab} = \omega_\mu(e)\varepsilon^{ab}, \quad i,j,a,b = 1,2, \qquad \delta g_{ij} = \partial_i u_j + \partial_j u_i \qquad \text{strain}$$

$$S[A,g] = \frac{1}{2\pi}\int -\frac{1}{2\nu} ada + j\cdot(A + s\,\omega) = \frac{\nu}{4\pi}\int AdA + 2s\,Ad\omega + s^2\,\omega d\omega$$

$$\rho = \frac{\delta S}{A_0} = \frac{\nu}{2\pi}\left(B + \frac{s}{2}\mathcal{R}\right)$$

Wen-Zee shift

$$N = \nu N_\phi + \nu s \chi$$

$$T_{ij} = -2\frac{\delta S}{\delta g^{ij}} = \frac{\eta_H}{2}\varepsilon_{ik}\dot{g}_{jk} + (i \leftrightarrow j)$$

Hall viscosity

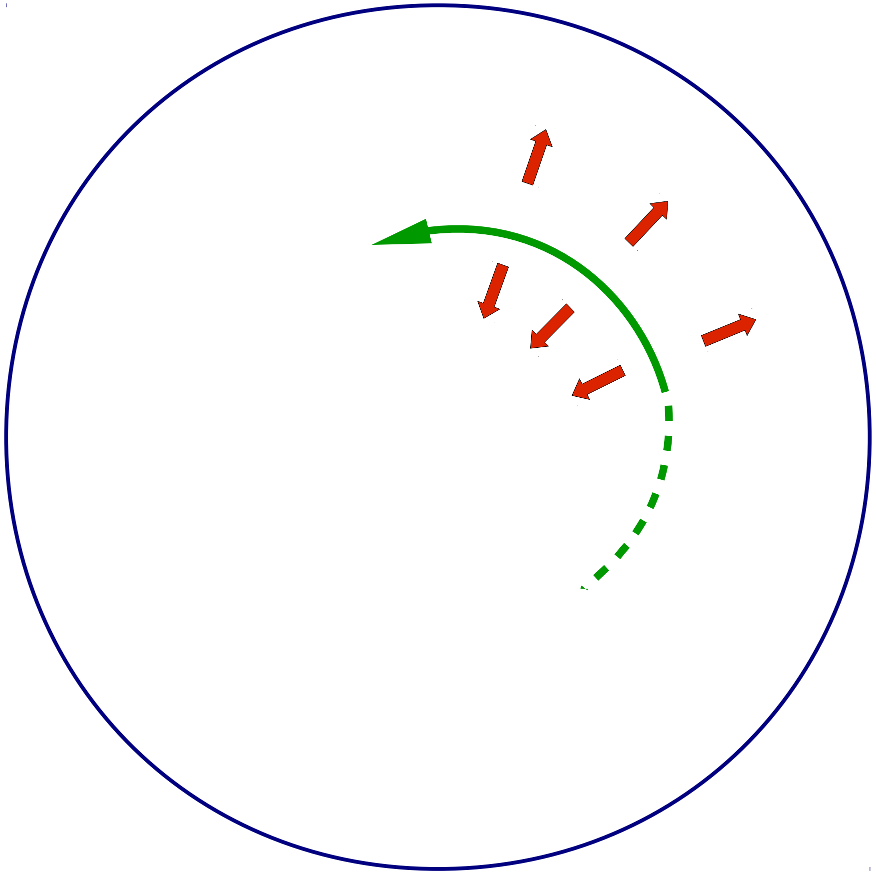
$$\eta_H = \frac{\rho_0 s}{2}$$

- η_H further universal transport coefficient (Avron et al., Read et al.)
- s intrinsic angular momentum, $s = \frac{p}{2}, \frac{2n-1}{2}$ on resp. Laughlin & n-th Landau L.
- Checks have been done and other quantities have been computed

(Abanov, Gromov et al.: Fradkin et al.; Read et al.; Son et al.; Wiegmann et al.)

