Squeezing hadronic matter

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Summary

- Introduction and basics in Superconductivity
- Effective theory
- BCS theory
- Color Superconductivity: CFL and 2SC phases
- Effective theories and perturbative calculations
- LOFF phase
- Phenomenology

Introduction

- Motivations
- Basics facts in superconductivity
- Cooper pairs

Motivations

- Important to explore the entire QCD phase diagram: Understanding of

Hadrons → QCD-vacuum

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Understanding of its modifications

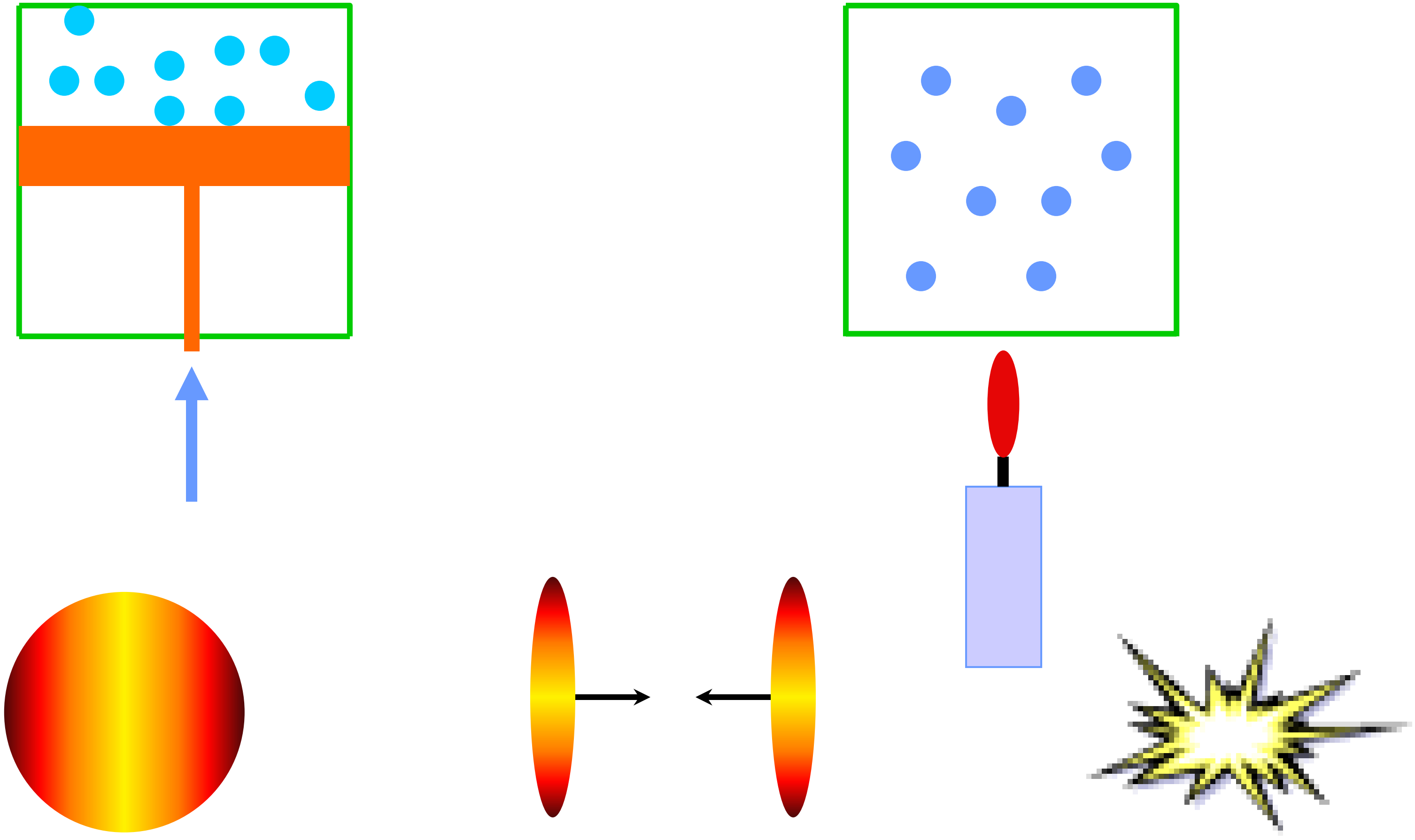
- Extreme Conditions in the Universe:

  Neutron Stars, Big Bang

- QCD simplifies in extreme conditions:

Study QCD when quarks and gluons are the relevant degrees of freedom

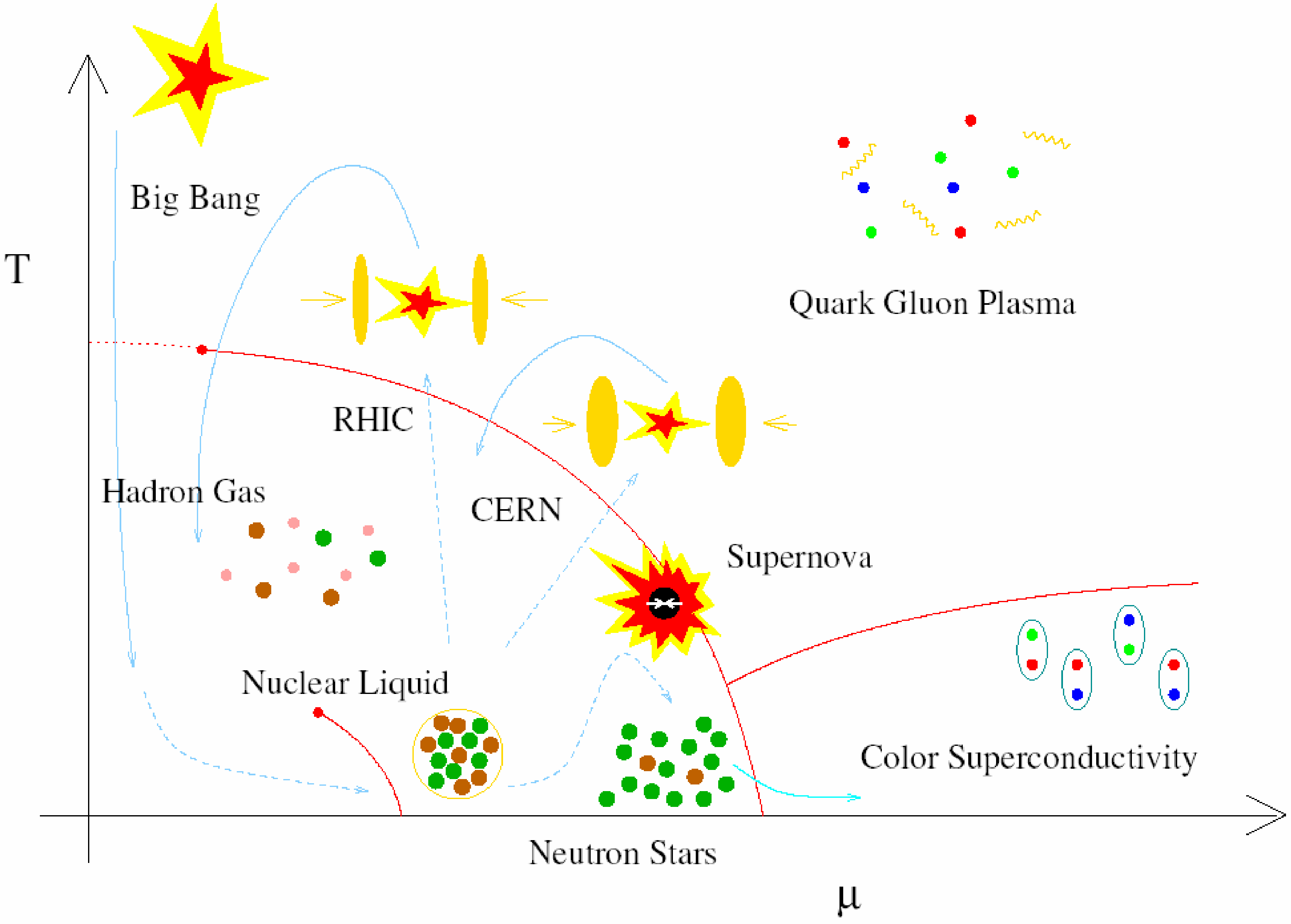
Studying the QCD vacuum under different and extreme conditions may help our understanding



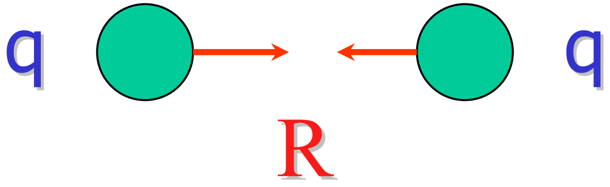
Neutron star

Heavy ion collision

Big Bang


Big Bang
T
Quark Gluon Plasma
RHIC
Hadron Gas
CERN
Supernova
Nuclear Liquid
Color Superconductivity
Neutron Stars
μ

Limiting case $\rho \to \infty \quad (R \to 0)$

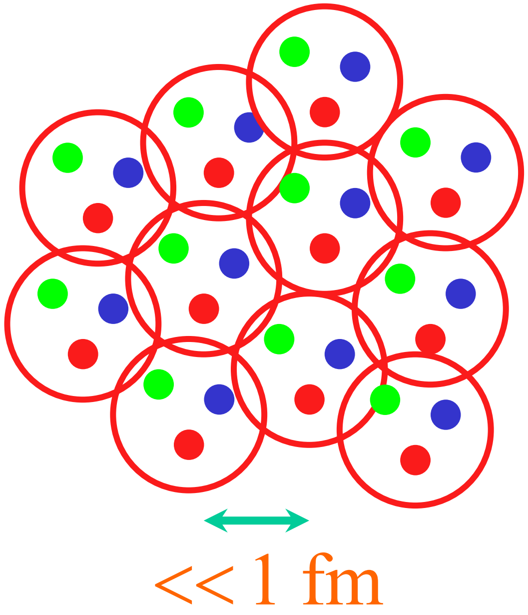


Free quarks

Asymptotic freedom:

