Stability of 2D Topological Insulators

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Outline

- Chiral and non-chiral Topological States of Matter
- Stability of edge states of 2D Topological Insulators: $\mathbb{Z}_2$ anomaly
- Partition function: electromagnetic and gravitational (discrete) responses
- Stability of TI with interacting & non-Abelian edges
Topological States of Matter

- System with bulk gap but non-trivial at energies below the gap
- Global effects and global degrees of freedom (edge states, g.s. degeneracy)
- Described by topological field theory: Chern-Simons theory etc.
- Quantum Hall effect is chiral (B field, chiral edge states)
- Quantum spin Hall effect is non-chiral (edge states of both chiralities)
- Other systems: Chern Insulators, Topological Insulators, Topological Superconductors, etc.
- Topological Band Insulators (free fermions) have been observed in 2 & 3 D

<< fever >>
Questions/Answers

- Q: systems of interacting fermions? (e.g. fractional insulators)
- A: use quantum Hall modelling and CFTs
- Q: but non-chiral edge states are stable?
- A: generically NO
- Q: stability with Time Reversal symmetry?
- A: YES/NO; there is a $\mathbb{Z}_2$ symmetry; if this is anomalous, they are stable
**Chiral Topological States**

**Quantum Hall effect**

- chiral edge states
- no Time-Reversal symmetry (TR)
- Laughlin's argument. $\nu = \frac{1}{3}$
  
  $\Phi \to \Phi + \Phi_0, \quad H[\Phi + \Phi_0] = H[\Phi]$  
  
  $Q_R \to Q_R + \Delta Q = \nu, \quad \Delta Q = \int dt dx \, \partial_t J^0_R = \nu \int F = \nu n$ chiral anomaly

- $\Phi \to \Phi + n\Phi_0$ spectral flow between charge sectors $\{0\} \to \{\frac{1}{3}\} \to \{\frac{2}{3}\} \to \{0\}$

- **edge chiral anomaly = response of topological bulk to e.m. background**

- **chiral edge states cannot be gapped** $\leftrightarrow$ **topological phase is stable**

- anomalous response extended to other systems and anomalies in any $D=1,2,3,\ldots$

(S. Ryu, J. Moore, A. Ludwig '10)
Non-chiral topological states

Quantum Spin Hall Effect

- take two $\nu = 1$ Hall states of spins
- system is Time Reversal invariant:
  $T : \psi_{k\uparrow} \rightarrow \psi_{-k\downarrow}, \quad \psi_{k\downarrow} \rightarrow -\psi_{-k\uparrow}$
- non-chiral CFT with $U(1)_Q \times U(1)_S$ symmetry
- adding flux pumps spin $U(1)_S$ anomaly

\[ \Phi_0 \]

$\Delta S = \Delta Q^\uparrow + \Delta Q^\downarrow = 1 - 1 = 0$
$\Delta S = \frac{1}{2} - (-\frac{1}{2}) = 1$
$\Delta S = \Delta Q^\uparrow = \nu^\uparrow = 1$

- in Top. Insulators $U(1)_S$ is explicitly broken by Spin-Orbit Coupling etc.
- no currents $\sigma_H = \sigma_{sH} = \kappa_H = 0$
- but TR symmetry keeps $\mathbb{Z}_2$ symmetry $(-1)^{2S}$

Kramers theorem
Symmetry Protected Topological Phases

- QSHE edge theory is used to describe Topological Insulator with Time-Reversal symmetry $U(1)_S \rightarrow \mathbb{Z}_2$ of $(-1)^{2S}$

  Main issue: stability of TI ↔ stability of non-chiral edge states

- e.g. TR symmetry forbids mass term in CFT with odd number of free fermions

  $\mathcal{T} : H_{\text{int.}} = m \int \psi_\uparrow^\dagger \psi_\downarrow + h.c. \rightarrow -H_{\text{int.}}$ 

  $\mathbb{Z}_2$ classification (free fermions)
Flux insertion argument

- TR symmetry: \( \mathcal{T} H [\Phi] \mathcal{T}^{-1} = H [-\Phi] \) & \( H [\Phi + \Phi_0] = H [\Phi] \)