Hall viscosity

$$T_{ij} = \frac{\eta_H}{2} \varepsilon_{(ik} \dot{g}_{j)k}$$



- Constant stirring creates an orthogonal static force, non dissipative

Wen-Zee-Fröhlich action

$$g_{ij} = e^a_i e^a_j, \quad \omega^{ab}_\mu = \omega_\mu(e)\varepsilon^{ab}, \quad i,j,a,b = 1,2, \qquad \delta g_{ij} = \partial_i u_j + \partial_j u_i$$ strain

$$S[A,g] = \frac{1}{2\pi}\int -\frac{1}{2\nu} ada + j\cdot(A + s\,\omega) = \frac{\nu}{4\pi}\int AdA + 2s\,Ad\omega + s^2\,\omega d\omega$$

$$\rho = \frac{\delta S}{A_0} = \frac{\nu}{2\pi}\left(B + \frac{s}{2}\mathcal{R}\right)$$

$S_{WZ}[A,g]$ $\qquad$ $S_{GRWZ}[g]$

(discuss it later)

$$T_{ij} = -2\frac{\delta S}{\delta g^{ij}} = \frac{\eta_H}{2}\varepsilon_{ik}\dot{g}_{jk} + (i \leftrightarrow j)$$

- S is invariant under time-independent diffeomorphisms only
- Hall viscosity vanishes for conformal metrics $\qquad g_{jk} = \sqrt{g}\,\delta_{jk}$

→ time-dep. area-preserving diffeomorphisms $\qquad \delta x^i = \varepsilon^{ij}\partial_j w(t,x), \quad \delta g = 0$

Another derivation based on the symmetry of

incompressible fluids under area-preserving diffs,

the W-infinity symmetry

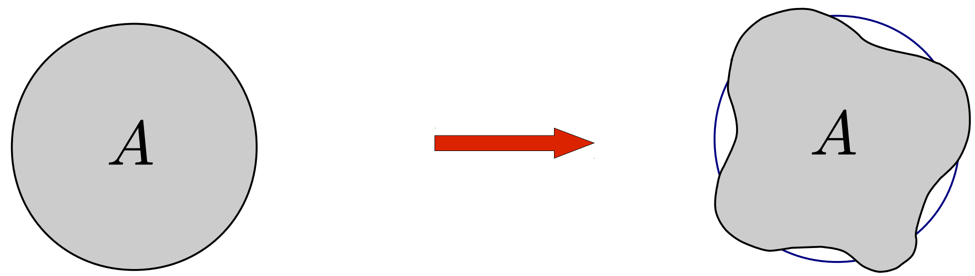
→ Hall states have excitations

of dipoles (Wen-Zee) and higher multipoles

Quantum incompressible fluids

- Area-preserving diffeomorphisms of incompressible fluids

$$\int d^2x \; \rho(x) = N = \rho_o A$$ → A = constant



- Fluctuations of the fluid are described by generators of the symmetry

classical $\delta\rho(z,\bar{z}) = \{\rho, w\}_P$ Poisson brackets $\delta z = \{z, w\}_P$

quantum $\delta\rho(z,\bar{z}) = i\langle\Omega|[\widehat{\rho},\widehat{w}]|\Omega\rangle = \{\rho, w\}_M$ Moyal $\rho(z,\bar{z}) = \langle\Omega|\,\widehat{\rho}\,|\Omega\rangle$

→ W_∞ algebra (in momentum space GMP sin-algebra)

- generators at the edge $z = Re^{i\theta}$ are higher-spin currents:

$$W^0 = \psi^\dagger\psi, \qquad W^1 = \psi^\dagger\partial_\theta\psi \sim H, \qquad W^2 = \psi^\dagger\partial_\theta^2\psi, \cdots$$

- CFT fully developed and matches Jain hierarchy: W_∞ minimal models

(A.C., Trugenberger, Zemba '96)

- Bulk fluctuations in lowest Landau level are non-local:

$$\delta\rho(z,\bar z) = i\langle\Omega|\,[\widehat{\rho},\widehat{w}]\,|\Omega\rangle = i\sum_{n=1}^{\infty}\frac{\hbar^n}{B^n n!}\left(\partial_{\bar z}^n\rho\,\partial_z^n w - \partial_{\bar z}^n w\,\partial_z^n\rho\right)$$

(Iso, Karabali, Sakita)

- can be expressed in terms of fields of increasing spin, traceless & symmetric

$$\delta\rho = \frac{i}{B}\partial_{\bar z}\left(\rho\partial_z w\right) + \frac{i}{2B^2}\partial_{\bar z}^2\left(\rho\partial_z^2 w\right) + \cdots + \text{h.c.}$$

$$= i\partial_{\bar z}\,a_z + \frac{i}{B}\partial_{\bar z}^2\,b_{zz} + \cdots + \text{h.c.}$$

- Recover Wen hydrodynamic field a_μ plus $\frac{1}{B}$ correction $b_{\mu k}$ $(\mu=0,1,2,\ k=1,2)$

$$a_\mu = (a_0, a_z, a_{\bar z}),\qquad b_{\mu k} = (b_{0z}, b_{0\bar z}, b_{zz}, b_{\bar z\bar z}, b_{\bar z z}, b_{z\bar z})$$