When n_B >> 1 fm^{-3}

free quarks expected

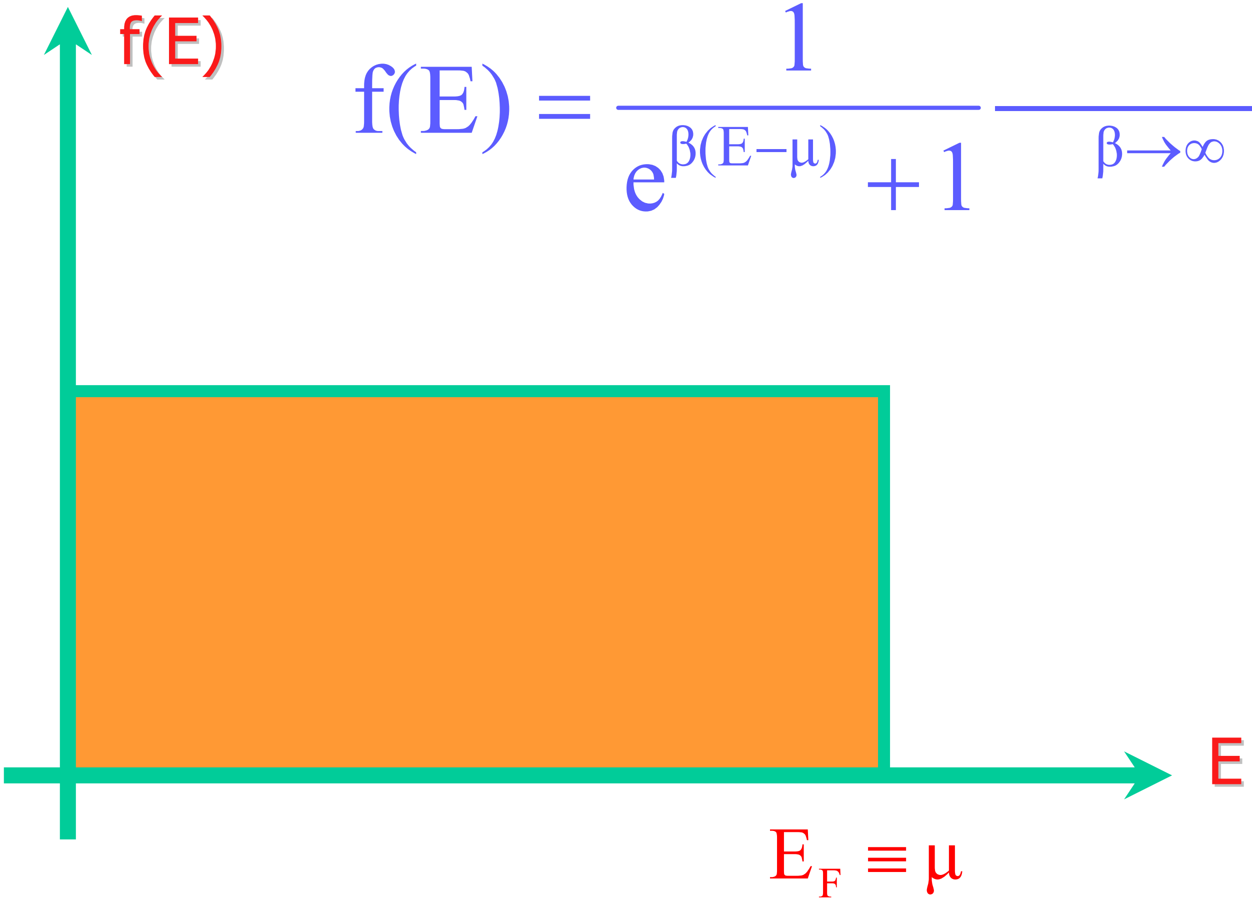


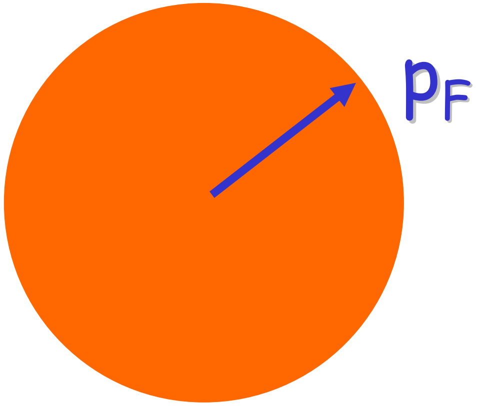
Free Fermi gas and BCS

(high-density QCD)

For $T \to 0 \quad (\beta = 1/kT \to \infty)$

$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1} \xrightarrow[\beta\to\infty]{} \theta(\mu - E)$$





- High density means high p_F
- Typical scattering at momenta of order of p_F

For $p_F >> \Lambda_{QCD}$

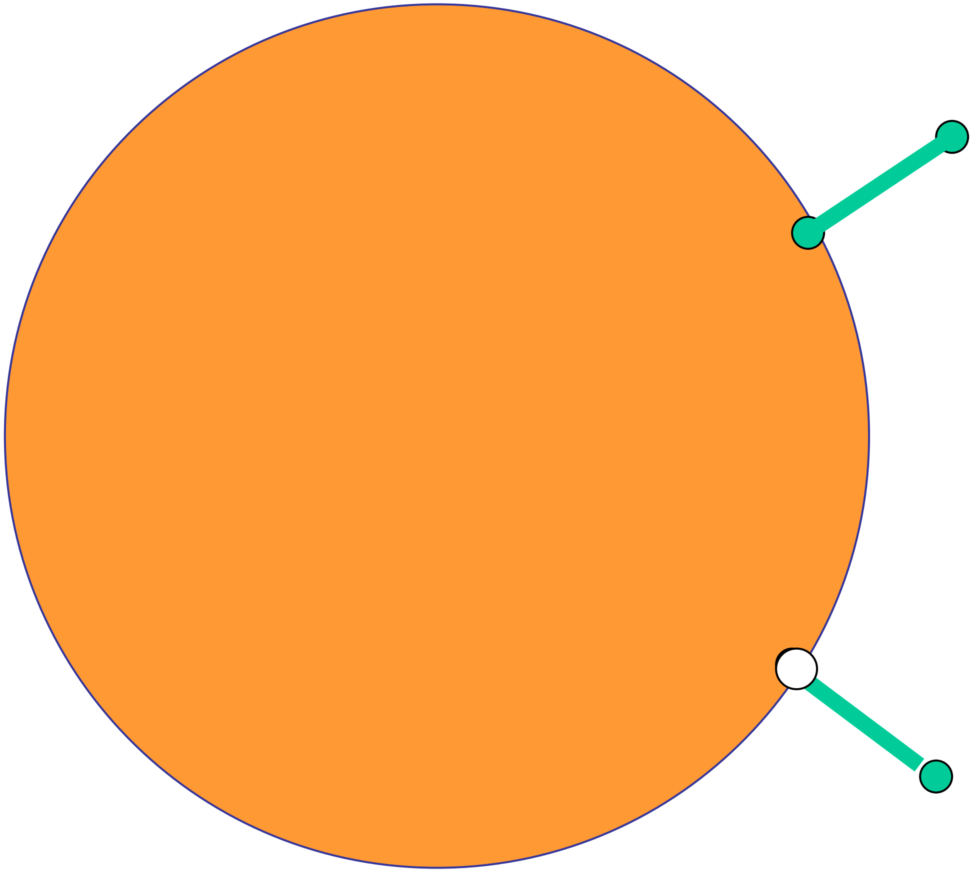
- No chiral breaking
- No confinement
- No generation of masses

→ Trivial theory ?

Grand potential unchanged: $(F = E - \mu N)$

- Adding a particle to the Fermi surface
- Taking out a particle (creating a hole)

$$F \rightarrow (E \pm E_F) - \mu(N \pm 1) = F$$



For an arbitrary attractive interaction it is convenient to form pairs particle-particle or hole-hole (Cooper pairs)

$$E + (\pm 2E_F - E_B) - \mu(N \pm 2) = F - E_B$$

In matter SC only under particular conditions (phonon interaction should overcome the Coulomb force)

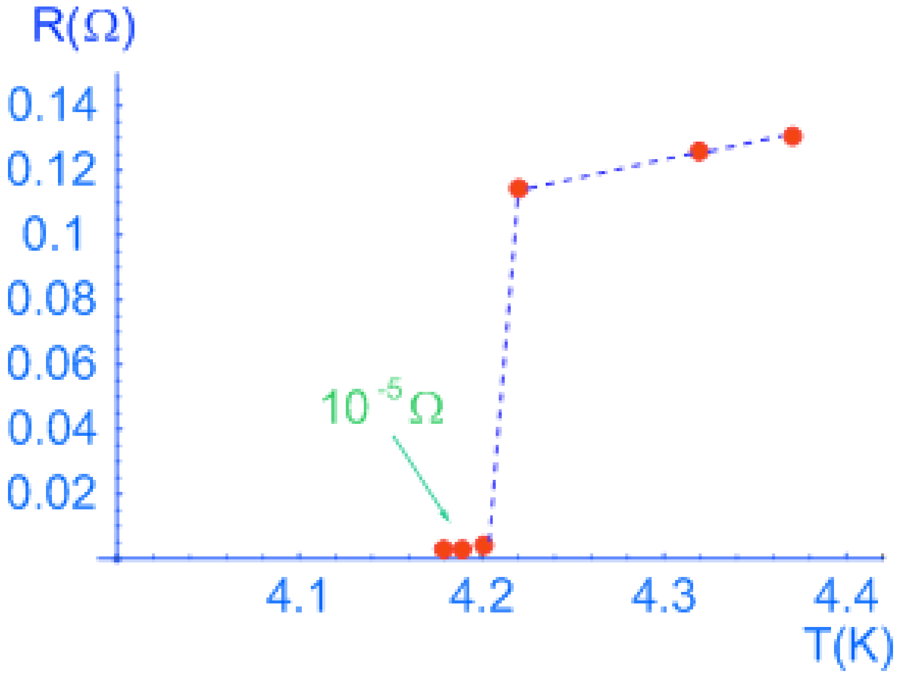
$$\frac{T_c(\text{electr.})}{E(\text{electr.})} \approx \frac{1 \div 10\ {}^0\text{K}}{10^4 \div 10^5\ {}^0\text{K}} \approx 10^{-3} \div 10^{-4}$$