- TR invariant points: \( \Phi = 0, \frac{\Phi_0}{2}, \Phi_0, \frac{3\Phi_0}{2}, \ldots \)

- Kane et al. define a TR-invariant \( \mathbb{Z}_2 \) polarization (bulk quantity) that:
  - is topological and conserved by TR invariance
  - is equal to parity of edge spin \( (-1)^{2S} = (-1)^{N_\uparrow + N_\downarrow} \)
  - if \( (-1)^{2S} = -1 \) there exits a pair of edge states degenerate by Kramers theorem

\[
\Delta S = \frac{1}{2}
\]
\( \Phi = 0 : \quad (-1)^{2S} |\Omega\rangle = |\Omega\rangle \)
\( \Phi = \frac{\Phi_0}{2} : \quad (-1)^{2S} |\Omega\rangle = -|\Omega\rangle \)

\[ |\text{ex}\rangle \quad \text{gapless edge state} \]
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|ex\rangle \quad \text{gapless edge state}

**Conclusions**

- Topological phase is protected by TR symmetry if \( \exists \text{ edge Kramers pair } (N_f \text{ odd}) \)
- Spin parity is anomalous, discrete remnant of spin anomaly \( U(1)_S \rightarrow \mathbb{Z}_2 \)

Fu-Kane argument is Laughlin’s argument for \( \mathbb{Z}_2 \) anomaly: \( (-1)^{2\Delta S} = -1 \)
\[ \Phi = 0 : \quad (-1)^{2S} |\Omega\rangle = |\Omega\rangle \]
\[ \Phi = \frac{\Phi_0}{2} : \quad (-1)^{2S} |\Omega\rangle = -|\Omega\rangle \]

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**Conclusions**

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**Questions**

- Can we extend this to Topological Insulators with interacting fermions?
- Can we use modular non-invariance, i.e. discrete gravitational anomaly of the partition function as another probe?

(S. Ryu, S.-C. Zhang '12)
Answers in this talk

- partition functions of edge states for QHE, QSHE and TI are completely understood \((\text{AC, Zemba '97; AC, Georgiev, Todorov '01; AC, Viola '11})\)
- we can study flux insertions and discuss stability
- modular transformations: e.m. & grav. responses

**TI:** \(\mathbb{Z}_2\) classification extends to interacting & non-Abelian edges

\[
(-1)^{2\Delta S} = \begin{cases} +1 & \text{unstable, } \mathbb{Z} \text{ modular invariant} \\ -1 & \text{stable, } \mathbb{Z} \text{ not modular invariant} \end{cases}
\]

\[
2\Delta S = \frac{\sigma_{sH}}{e^*} = \frac{\nu^\uparrow}{e^*}
\]

spin-Hall conduct. = chiral Hall conduct. minimal fractional charge

\((\text{Levin, Stern})\)
QHE partition function

Consider states of outer edge for $\nu = \frac{1}{p}$, $p$ odd described by $c = 1$ CFT

$$E \sim P \sim v \frac{L_0}{R}$$

$\tau = v \frac{i\beta}{2\pi R} + t$, modular parameter

$\zeta = \frac{\beta}{2\pi} (iV_0 + \mu)$

edge energy & momentum

electric & chemical pot.

Partition function for one charge sector $Q = \frac{\lambda}{p} + n$, $n \in \mathbb{Z}$

sum of characters for representations of $\widehat{U(1)}$ current algebra

$$K_{\lambda}(\tau, \zeta; p) = \text{Tr}_{\mathcal{H}(\lambda)} [\exp (i2\pi \tau L_0 + i2\pi \zeta Q)]$$

$$= \frac{1}{\eta(\tau)} \sum_n \exp \left( i2\pi \left( \tau \frac{(np + \lambda)^2}{2p} + \zeta \frac{np + \lambda}{p} \right) \right)$$

theta function

Dedekind function

$K_{\lambda+p} = K_{\lambda}$
Modular transformations

\[ K_\lambda(\tau, \zeta; p) = \frac{1}{\eta(\tau)} \sum_n \exp \left( i2\pi \left( \frac{\tau (np + \lambda)^2}{2p} + \zeta \frac{np + \lambda}{p} \right) \right), \quad Q = \frac{\lambda}{p} + \mathbb{Z} \]