+ gauge symmetry

$$j^\mu = j^\mu_{(1)} + j^\mu_{(2)} + \cdots,\qquad j^\mu_{(1)} = \varepsilon^{\mu\nu\rho}\partial_\nu\,a_\rho,\qquad a_\rho \to a_\rho + \partial_\rho f$$

$$j^\mu_{(2)} = \frac{1}{B}\varepsilon^{\mu\nu\rho}\partial_\nu\partial_k\,b_{\rho k},\qquad b_{\rho k}\to b_{\rho k} + \partial_\rho v_k$$

- Expressions determined by current conservation and gauge symmetry

$$j^\mu = j^\mu_{(1)} + j^\mu_{(2)} + \cdots ,$$

$$j^\mu_{(1)} = \varepsilon^{\mu\nu\rho}\partial_\nu\, a_\rho, \qquad a_\rho \to a_\rho + \partial_\rho f$$

$$j^\mu_{(2)} = \frac{1}{B}\varepsilon^{\mu\nu\rho}\partial_\nu\partial_k\, b_{\rho k}, \quad b_{\rho k} \to b_{\rho k} + \partial_\rho v_k$$

- a_μ $(b_{\mu k})$ have 1 (2) degrees of freedom
- Assume Chern-Simons dynamics for the $b_{\mu k}$ field too (Gaberdiel et al.)

$$S[A] = S_{(1)}[A] + S_{(2)}[A] + \cdots$$

$$S_{(2)}\,[A] = \int -\frac{1}{B2\gamma} b_k d\, b_k + A \cdot j_{(2)} = -\frac{\gamma}{2B}\int (\Delta A)\, dA$$

$$b_k = b_{\mu k}\; dx^\mu$$

$\Rightarrow$ $O\left(\frac{k^2}{B}\right)$ correction to density and Hall conductivity (Hoyos, Son)

Multipole expansion

$$\delta\rho = \varepsilon^{ij}\partial_i a_j + \frac{1}{B}\varepsilon^{ij}\partial_i\partial_k b_{jk} + \cdots$$

$$\delta\rho_{charge} = q\delta(\vec{x}) \quad \Leftarrow \quad q = \oint dx^i a_i$$

$$\delta\rho_{dipole} = \frac{1}{B}\, p^k\partial_k\delta\left(\vec{x}\right) \quad \Leftarrow \quad p_k = \oint dx^i b_{ik}$$

Coupling to gravity

- Spin-two field allows independent coupling to the metric: the stress tensor is

$$t^{\mu k} = \varepsilon^{k\ell}\,\varepsilon^{\mu\nu\rho}\partial_\nu b_{\rho\ell}, \qquad \partial_\mu t^{\mu k} = 0, \qquad t^{jk} = -\dot{b}_{jk} + O(b_{0n})$$

- Stress tensor is conserved and symmetric in space indices (Non-Relativistic)

$$\delta\rho = \varepsilon^{ij}\partial_i a_j, \quad \delta Q = \int_D d^2x\,\delta\rho = \oint_{\partial D} dx^i a_i$$ net charge fluctuation at boundary

$$\delta P^k = \int_D d^2x\, t^{0k} = \varepsilon^{k\ell}\oint_{\partial D} dx^i b_{i\ell} = \varepsilon^{k\ell}u_\ell$$ net momentum fluctuation

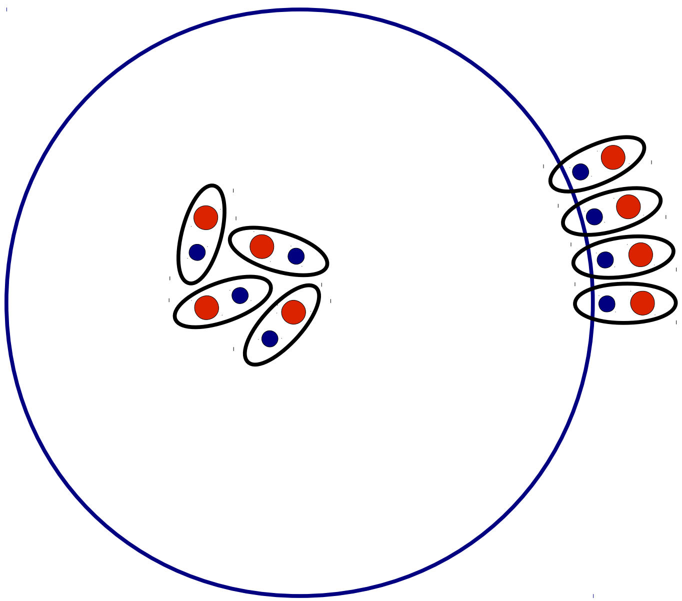
- Insert metric coupling in the second order action