In QCD attractive interaction (antitriplet channel) $\dfrac{T_c(\text{quarks})}{E(\text{quarks})} \approx \dfrac{50\ \text{MeV}}{100\ \text{MeV}} \approx 1$

SC much more efficient in QCD

Basics facts in superconductivity

- 1911 – Resistance experiments in mercury lead and thin by Kamerlingh Onnes in Leiden: existence of a critical temperature $T_c \sim 4\text{-}10\ ^0K$

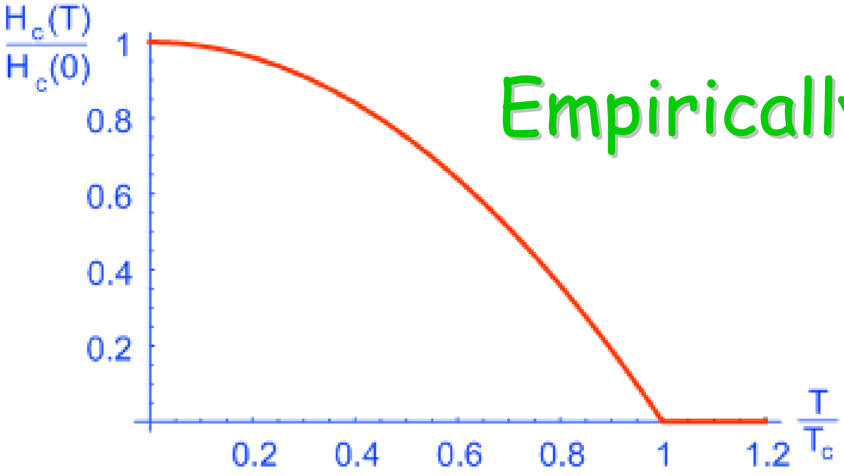


In a superconductor resistivity $< 10^{-23}$ ohm cm

- 1933 - Meissner and Ochsenfeld discover perfect diamagnetism. Exclusion of B except for a penetration depth of ~ 500 Angstrom.

Surprising since from Maxwell, for $E = 0$, B frozen ⟵ $\vec{\nabla} \wedge \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

Destruction of superconductivity for $H = H_c$ $f_s(T) + \frac{H_c^2(T)}{8\pi} = f_n(T)$



Empirically:

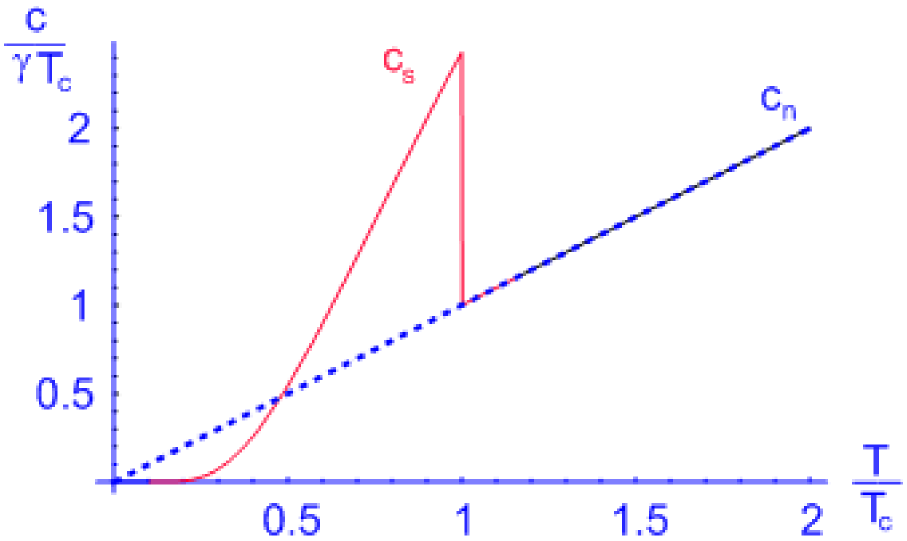
$$H_c(T) \approx H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

- 1950 - Role of the phonons (Frolich).

Isotope effect (Maxwell & Reynolds)

$$T_c \approx H_c(0) \approx \frac{1}{M^{\alpha}}, \qquad \alpha \approx 0.45 - 0.5$$

- 1954 - Discontinuity in the specific heat (Corak)

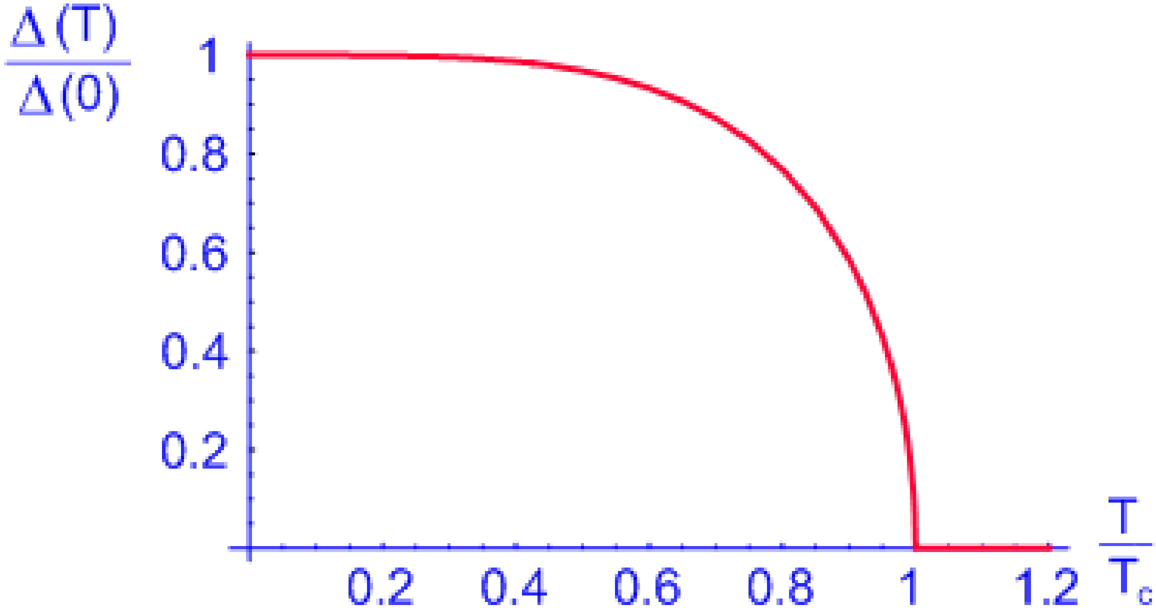


$$c_s \approx a\gamma T_c e^{-bT_c/T}$$

$$c_n \approx \gamma T$$

Excitation energy ~ $1.5\ T_c$

Implication is that there is a gap in the spectrum. This was measured by Glover and Tinkham in 1956



- ☐ **Two fluid models:** phenomenological expressions for the free energy in the normal and in the sueprconducting state (Gorter and Casimir 1934)

- ☐ **London & London theory, 1935:** still a two-fluid models based on

$$\boxed{\frac{\partial \vec{J}_s}{\partial t} = \frac{n_s e^2}{m}\vec{E}, \qquad (\vec{J}_s = -e n_s \vec{v}_s)} \longleftarrow$$

Newton equation

$$\vec{J}_n = \sigma_n \vec{E}, \qquad (\vec{J}_n = -e n_n \vec{v}_n)$$

$$\boxed{\vec{\nabla} \wedge \vec{J}_s = -\frac{n_s e^2}{mc}\vec{B}}$$

+ Maxwell

$$\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c}\vec{J}_s$$

$$\boxed{\nabla^2 \vec{B} = \frac{4\pi n_s e^2}{mc^2}\vec{B} = \frac{1}{\lambda_L^2}\vec{B}} \Longrightarrow \boxed{B(x) = B(0)e^{-x/\lambda_L}}$$

- 1950 - Ginzburg-Landau theory. In the context of Landau theory of second order transitions, valid only around T_c, not appreciated at that time. Recognized of paramount importance after BCS. Based on the construction of an effective theory (modern terms)

$$\boxed{n_s = |\psi(\vec{r})|^2}$$

$$\boxed{\begin{gathered} F_s(T) - F_n(T) = \\ = \int d^3\vec{r} \left(-\frac{1}{2m} \psi^*(\vec{r}) \,|\vec{\nabla} + ie^* \vec{A}|^2 \,\psi(\vec{r}) + \alpha(T)\,|\psi(\vec{r})|^2 + \frac{1}{2}\beta(T)\,|\psi(\vec{r})|^4 \right) \end{gathered}}$$

Cooper pairs

1956 - Cooper proved that two fermions may form a bound state for an arbitrary attractive interaction in a simple model

Only two particle interactions considered. Interactions with the sea neglected but from Fermi statistics