discrete coordinate changes respecting double periodicity

\[ S : \tau \rightarrow -1/\tau \]
\[ T : \tau \rightarrow \tau + 1 \]

e.m. background changes:

\[ U : \zeta \rightarrow \zeta + 1 \quad \text{adds the weight} \quad e^{i2\pi Q} \]
\[ V : \zeta \rightarrow \zeta + \tau \quad \text{adds a flux quantum} \quad \Phi \rightarrow \Phi + \Phi_0, \]

\[ S : \quad K_\lambda \left( \frac{-1}{\tau}, \frac{-\zeta}{\tau} \right) \sim \sum_{\mu=1}^{p} S_{\lambda \mu} K_\mu(\tau, \zeta) \quad \text{unitary S matrix, completeness} \]
\[ T^2 : \quad K_\lambda(\tau + 2, \zeta) \sim K_\lambda(\tau, \zeta) \quad \text{odd-integer electron statistics} \]
\[ U : \quad K_\lambda(\tau, \zeta + 1) \sim K_\lambda(\tau, \zeta) \quad \text{integer electron charge} \]
\[ V : \quad K_\lambda(\tau, \zeta + \tau) \sim K_{\lambda+1}(\tau, z) \quad \Phi_0 \quad \text{flux insertion:} \quad Q \rightarrow Q + \nu \]
Partition function for a single edge:
- combine two chiralities \( K^\uparrow_{\lambda} K^\downarrow_{\mu} \)

\[
Z^{NS}(\tau, \zeta) = \sum_{\lambda=1}^{p} K^\uparrow_{\lambda} K^\downarrow_{-\lambda}, \quad S, T^2, U, V \quad \text{invariant}
\]

In fermionic systems there are always four sectors of the spectrum:

\( NS, \tilde{NS}, R, \tilde{R} \), resp. \( (AA), (AP), (PA), (PP) \)

Ramond sector describes half-flux insertions:

\[
V^{\frac{1}{2}} : Z^{NS}(\tau, \zeta) \rightarrow Z^{NS}(\tau, \zeta + \frac{\tau}{2}) \sim Z^R(\zeta, \tau)
\]

add half flux

Each sector has \( \lambda = 1, \ldots, p \) vacua for the would-be fractional charges

\[
Z^R = \sum_{\lambda=1}^{p} \tilde{K}^\uparrow_{\lambda} \tilde{K}^\downarrow_{-\lambda}, \quad \tilde{K}_\lambda(\tau, \zeta) \sim K_\lambda(\tau, \zeta + \frac{\tau}{2})
\]
Modular invariant partition function of a single TI edge can be found

\[ Z_{\text{Ising}} = Z^{\text{NS}} + Z^{\text{\bar{NS}}} + Z^R + Z^{\bar{R}}, \quad S, T, U, V^{\frac{1}{2}} \quad \text{invariant} \]

But: is \( Z_{\text{Ising}} \) consistent with TR symmetry?
Stability and modular (non)invariance

- $p$ fluxes are needed to create one electron in the same charge sector

$$V^p : K^\uparrow_\lambda \rightarrow K^\uparrow_{\lambda+p} = K^\uparrow_\lambda, \quad \Delta Q^\uparrow = \frac{p}{p} = 1 \quad \nu = \frac{1}{p}$$

- Flux argument: add $\frac{p}{2}$ fluxes and check if $(-1)^{2\Delta S} = -1$ i.e. Kramers pair

$$V^{p^2} : K_0 \rightarrow K_{p/2} \sim K^R_0, \quad |\Omega\rangle_{NS} \rightarrow |\Omega\rangle_R, \quad \Delta Q^\uparrow = \Delta S = \frac{1}{2}$$

$$(-1)^{2S} |\Omega\rangle_{NS} = |\Omega\rangle_{NS} \quad \rightarrow \quad (-1)^{2S} |\Omega\rangle_R = - |\Omega\rangle_R \quad \text{stable TI}$$

- Spin parities of Neveu-Schwarz and Ramond ground states are different

$\mathbb{Z}_2$ spin-parity anomaly recovered

$Z_{\text{Ising}} = Z^{NS} + Z^{\tilde{N}S} + Z^R + Z^{\tilde{R}}$ not consistent with TR symmetry