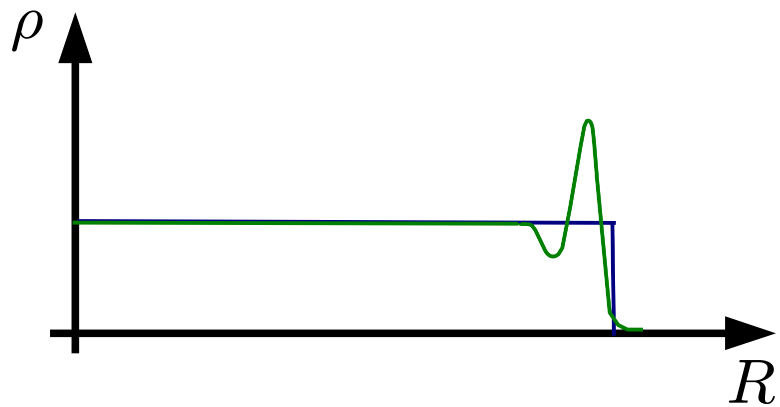
$$S_{(2)}[A,g] = \int -\frac{1}{B2\gamma} b_k d\, b_k + A\cdot j_{(2)} + \lambda g_{ij}t^{ij} = \frac{\nu s}{4\pi}\int -\frac{1}{B}\Delta A dA + 2Ad\omega$$

- Obtain: - earlier correction to $\sigma_H \sim B^{-1}$

  - Wen-Zee action (quadratic approx) $S_{WZ} = \frac{\nu s}{2\pi}\int A d\omega \sim B^0 + B^1$

Dipoles





- composite fermion = dipole of unbalanced charges: ● e, • h

➡ charge fluctuation at the boundary

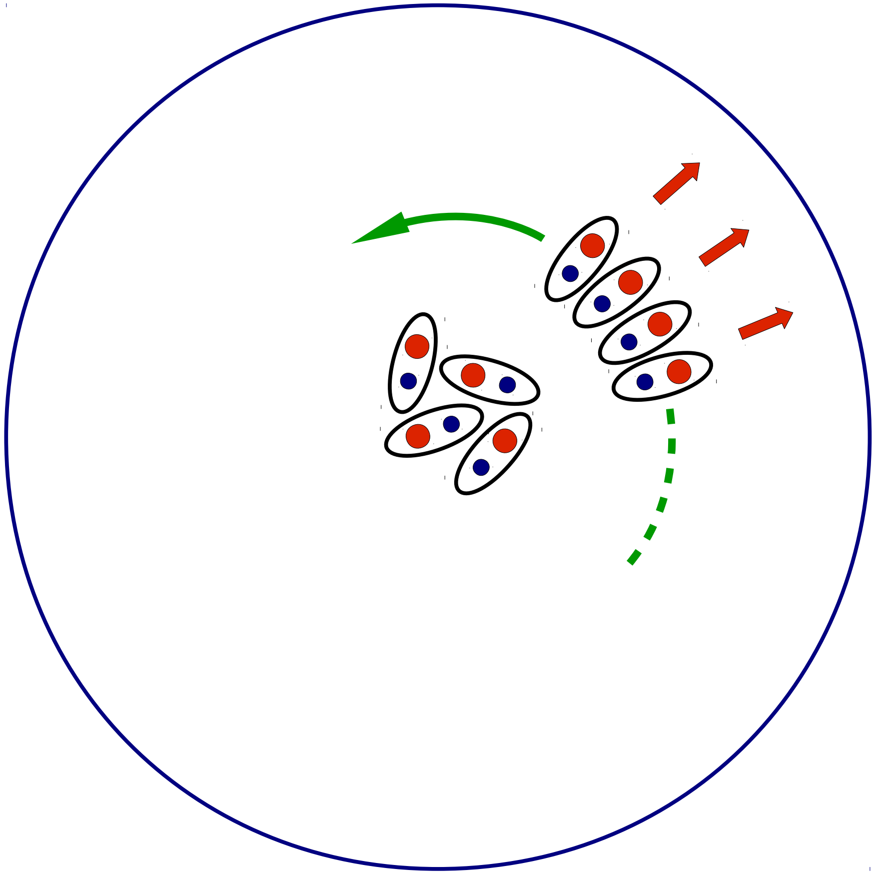
$$\int \rho(x)\, d^2x = N, \quad \int \frac{x^2}{\ell^2}\rho(x)\, d^2x = \frac{N^2}{2\,\nu} - N(s-1)$$