Assume for the ground state:

$$\psi_0(\vec{r}_1-\vec{r}_2)=\underbrace{(\alpha_1\beta_2-\alpha_2\beta_1)}_{\text{spin}}\sum_k g_k \cos\underbrace{(\vec{k}\cdot(\vec{r}_1-\vec{r}_2))}_{\text{zero total momentum}}$$

$$\left[-\frac{1}{2m}(\nabla_1^2+\nabla_1^2)+V(\vec{r}_1-\vec{r}_2)\right]\psi_0(\vec{r}_1-\vec{r}_2)=E\psi_0(\vec{r}_1-\vec{r}_2)$$

$$(E-2\varepsilon_k)g_k=\sum_{k'>k_F}V_{k,k'}g_{k'},\quad \varepsilon_k=\frac{|\vec{k}|^2}{2m}$$

$$V_{k,k'} = \frac{1}{L^3}\int V(\vec{r})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}d^3\vec{r}$$

Cooper assumed that only interactions close to Fermi surface are relevant (see later)

$$\boxed{V_{k,k'} = \begin{cases} -G, & k_F \le k \le k_c \\ 0, & \text{otherwise} \end{cases}}$$

cutoff: $\varepsilon_{k_c} = E_F + \delta$

$$(E - 2\varepsilon_k)g_k = -G\sum_{k'>k_F} g_{k'}$$

Summing over k:

$$\boxed{\frac{1}{G} = \sum_{k>k_F}\frac{1}{2\varepsilon_k - E}}$$

$$\frac{1}{G} = \int_{k_F}^{k_c} \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\varepsilon_k - E} = \int_{E_F}^{E_F+\delta} \frac{d\Omega}{(2\pi)^3} k^2 \frac{dk}{d\varepsilon_k}\,\frac{d\varepsilon}{2\varepsilon - E}$$

Defining the density of the states at the Fermi surface:

$$\rho = \rho(k_F) = 2\int \frac{d\Omega}{(2\pi)^3} k_F^2 \left.\frac{dk}{d\varepsilon_k}\right|_{k_F}$$

For a sphere:

$$\rho = \frac{k_F^2}{\pi^2 v_F}$$

$$\frac{1}{G} \approx \frac{1}{4}\rho \log \frac{2E_F - E + 2\delta}{2E_F - E}$$

$$E = 2E_F - 2\delta \frac{e^{-4/\rho G}}{1 - e^{-4/\rho G}}$$

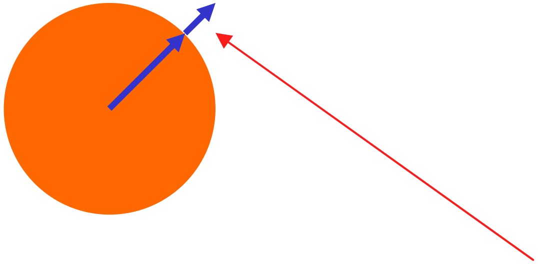
For most superconductors $\rho G < 0.3$

Weak coupling approximation:

$$\boxed{E = 2E_F - 2\delta e^{-4/\rho G}} \longrightarrow E_B$$

Very important: result not analytic in G

Close to the Fermi surface



$$\varepsilon_k = \mu + (\varepsilon_k - \mu) \approx \mu + \left.\frac{\partial \varepsilon_k}{\partial \vec{k}}\right|_{k=k_F} \cdot (\vec{k} - \vec{k}_F) = \mu + \vec{v}_F(\vec{k}) \cdot \vec{\ell}$$

$$N = \sum_{k>k_F} g_k \qquad \psi_0(\vec{r}) = N \sum_{k>k_F} \frac{\cos(\vec{k}\cdot\vec{r})}{2\varepsilon_k - E}$$

$$\xi_k = \varepsilon_k - E_F \qquad \boxed{\psi_0(\vec{r}) = N \sum_{k>k_F} \frac{\cos(\vec{k}\cdot\vec{r})}{2\xi_k + E_B}}$$

Wave function maximum in momentum space close to

$$\boxed{\xi_k = 0}$$

Paired electrons within E_B from E_F:

$$\boxed{E_B << \delta}$$

Only d.o.f. close to E_F relevant!!

Effective theory

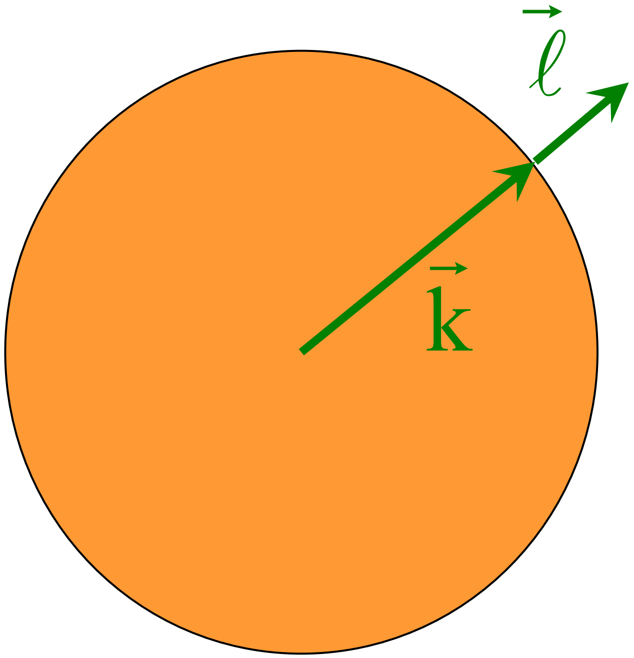
- Field theory at the Fermi surface
- The free fermion gas
- One-loop corrections

Field theory at the Fermi surface

(Polchinski, TASI 1992, hep-th/9210046)

Renormalization group analysis a la Wilson

How do fields behave scaling down the energies toward ε_F by a factor $s<1$?



$$\vec{p} = \vec{k} + \vec{\ell}$$

Scaling:

$$E \Rightarrow sE$$
$$\vec{\ell} \Rightarrow s\vec{\ell}$$
$$\vec{k} \Rightarrow \vec{k}$$

Using the invariance under phase transformations, construction of the most general action for the effective degrees of freedom: particles and holes close to the Fermi surface (non-relativistic description)

$$\int \mathrm{dt}\,\mathrm{d}^3\vec{\mathrm{p}}\left[\mathrm{i}\psi_\sigma^\dagger\left(\vec{\mathrm{p}}\right)\partial_\mathrm{t}\psi_\sigma\left(\vec{\mathrm{p}}\right)-\left(\varepsilon\left(\vec{\mathrm{p}}\right)-\varepsilon_\mathrm{F}\right)\psi_\sigma^\dagger\left(\vec{\mathrm{p}}\right)\psi_\sigma\left(\vec{\mathrm{p}}\right)\right]$$

Expanding around ε_F:

$$\varepsilon\left(\vec{\mathrm{p}}\right)-\varepsilon_\mathrm{F}=\left|\frac{\partial\varepsilon\left(\vec{\mathrm{p}}\right)}{\partial\vec{\mathrm{p}}}\right|_{\ell=0}\cdot\vec{\ell}+O\left(\ell^2\right)\equiv\mathrm{v}_\mathrm{F}\ell+\cdots$$

$$S=\int dt\, d^2\vec{k}\, d\vec{\ell}\left[i\psi_\sigma^\dagger(\vec{p})\,\partial_t\psi_\sigma(\vec{p})-\ell v_F\psi_\sigma^\dagger(\vec{p})\,\psi_\sigma(\vec{p})\right]$$

Scaling: ⟶ $S\rightarrow s^{2d_\psi+1}S$

$$\ell\rightarrow s\ell$$

$$dt\rightarrow s^{-1}dt$$

$$d\vec{k}\rightarrow d\vec{k}$$

$$d\vec{\ell}\rightarrow sd\vec{\ell}$$

$$\partial_t\rightarrow s\partial_t$$

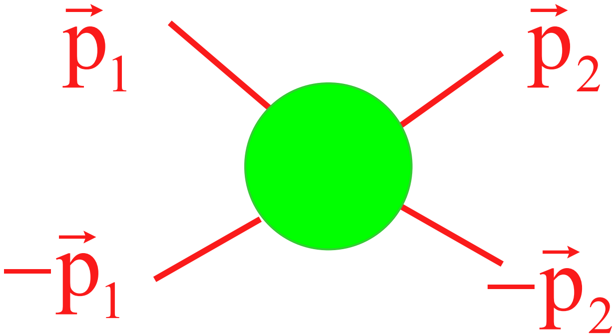
requiring the action S to be invariant

$$\psi\rightarrow s^{-1/2}\psi$$

The result of the analysis is that all possible interaction terms are irrelevant (go to zero going toward the Fermi surface) except a marginal (independent on s) quartic interaction of the form:

$$V\sum_{\sigma,\sigma'}\int dt\,d^3\vec{p}_1 d^3\vec{p}_2\,\psi^\dagger_\sigma(\vec{p}_1)\psi_\sigma(\vec{p}_2)\psi^\dagger_{\sigma'}(-\vec{p}_1)\psi_{\sigma'}(-\vec{p}_2)$$

corresponding to a Cooper-like interaction



Higher order interactions irrelevant

Free theory BUT check quantum corrections to the marginal interactions among the Cooper pairs

The free fermion gas

Eq. of motion: $(i\partial_t - \ell v_F)\psi_\sigma(\vec{p},t) = 0$

Propagator: $(i\partial_t - \ell v_F)G_{\sigma,\sigma'}(\vec{p},t) = \delta_{\sigma,\sigma'}\delta(t)$

$$\boxed{\begin{gathered} G_{\sigma,\sigma'}(\vec{p},t) = \delta_{\sigma,\sigma'}G(\vec{p},t) = \\ = -i\delta_{\sigma,\sigma'}\left[\theta(t)\theta(\ell) - \theta(-t)\theta(-\ell)\right]e^{-i\ell v_F t} \end{gathered}}$$