$(-1)^{2S} = 1 \quad 1 \quad -1 \quad -1$

TR symmetry + anomaly $\rightarrow$ no modular invariance $\rightarrow$ stable insulator

TR symmetry + modular invariance $\rightarrow$ no anomaly $\rightarrow$ trivial insulator
General stability analysis

- Edge theory involves neutral excitations (possibly non-Abelian)
- Electron is Abelian: “simple current” modular invariant (A.C., Viola '11)
- Fractional charge sectors $\Theta_{\lambda}(\tau, \zeta)$ parametrized by two integers $(k, p)$
  $$\Theta_{\lambda}^\alpha(\tau, \zeta) = \sum_{a=1}^{k} K_{\lambda+ap}(\tau, k\zeta; kp) \chi_{\lambda+ap}^\alpha(\tau, 0) = \{\text{g.s.}\} + \{1 \text{ el.}\} + \{2 \text{ el.}\} + \ldots$$

- Minimal charge: $Q = \frac{k\lambda}{kp}$, $\lambda = 1$, $e^* = \frac{1}{p}$
- Hall current: $V : \zeta \rightarrow \zeta + \tau$, $\lambda \rightarrow \lambda + k$, $\Delta Q = \nu^\uparrow = \frac{k}{p}$
- Construct TI partition function as before
  $$Z^{NS} = \sum_{\lambda\alpha} \Theta_{\lambda}^\alpha \Theta_{-\lambda}^\alpha$$

- Stability: add fluxes to create an excitation in the same charge sector
  $$V^{\frac{p}{2}} : \Delta S = \Delta Q^\uparrow = \frac{p}{2} \nu^\uparrow = \frac{k}{2}$$
  Kramers pairs if $k$ odd → stable TI
Levin-Stern index \[ 2\Delta S = \frac{\nu^\uparrow}{e^*}, \quad (-1)^{2\Delta S} = (-1)^k \quad \text{fully general} \]

**Analysis of modular invariance vs. stability can be extended to any edge CFT**

- \( k \) even = unstable, TR-inv. modular invariant
  \[ Z_{\text{Ising}} = Z^{NS} + Z^{\bar{NS}} + Z^R + Z^{\bar{R}} \]
- \( k \) odd = stable, modular vector
  \[ Z = \left( Z^{NS}, Z^{\bar{NS}}, Z^R, Z^{\bar{R}} \right) \]
Levin-Stern index \[2\Delta S = \frac{\nu^\uparrow}{e^*}, \quad (-1)^{2\Delta S} = (-1)^k \quad \text{fully general}\]

Analysis of modular invariance vs. stability can be extended to any edge CFT

\[k \text{ even} = \text{unstable, TR-inv. modular invariant} \quad Z_{\text{Ising}} = Z^{NS} + Z^{\overline{NS}} + Z^R + Z^{\overline{R}}\]

\[k \text{ odd} = \text{stable, modular vector} \quad Z = \left(Z^{NS}, Z^{\overline{NS}}, Z^R, Z^{\overline{R}}\right)\]

Examples

- **Jain-like TI** \[\nu^\uparrow = \frac{k}{2nk + 1}, \quad e^* = \frac{1}{2nk + 1}, \quad 2\Delta S = \frac{\nu^\uparrow}{e^*} = k \quad \text{stable, unstable}\]

- **(331) & Pfaffian TI** \[\nu^\uparrow = \frac{1}{2}, \quad e^* = \frac{1}{4}, \quad 2\Delta S = 2 \quad \text{unstable}\]

- **Abelian TI** \[K = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \quad \nu^\uparrow = \frac{3}{7}, \quad e^* = \frac{1}{7}, \quad 2\Delta S = 3 \quad \text{stable}\]

- **Read-Rezayi TI** \[\nu^\uparrow = \frac{k}{kM + 2}, \quad e^* = \frac{1}{kM + 2}, \quad 2\Delta S = k \quad \text{stable, unstable}\]
Remarks

- general expression of partition function allows to extend Levin-Stern stability criterium to any TI with interacting fermions

\[ \mathbb{Z}_2 \text{ classification of TI protected by TR invariance} \]