exact sum rule $\quad \nu = \frac{1}{p}, \; s = \frac{p}{2}$

$\frac{1}{N} \sim \frac{1}{B}$ correction from $S_{(2)}$

Hall viscosity

$$T_{ij} = \frac{\eta_H}{2}\varepsilon_{(ik}\dot{g}_{j)k}$$



- Stirring creates a local ordering of dipoles → inversion layer in the density

Wen-Zee vs. W-infinity coupling

- Wen-Zee interaction is the standard coupling of spin to gravity

$$S^{\mu}_{ab} = \bar{\psi}\gamma^{\mu}\frac{1}{4}[\gamma_a, \gamma_b]\,\psi, \qquad S^{\mu}_{ab}\,\omega^{ab}_{\mu} \;\to\; J^{\mu}\,\omega^{12}_{\mu}$$

D=2+1 & Non-Relativistic

- Minimal coupling of electrons is problematic in the lowest Landau Level

$$P^{i} = \frac{m}{e}J^{i} \;\to\; ?? \quad m \to 0$$

cf. Generalized Galilean symmetry

- W-infinity provides independent sources for P^{i} and J^{i} (and . . .)

$$S_{\mathrm{int.}} = \int A_i\, J^{i}(a,b,c,\cdots) + g_{ij}\,T^{ij}(b,c,\cdots) + \gamma_{ijk}\,S^{ijk}(c,\cdots) + \cdots$$

- spin equivalent to angular momentum to leading order (up to technicalities)
- Wen-Zee action is obtained to second order but there are higher terms

Universal and non-universal terms

$$S[a, b_k, c_{k\ell}] = -\frac{1}{4\pi\nu}\int ada + \frac{1}{sB}b_k db_k + \frac{1}{\alpha B^2}c_{k\ell}dc_{k\ell} + (A, g)\ \text{couplings}$$

$$\Longrightarrow \quad S[A, g] = \frac{\nu}{4\pi}\int \left(1 - s\frac{\Delta}{B} + \alpha\frac{\Delta^2}{B^2}\right) AdA + 2s\left(1 - \beta\frac{\Delta}{B}\right) Ad\omega$$

- Couplings ν, s, α of Chern-Simons actions are universal by matching to observables of CFT on boundary $S[a, b_k, c_{k\ell}] + \Delta S[a_k = \partial_k \phi,\ b_{kj} = \partial_k v_j, \cdots]$
- However $\frac{\Delta}{B}, \frac{\mathcal{R}}{B}$ terms are local corrections and can be altered at will

  ⟹ only first term in each series is universal
- Effective action is a bookkeeping method for disentangling universal and non-universal transport coefficients & quantities

Third order (in progress)

- W-infinity deformation suggests the spin-3 field, with 2 physical components

$$c_{\mu,k\ell} = \left(c_{0zz},\, c_{0\bar z\bar z},\, c_{zzz},\, c_{\bar z\bar z\bar z},\, c_{\bar z zz},\, c_{z\bar z\bar z}\right)$$

- corrections to electromagnetic current and stress tensor (not-unique)

$$j^{\mu}_{(3)} = \frac{1}{B^2}\varepsilon^{\mu\nu\rho}\partial_\nu\left(\partial_k\partial_\ell\, c_{\rho k\ell}\right), \qquad c_{\rho k\ell} \;\to\; c_{\rho k\ell} + \partial_\rho v_{k\ell} \qquad (k\ell)\text{-traceless \& symm}$$

$$t^{\mu k}_{(2)} = \frac{1}{B}\varepsilon^{kn}\varepsilon^{\mu\nu\rho}\partial_\nu\left(\partial_\ell\, c_{\rho n\ell}\right)$$

- and Chern-Simons dynamics

$$S_{(3)}\left[A,g\right] = \int -\frac{1}{2\alpha B^2}c_{k\ell}d\,c_{k\ell} + A\cdot j_{(3)} + \lambda g\cdot t_{(2)} \sim \int \frac{\Delta^2}{B^2}A\,dA + \frac{\Delta}{B}A\,d\omega$$

- Derivative corrections, including part of gravit. Wen-Zee term $\int \omega d\omega$
- Universality? Need new coupling to three-index background γ_{ijk} (?)

Conclusion

- Effective action of quantum Hall states can be derived systematically by 1/B expansion
- Building principle is the W-infinity (i.e. GMP) symmetry of quantum incompressible fluids
- Multipole expansion of spatially extended low-energy excitations:

  "composite fermion" (Jain), "dipole" (Haldane), "electron+vortex" (Wiegmann)
- Universal quantities can be identified
- Many aspects to be fully developed