Using: $\theta(t) = \frac{i}{2\pi}\int d\omega \frac{e^{-i\omega t}}{\omega + i\varepsilon}$

$$G(\vec{p},t) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\left[\frac{\theta(\ell)}{\omega-\ell v_F+i\varepsilon}+\frac{\theta(-\ell)}{\omega-\ell v_F-i\varepsilon}\right]$$

or:

$$G(\vec{p},t) \equiv \frac{1}{2\pi}\int dp_0\, e^{-ip_0 t} G(p_0,\vec{p})$$

$$G(p) = \frac{1}{(1+i\varepsilon)p_0-\ell v_F}$$

Fermi field decomposition

$$\psi_\sigma(x) = \sum_{\vec{p}} b_\sigma(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} = \sum_{\vec{p}} b_\sigma(\vec{p})e^{-ipx}$$

$$x^\mu = (t,\vec{x}), \quad p^\mu = (\ell v_F,\vec{p})$$

with:

$$b_\sigma(\vec{p})\left|0\right\rangle = 0 \quad \text{for} \quad |\vec{p}| > p_F$$

$$b_\sigma^\dagger(\vec{p})\left|0\right\rangle = 0 \quad \text{for} \quad |\vec{p}| < p_F$$

$$[b_\sigma(\vec{p}), b_\sigma^\dagger(\vec{p})]_+ = \delta_{\vec{p},\vec{p}'}\delta_{\sigma,\sigma'}$$

$$[\psi_\sigma(\vec{x},t), \psi_\sigma^\dagger(\vec{x}',t)]_+ = \delta_{\sigma,\sigma'}\delta^3(\vec{x}-\vec{x}')$$

The following representation holds:

$$G_{\sigma,\sigma'}(x) = -i\delta_{\sigma,\sigma'}\sum_{\vec{p}}\left\langle 0\right|T(b_\sigma(\vec{p},t)b_\sigma^\dagger(\vec{p},0)\left|0\right\rangle e^{i\vec{p}\cdot\vec{x}} = \delta_{\sigma,\sigma'}\sum_{\vec{p}} G(\vec{p},t)$$

In fact, using

$$\left\langle 0\right|b_\sigma^\dagger(\vec{p})b_\sigma(\vec{p})\left|0\right\rangle = \theta(p_F - p) = \theta(-\ell)$$

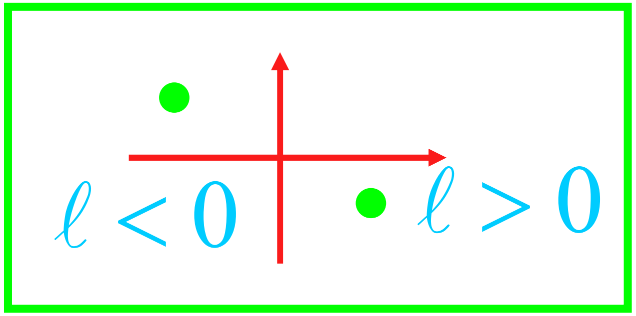
$$\left\langle 0\right|b_\sigma(\vec{p})b_\sigma^\dagger(\vec{p})\left|0\right\rangle = 1 - \theta(p_F - p) = \theta(p - p_F) = \theta(\ell)$$

$$\boxed{G(\vec{p},t) = \begin{cases} -i\theta(\ell)e^{-i\ell v_F t}, & t>0 \\ i\theta(-\ell)e^{-i\ell v_F t}, & t<0 \end{cases}}$$

The following property is useful:

$$\lim_{\delta\to 0^+} G_{\sigma,\sigma}(\vec{0},-\delta) = -i \lim_{\delta\to 0^+} \langle 0 | T(\psi_\sigma(\vec{0},-\delta)\psi^\dagger_\sigma(0)|0\rangle =$$

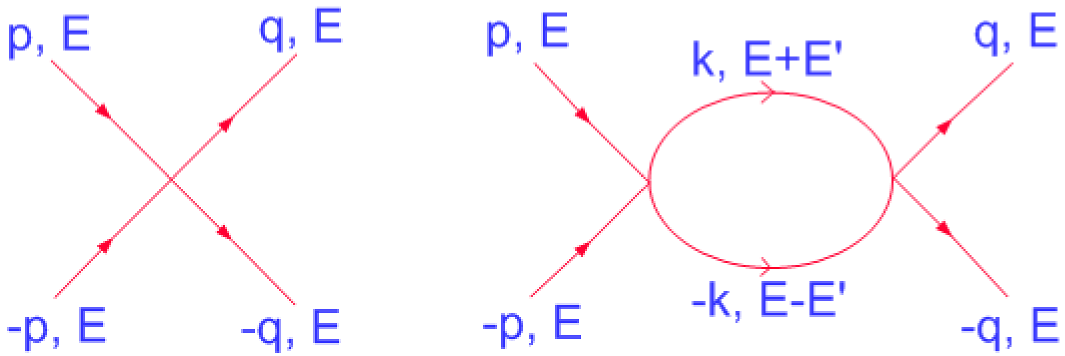
$$= i\langle 0|\psi^\dagger_\sigma\psi_\sigma|0\rangle \equiv i\rho_F$$



$$\rho_F = -2i\lim_{\delta\to 0^+}\sum_\sigma G_{\sigma,\sigma}(\vec{0},-\delta) = -2i\lim_{\delta\to 0^+}\int\frac{d^4p}{(2\pi)^4}e^{ip_0\delta}\frac{1}{(1+i\varepsilon)p_0-\ell v_F}$$

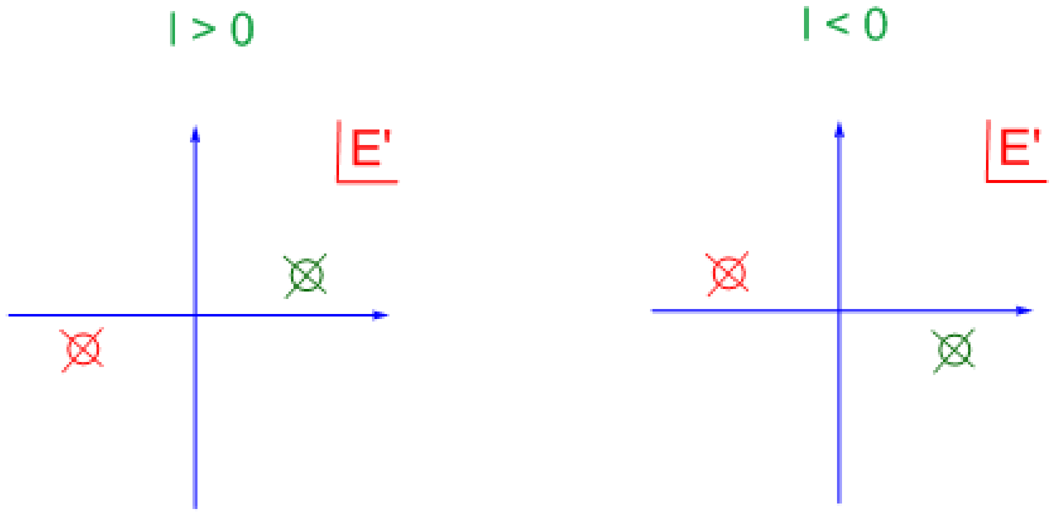
$$\rho_F = 2\int\frac{d^3\vec{p}}{(2\pi)^3}\theta(-\ell) = 2\int\frac{d^3\vec{p}}{(2\pi)^3}\theta(p_F-p) = \frac{p_F^3}{3\pi^2}$$

One-loop corrections



$$\boxed{\frac{1}{(1+i\varepsilon)p_0 - \ell v_F}}$$

$$G(E) = G - G^2 \int \frac{dE' d^2\vec{k} d\ell}{(2\pi)^4} \frac{1}{((E+E')(1+i\varepsilon) - v_F \ell)((E-E')(1+i\varepsilon) - v_F \ell)}$$



Closing in the upper plane we get

$$\boxed{G(E) = G - \frac{1}{2} G^2 \rho \log(\delta/E) + O(G^3)}$$

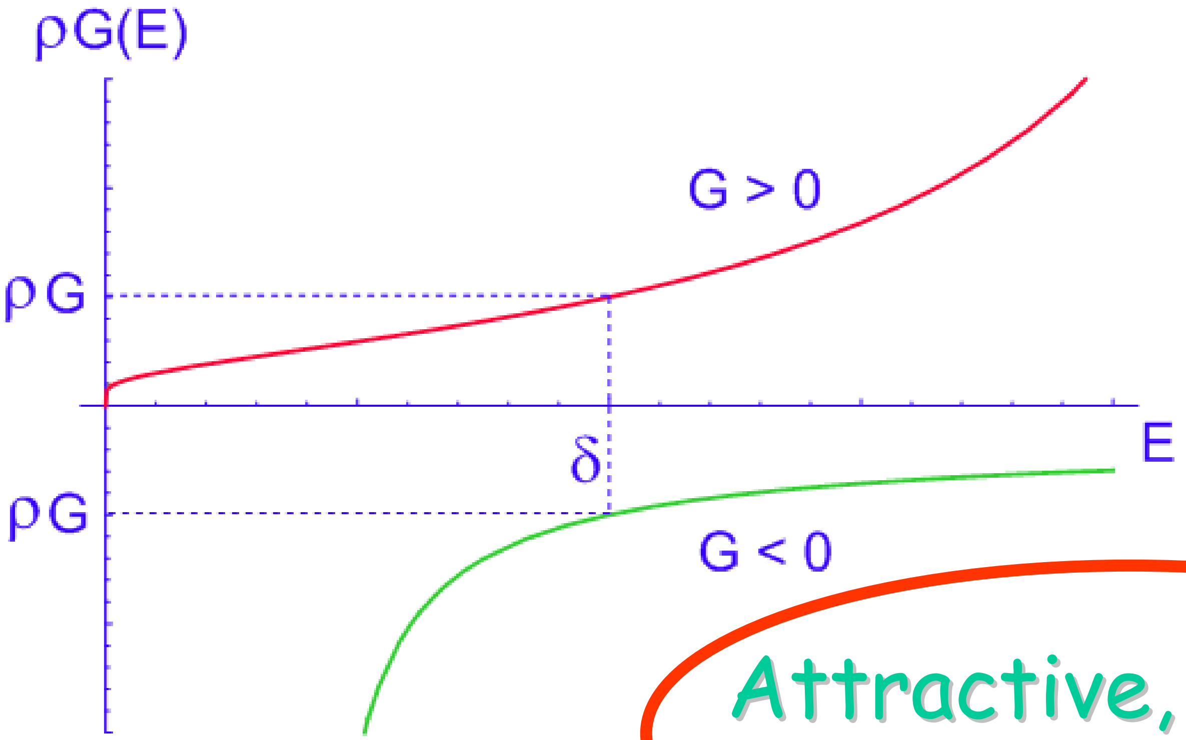
$$\boxed{\rho = 2 \int \frac{d^2\vec{k}}{(2\pi)^3} \frac{1}{v_F(\vec{k})}}$$