- neutral states are left invariant by flux insertions

- unprotected edge states do become fully gapped?
  - Abelian states: yes, by careful analysis of possible TR-invariant interactions
    (Levin, Stern; Neupert et al.; Y-M Lu, Vishwanath)
  - non-Abelian states: yes, use projection from “parent” Abelian states
    e.g. (331) -> Pfaffian because \([\text{projection, TR-symm.}]=0\) (A.C., to appear)
Conclusions

- $\mathbb{Z}_2$ spin parity anomaly characterizes Topological Insulators protected by Time Reversal symmetry, (cf. Ringel, Stern; Koch-Janusz, Ringel)

- anomaly signalled by index $(-1)^{2\Delta S} = -1$, $2\Delta S = \frac{\nu^\uparrow}{e^*} = \frac{k}{p}$

- Pfaffian TI is unstable

- To do:
  - explicit form of interactions gapping the edge of non-Abelian states (done)
  - stability of Topological Superconductors: Ryu-Zhang stability criterium
  - stability 3D systems and 2D-3D systems (cf. arxiv: 1306.3238, 3250, 3286)
Gapping interactions for Pfaffian TI

- Gapping interactions for Abelian states defined by $K$ matrix
  \[ U_\alpha = \exp \left( i \Lambda_\alpha^T K \Phi_\uparrow - i \Lambda_\alpha^T K \Phi_\downarrow \right) \ + \ \text{h.c.} \quad \alpha = 1, \ldots, n = c \]

- For (331) state, they can be written in terms of Weyl fermions fields
  \[ U_1 = \Psi_{\uparrow 1} \Psi_{\uparrow 2} \Psi_{\downarrow 1} \Psi_{\downarrow 2} \ + \ \text{h.c.} \]
  \[ U_2 = \Psi_{\uparrow 1} \Psi_{\uparrow 2} \Psi_{\downarrow 1} \Psi_{\downarrow 2} \ + \ \text{h.c.} \]

- Projection (331) → Pfaffian, i.e. to identical layers  \[ \Psi_{\uparrow 1} \sim \Psi_{\uparrow 2} \rightarrow V \chi \]
  \[ U_1 =: \chi \partial \chi : : \bar{\chi} \bar{\partial} \bar{\chi} : \quad \text{neutral} \]
  \[ U_2 =: V^2 \bar{V}^2 : , \quad V = e^{i \sqrt{2} \phi} \quad \text{charged field} \]

- $U_1$ couples to fermion field, $U_2$ to charges modes, giving both mass

- Analysis extends to Read-Rezayi states and Ardonne-Schoutens NASS states
Topological Superconductors

- stability of TS ↔ gapless non-chiral Majorana edge states
- $N_f$ free Majorana, they are neutral ↔ no flux insertion argument
- $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral spin parity $(-1)^{2S^\dagger}(-1)^{2S^\uparrow} = (-1)^{N^\uparrow}(-1)^{N^\downarrow}$ no spin flip
- $\mathbb{Z}$ classification (free fermions)

- $N_f = 8$ unstable by non-trivial quartic interaction: $\mathbb{Z} \to \mathbb{Z}_{8}$ (Kitaev, many people)
  - proposal: study partition function and gravitational response (Ryu, S-C-Zhang)
  - Standard invariant for any $N_f$
    

$$Z_{\text{Ising}} = Z^{NS} + Z^{\overline{NS}} + Z^R + Z^{\overline{R}}$$
  - Is it consistent with $\mathbb{Z}_2 \times \mathbb{Z}_2$ parity?

- Yes, for $N_f = 8$: mod. trasf. $ST \sim V^{\frac{1}{2}}$ creates $\Delta S^\uparrow = \Delta S^\downarrow = 1$ in R sector OK
- Ryu-Zhang: test modular invariance of subsector $(-1)^{N_R} = (-1)^{N_L} = 1$
  - GSO projection $N_f = 8$

- general analysis of modular invariance + discrete symm. not understood yet
- many modular invariants are possible without charge matching $Z = \sum \mathcal{N}_\lambda \mathcal{K}^\uparrow \overline{\mathcal{K}}^\downarrow$