δ, UV cutoff

From RG equations:

$$\frac{dG(E)}{dE} = \frac{1}{2E} \rho G(E)^2$$

$$\rho G(E) = \frac{\rho G}{1 + \frac{\rho G}{2}\log(\delta/E)}$$



$E \to 0$

BCS instability

Attractive, stronger for $E \to 0$

BCS theory

- A toy model
- BCS theory
- Functional approach
- The critical temperature
- The relevance of gauge invariance

A toy model

Solution to BCS instability

↓

Formation of condensates

Studied with variational methods, Schwinger-Dyson, CJT, etc.

Idea of quasi-particles through a toy model (Hubbard toy-model)

2 Fermi oscillators:

$$\boxed{H = \varepsilon\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) + G a_1^\dagger a_2^\dagger a_1 a_2}$$

Trial wave function:

$$\left|\Psi\right\rangle_{\text{trial}} = (\cos\theta + \sin\theta\, a_1^\dagger a_2^\dagger)\left|0\right\rangle$$

$$\Gamma = \left\langle\Psi\left|a_1 a_2\right|\Psi\right\rangle = -\sin\theta\cos\theta$$

Decompose:

$$\boxed{H = H_0 + H_{res}}$$

$$H_0 = \varepsilon\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) - G\Gamma\left(a_1 a_2 - a_1^\dagger a_2^\dagger\right) + G\Gamma^2$$

$$H_{res} = G a_1^\dagger a_2^\dagger a_1 a_2 + G\Gamma\left(a_1 a_2 - a_1^\dagger a_2^\dagger\right) - G\Gamma^2 =$$

$$= G\left(a_1^\dagger a_2^\dagger + \Gamma\right)\left(a_1 a_2 - \Gamma\right)$$

Mean field theory assumes $H_{res} = 0$

$$\langle \Psi | H_0 | \Psi \rangle = 2\varepsilon \sin^2\theta - G\Gamma^2$$

Minimize w.r.t. θ

$$\boxed{2\varepsilon \sin 2\theta + 2G\Gamma \cos 2\theta = 0 \Rightarrow \tan 2\theta = -\frac{G\Gamma}{\varepsilon}}$$

From the expression for Γ:

$$\Gamma = -\frac{1}{2}\sin 2\theta = \frac{1}{2}\frac{G\Gamma}{\sqrt{\varepsilon^2 + G^2\Gamma^2}}$$

Gap equation

$$\boxed{1 = \frac{1}{2} \frac{G}{\sqrt{\varepsilon^2 + \Delta^2}}}$$

$$\Delta = G\Gamma$$

$|\Psi\rangle_{\text{trial}}$ Is the fundamental state in the broken phase where the condensate Γ is formed

In fact, via Bogolubov transformation

$$A_1 = a_1 \cos\theta - a_2^\dagger \sin\theta$$

$$A_2 = a_1^\dagger \sin\theta + a_2 \cos\theta$$

one gets: $$A_{1,2}\left|\Psi\right\rangle_{\text{trial}} = 0$$

$$H_0 = \left(\varepsilon - \sqrt{\varepsilon^2 + \Delta^2}\right) + \sqrt{\varepsilon^2 + \Delta^2}\left(A_1^\dagger A_1 + A_2^\dagger A_2\right)$$

Energy of quasi-particles (created by $A^+_{1,2}$)

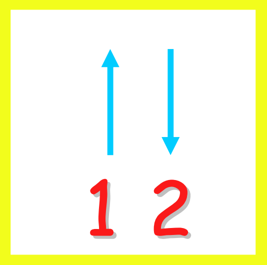
$$\boxed{E = \sqrt{\varepsilon^2 + \Delta^2}}$$

BCS theory

$$\boxed{\tilde{H} = H - \mu N = \sum_{k\sigma} \xi_k b^\dagger_\sigma(k) b_\sigma(k) + \sum_{kq} V_{kq} b^\dagger_1(k) b^\dagger_2(-k) b_2(-q) b_1(q)}$$

$$\xi_k = \varepsilon_k - E_F = \varepsilon_k - \mu$$

$$\boxed{\tilde{H} = H_0 + H_{res}}$$



$$\boxed{H_0 = \sum_{k\sigma} \xi_k b^\dagger_\sigma(k) b_\sigma(k) + \sum_{kq} V_{kq} \left[b^\dagger_1(k) b^\dagger_2(-k) \Gamma_q + b_2(-q) b_1(q) \Gamma^*_k - \Gamma_q \Gamma^*_k \right]}$$

$$\boxed{H_{res} = \sum_{kq} V_{kq} \left(b^\dagger_1(k) b^\dagger_2(-k) - \Gamma^*_k \right) \left(b_2(-q) b_1(q) - \Gamma_q \right)}$$

$$\Gamma_k = \left\langle b_2(-k) b_1(k) \right\rangle$$

$$\boxed{H_0 = \sum_{k\sigma} \xi_k b_\sigma^\dagger(k) b_\sigma(k) + \sum_k \left[\Delta_k b_1^\dagger(k) b_2^\dagger(-k) + \Delta_k^* b_2(-k) b_1(k) - \Delta_k \Gamma_k^* \right]}$$

$$\Delta_k = -\sum_q V_{kq} \Gamma_q$$

Bogolubov-Valatin transformation:

$$b_1(k) = u_k^* A_1(k) + v_k A_2^\dagger(k),$$

$$b_2^+(-k) = -v_k^* A_1(k) + u_k A_2^\dagger(k)$$

$$|u_k|^2 + |v_k|^2 = 1$$

To bring H_0 in canonical form we choose

$$\left|u_k\right|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right), \quad \left|v_k\right|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$$

$$\boxed{E_k = \sqrt{\xi_k^2 + \left|\Delta_k\right|^2}}$$

$$\boxed{H_0 = \sum_{k\sigma} E_k A_\sigma^\dagger(k) A_\sigma(k) + \left\langle H_0 \right\rangle}$$

$$\boxed{\left|0\right\rangle_{BCS} = \prod_k \left(u_k + v_k b_1^\dagger(k) b_2^\dagger(-k)\right)\left|0\right\rangle}$$

$$\boxed{A_1(k)\left|0\right\rangle_{BCS} = A_2(k)\left|0\right\rangle_{BCS} = 0}$$

$$\Gamma_k = \left\langle b_2(-k)b_1(k) \right\rangle = u_k^* v_k \left\langle (1 - A_1^\dagger(k)A_1(k) - A_2^\dagger(k)A_2(k) \right\rangle =$$

$$= u_k^* v_k = \frac{1}{2}\frac{\Delta_k}{E_k}$$

$$\boxed{\Gamma_k = \frac{1}{2}\frac{\Delta_k}{E_k}}$$

$$\Delta_k = -\sum_q V_{kq}\Gamma_q$$

$$\boxed{\Delta_k = -\frac{1}{2}\sum_q V_{kq}\frac{\Delta_q}{E_q}}$$

Gap equation

As for the Cooper case choose:

$$\boxed{V_{k,k'} = \begin{cases} -G, & |\xi_k|, |\xi_k| < \delta \\ 0, & \text{otherwise} \end{cases}} \qquad \Delta_k \approx \Delta$$

$$\langle H_0 \rangle = 2 \sum_{k>k_F} \underbrace{\left(\xi_k - \frac{\xi_k^2}{E_k} \right)}_{\text{Kinetic energy}} - \underbrace{\frac{\Delta^2}{G}}_{\text{Interaction term}}$$

$$\langle H_0 \rangle = \rho \int_0^{\delta} d\xi \left(\xi - \frac{\xi^2}{\sqrt{\xi^2 + \Delta^2}} \right) - \frac{\Delta^2}{G} =$$

$$= \rho \left[\delta^2 - \delta\sqrt{\delta^2 + \Delta^2} + \Delta^2 \overbrace{\log \frac{\delta + \sqrt{\delta^2 + \Delta^2}}{\Delta}}^{\frac{2}{\rho G}} \right] - \frac{\Delta^2}{G}$$

$$\Delta = \frac{1}{2}\rho G\int_0^{\delta} d\xi \frac{\Delta}{\sqrt{\xi^2+\Delta^2}} = \frac{1}{2}\rho G\Delta \log\frac{\delta+\sqrt{\delta^2+\Delta^2}}{\Delta}$$

$$\langle H_0\rangle = \frac{\rho}{2}\left[\delta^2 - \delta\sqrt{\delta^2+\Delta^2} + \cancel{\frac{2\Delta^2}{\rho G}}\right] - \cancel{\frac{\Delta^2}{G}} =$$

$$= \frac{\rho}{2}\left[\delta^2 - \delta\sqrt{\delta^2+\Delta^2}\right] \approx -\frac{1}{4}\rho\Delta^2$$

$$\rho G << 1, \text{ or } \Delta << \delta$$

Pair condensation energy

$$\boxed{\langle H_0\rangle \approx -\frac{1}{4}\rho\Delta^2}$$

$$\boxed{\Delta \approx 2\delta e^{-2/\rho G}}$$

$T \neq 0$

$$\langle O \rangle_T = \frac{\mathrm{Tr}\left[e^{-H/T} O\right]}{\mathrm{Tr}\left[e^{-H/T}\right]}$$

For a single Fermi oscillator $H = Eb^\dagger b$

$$\mathrm{Tr}[e^{-Eb^\dagger b/T}] = 1 + e^{-E/T}$$

$$\mathrm{Tr}[b^\dagger b e^{-Eb^\dagger b/T}] = e^{-E/T}$$

$$\boxed{\left\langle b^\dagger b \right\rangle_T = f(E) = \frac{1}{e^{E/T} + 1}}$$

Fermi distribution

$$\Gamma_k = u_k^* v_k \left\langle (1 - A_1^\dagger(k) A_1(k) - A_2^\dagger(k) A_2(k) \right\rangle_T = u_k^* v_k (1 - 2f(E_k))$$

$$\Delta_k = -\sum_q V_{kq} u_q^* v_q (1 - 2f(E_q)) = -\sum_q V_{kq} \frac{\Delta_q}{2E_q} \tanh\frac{E_q}{2T}$$

$$\boxed{1 = \frac{1}{4} \rho G \int_{-\delta}^{+\delta} \frac{d\xi}{E} \tanh\frac{E}{2T}, \quad E = \sqrt{\xi^2 + \Delta^2}}$$

Functional approach

$$S\left[\psi,\psi^{\dagger}\right]=\int d^4x\left[\psi^{\dagger}(i\partial_t-\varepsilon(|\vec{\nabla}|)+\mu)\psi+\frac{G}{2}(\psi^{\dagger}\psi)^2\right]$$

Fierzing $(C=i\sigma_2)$

$$\psi_a^{\dagger}\psi_a\psi_b^{\dagger}\psi_b=-\psi_a^{\dagger}\psi_b^{\dagger}\psi_a\psi_b=$$

$$=-\frac{1}{4}\varepsilon_{ab}\varepsilon_{ab}\psi_c^{\dagger}\psi^{\dagger c}\psi_d\psi^d=-\frac{1}{2}\psi^{\dagger}C\psi^{*}\psi^{T}C\psi$$

$$S\left[\psi,\psi^{\dagger}\right]=\int d^4x\left[\psi^{\dagger}(i\partial_t-\varepsilon(|\vec{\nabla}|)+\mu)\psi-\frac{G}{4}(\psi^{\dagger}C\psi^{*})(\psi^{T}C\psi)\right]$$

Quantum theory $Z=\int D(\psi,\psi^{\dagger})e^{iS\left[\psi,\psi^{\dagger}\right]}$

$$\text{const.}=\int D(\Delta,\Delta^*)e^{-\frac{i}{G}\int d^4x\left[\Delta-\frac{G}{2}(\psi^T C\psi)\right]\left[\Delta^*+\frac{G}{2}(\psi^\dagger C\psi^*)\right]}$$

$$\frac{Z}{Z_0}=\frac{1}{Z_0}\int D(\psi,\psi^\dagger)D(\Delta,\Delta^*)e^{iS_0[\psi,\psi^\dagger]+i\int d^4x\left[-\frac{|\Delta|^2}{G}-\frac{1}{2}\Delta(\psi^\dagger C\psi^*)+\frac{1}{2}\Delta^*(\psi^T C\psi)\right]}$$

$$\chi=\frac{1}{\sqrt{2}}\begin{pmatrix}\psi\\ C\psi^*\end{pmatrix}$$

$$\boxed{S_0+\cdots=\int d^4x\left(\chi^\dagger S^{-1}\chi-\frac{|\Delta|^2}{G}\right)}$$

$$\boxed{S^{-1}(p)=\begin{bmatrix}p_0-\xi_p & -\Delta\\ -\Delta^* & p_0+\xi_p\end{bmatrix}}$$

Since ψ^* appears already in χ we are double-counting. Solution: integrate over the fermions with the “replica trick”:

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\left[\det(S^{-1})\right]^{1/2} e^{-i\int d^4x \frac{|\Delta|^2}{G}} \equiv e^{iS_{\mathrm{eff}}}$$

$$\boxed{S_{\mathrm{eff}}(\Delta,\Delta^*) = -\frac{i}{2}\mathrm{Tr}\left[\log(S_0 S^{-1})\right] - \int d^4x \frac{|\Delta|^2}{G}}$$

Evaluating the saddle point:

$$\Delta = iG\int \frac{d^4p}{(2\pi)^4}\frac{\Delta}{p_0^2 - \xi_p^2 - |\Delta|^2} \Longrightarrow \boxed{\Delta = \frac{G}{2}\int \frac{d^3p}{(2\pi)^3}\frac{\Delta}{\sqrt{\xi_p^2 + |\Delta|^2}}}$$

At T not 0, introducing the Matsubara frequencies

$$\omega_n = (2n+1)\pi T$$

$$\Delta = GT\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\omega_n^2+\xi_p^2+|\Delta|^2}$$

and using

$$\sum_{n=-\infty}^{+\infty}\frac{1}{\omega_n^2+\xi_p^2+|\Delta|^2} = \frac{1}{2E_pT}(\underbrace{1-2f(E_p)})$$

$$\boxed{\Delta = \frac{G}{2}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\sqrt{\xi_p^2+|\Delta|^2}}\tanh(E_p/2T)}$$

By saddle point:

$$\frac{Z}{Z_0}=\frac{1}{Z_0}\int D(\psi,\psi^\dagger)D(\Delta,\Delta^*)e^{iS_o[\psi,\psi^\dagger]+i\int d^4x\left[-\frac{|\Delta|^2}{G}-\frac{1}{2}\Delta(\psi^\dagger C\psi^*)+\frac{1}{2}\Delta^*(\psi^T C\psi)\right]}$$

$$\Delta=\frac{G}{2}\left\langle\psi^T C\psi\right\rangle$$

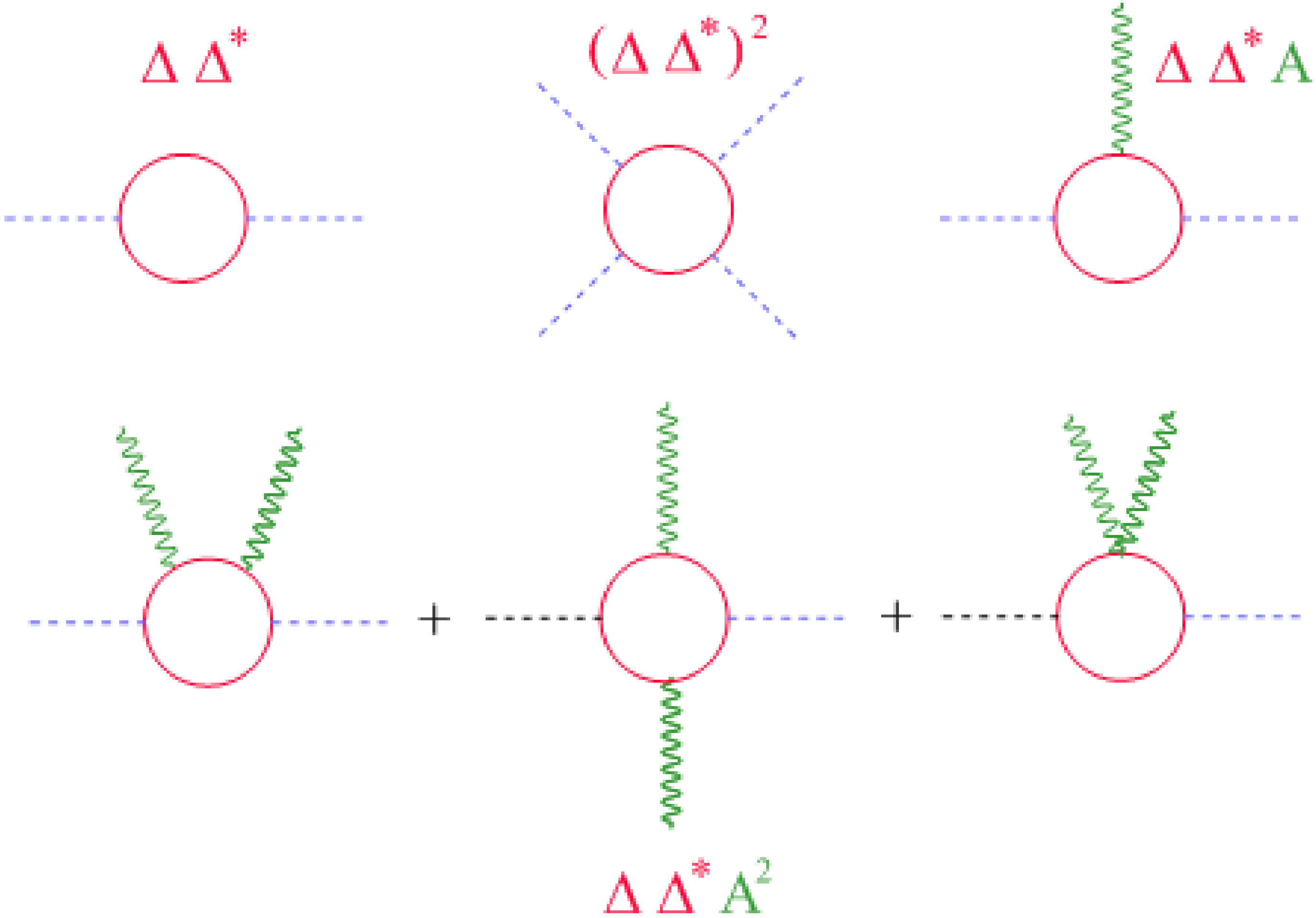
Introducing the em interaction in S_0 we see that Z is gauge invariant under

$$\psi\rightarrow\psi e^{i\alpha(x)},\quad \Delta\rightarrow\Delta e^{2i\alpha(x)}$$

Therefore also S_{eff} must be gauge invariant and it will depend on the space-time derivatives of Δ through

$$\boxed{D_\mu = \partial_\mu + 2ieA_\mu}$$

In fact, evaluating the diagrams (Gor'kov 1959):



got the result (with a convenient renormalization of the fields):

$$H = \int d^3\vec{r} \left(-\frac{1}{4m} \psi^*(\vec{r}) \,|\, (\vec{\nabla} + i\underset{\big\downarrow}{2e}\vec{A}) \,|^2\, \psi(\vec{r}) + \alpha \,|\, \psi(\vec{r}) \,|^2 + \frac{1}{2} \beta \,|\, \psi(\vec{r}) \,|^4 \right)$$

charge of the pair

This result gave full justification to the Landau treatment of superconductivity

The critical temperature

By definition at T_c the gap vanishes. One can perform a GL expansion of the grand potential

$$\Omega = \frac{1}{2}\alpha\Delta^2 + \frac{1}{4}\beta\Delta^4$$

with extrema: $\alpha\Delta + \beta\Delta^3 = 0$

α and β from the expansion of the gap equation up to normalization

$$\Delta = GT\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\omega_n^2 + \xi_p^2 + |\Delta|^2}$$

To get the normalization remember (in the weak coupling and relatively to the normal state):

$$\langle H_0 \rangle = \Omega = -\frac{1}{4}\rho\Delta^2$$

Starting from the gap equation: $\Delta - \frac{1}{2}\rho G \Delta \log\frac{2\delta}{\Delta} = 0$

Integrating over Δ and using the gap equation one finds:

$$-\frac{G\rho}{8}\Delta^2$$

Rule: Integrate the gap equation and multiply by 2/G

Expanding the gap equation: $(\omega_n = (2n+1)\pi T)$

$$\Delta - 2G\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta} d\xi\left[\frac{\Delta}{(\omega_n^2+\xi^2)} - \frac{\Delta^3}{(\omega_n^2+\xi^2)^2} + \cdots\right] = 0$$

One gets:

$$\alpha = \frac{2}{G}\left(1 - 2G\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\omega_n^2+\xi^2)}\right)$$

→

Integrating over ξ and summing over n up to N

$$\beta = 4\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\omega_n^2+\xi^2)^2}$$

$$\omega_N = \delta \Rightarrow N \approx \frac{\delta}{2\pi T}$$

$$\boxed{\alpha(T) = \rho \log \frac{\pi T}{\gamma \Delta_0}}$$

Requiring $\alpha(T_c) = 0$

$$\boxed{T_c = \frac{\gamma}{\pi} \Delta_0 \approx 0.56693\,\Delta}$$

Also

$$\boxed{\beta(T) \approx \frac{7\rho}{8\pi^2 T_c^2} \zeta(3)}$$

and, from the gap equation $\left(\alpha(T) \approx -\rho \left(1 - \frac{T}{T_c} \right) \right)$

$$\boxed{\Delta^2(T) = -\frac{\alpha(T)}{\beta(T)} \Rightarrow \Delta(T) \approx \frac{2\sqrt{2}\pi T_c}{\sqrt{7\zeta(3)}} \left(1 - \frac{T}{T_c} \right)^{1/2} \approx 3.06 T_c \left(1 - \frac{T}{T_c} \right)^{1/2}}$$

Origin of the attractive interaction

- Coulomb force repulsive, need of an attractive interaction
- Electron-phonon interaction (Frolich 1950)
- Simple description: Jellium model (Pines et al. 1958): electrons + ions treated as a fluid.
- Interaction:

$$\frac{4\pi e^2}{q^2+k_s^2} + \underbrace{\frac{4\pi e^2}{q^2+k_s^2}\,\frac{\omega_q^2}{\omega^2-\omega_q^2}}_{\text{may give attraction}}, \qquad \omega_q \approx v_s q$$

$$k_s^2 = \frac{6\pi n e^2}{E_{F_o}}$$

$$(1/k_s \approx 1 A)$$

Coulomb interaction screened by electrons and ions

The relevance of gauge invariance

(See Weinberg (1990))

In the BCS ground state:

$$\langle \mathrm{O} \rangle = \langle \varepsilon_{\alpha\beta} \psi^{\alpha} \psi^{\beta} \rangle \neq 0$$

The $\mathrm{U(1)}_{\mathrm{em}}$ is broken since $\mathrm{Q}_{\mathrm{em}}(\mathrm{O}) = - 2\mathrm{e}$.

Introduce an order parameter Φ transforming as the operator O:

$$\mathrm{A}_{\mu} \to \mathrm{A}_{\mu} + \partial_{\mu}\Lambda, \ \psi \to \mathrm{e}^{\mathrm{ie}\Lambda}\psi, \ \Phi \to \mathrm{e}^{2\mathrm{ie}\Lambda}\Phi$$

As usual the phase of O is the Goldstone field associated to the breaking of the global U(1). Decompose:

$$\Phi(x) = \rho(x)e^{2ie\phi(x)}$$

Order parameter

Goldstone

$\rho(x)$ is gauge invariant, whereas

$$\phi(x) \to \phi(x) + \Lambda(x)$$

- ϕ dependence through $\partial_\mu \phi$
- U(1) broken to Z_2 $\left(\Lambda = 0, \quad \text{and} \quad \Lambda = \frac{\pi}{e} \right)$

- Gauge invariant Fermi field $\tilde{\psi} = e^{-ie\phi}\psi$
- Effective theory in terms of $\tilde{\psi}, A_\mu, \partial_\mu \phi$
- From gauge invariance only combinations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,\; A_\mu - \partial_\mu \phi$$

$$\boxed{L = -\frac{1}{4}\int d^3\vec{x}\, F_{\mu\nu}F^{\mu\nu} + L_s(A_\mu - \partial_\mu \phi)}$$

Eqs. of motion for ϕ: $0 = \partial_\mu \dfrac{\delta L_s}{\delta \partial_\mu \phi} = -\partial_\mu \dfrac{\delta L_s}{\delta A_\mu} = -\partial_\mu J^\mu$

Assume that L_s gives a stable state in absence of A and ϕ. This implies that

$$A_\mu = \partial_\mu \phi$$

is a local minimum and that

$$\left.\frac{\delta^2 L_s}{\delta(A_\mu - \partial_\mu \phi)^2}\right|_{A_\mu = \partial_\mu \phi} \neq 0$$

Well inside the superconductor we will be at the minimum. The em field is a pure gauge and

$$\boxed{F_{\mu\nu} = 0 \Rightarrow \vec{B} = 0}$$ → Meissner effect

Close the minimum:

$$L_s(A_\mu - \partial_\mu \phi) \approx L_s(0) + \frac{1}{2} \underbrace{\left.\frac{\delta^2 L_s}{\delta(A_\mu - \partial_\mu \phi)^2}\right|_{A_\mu = \partial_\mu \phi}}_{\dim = E \times E^{-2} = L} (A_\mu - \partial_\mu \phi)^2$$

$$L_s \approx \frac{L^3}{\lambda^2} \left| \vec{A} - \vec{\nabla}\phi \right|^2$$

L^3 = volume, λ some typical length where the field is not a pure gauge

$$\left|\vec{A} - \vec{\nabla}\phi\right| \approx BL \quad \Rightarrow \quad L_s \approx \frac{B^2L^5}{\lambda^2}$$

Cost of expelling B $\qquad B^2L^3$

Convenience in expelling B if $\dfrac{B^2L^5}{\lambda^2} \gg B^2L^3$

$$\boxed{L \gg \lambda}$$

Since $\vec{J} \propto \vec{\nabla} \wedge \vec{B}$ the current flows at the surface in a region of thickness λ

Superconductivity

Current density conjugated to ϕ: $\frac{\delta L_s}{\delta \dot{\phi}} = -\frac{\delta L_s}{\delta A_0} = -J_0$

Hamilton equation: $\dot{\phi}(x) = \frac{\delta H_s}{\delta(-J_0(x))} = -V(x)$

In stationary conditions the voltage $V(x) = 0$, with J not zero

Close to the phase transition the Goldstone field ϕ is not the only long wave-length mode.

Consider again

$$\Phi(x) = \rho(x)e^{2ie\phi(x)}$$

and expand L_s for small Φ

$$L_s \approx \int d^3\vec{x}\left[-\frac{1}{2}\Phi^*\left|(\vec{\nabla}-2ie\vec{A})\right|^2\Phi-\frac{1}{2}\alpha\left|\Phi\right|^2-\frac{1}{4}\beta\left|\Phi\right|^4\right]$$

$$L_s \approx \int d^3\vec{x}\left[-\boxed{2e^2\rho^2}(\vec{\nabla}\phi-e\vec{A})^2-\frac{1}{2}(\vec{\nabla}\rho)^2-\frac{1}{2}\alpha\rho^2-\frac{1}{4}\beta\rho^4\right]$$

$$\lambda = \frac{1}{\sqrt{4e^2\langle\rho^2\rangle}} \qquad \langle\rho^2\rangle = -\frac{\alpha}{\beta}$$

Looking at the fluctuations: $\rho = \rho' + \langle \rho \rangle$

$$\vec{\nabla}^2 \rho' = -2\alpha\rho'$$

Coherence length: $\boxed{\xi = \frac{1}{\sqrt{-2\alpha}}}$

Notice that in the SM: $\lambda = \frac{1}{M_V^2}, \quad \xi = \frac{1}{M_H^2}$