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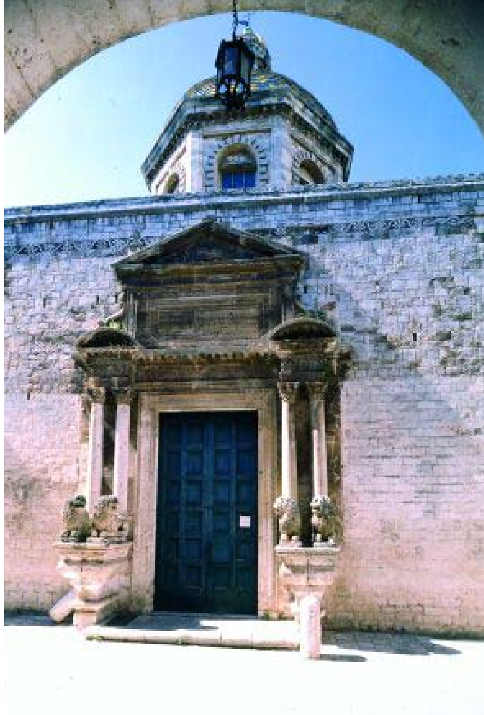

# Inhomogeneous color superconductivity


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# Summary

- Introduction
- Effective theory of CS
- Gap equation
- The inhomogeneous phase (LOFF): phase diagram and crystalline structure
- Phonons
- LOFF phase in compact stellar objects
- Outlook

# Introduction

★ $m_u, m_d, m_s << \mu$: CFL phase

$$\langle 0|\psi^{\alpha}_{aL}\psi^{\beta}_{bL}|0\rangle = -\langle 0|\psi^{\alpha}_{aR}\psi^{\beta}_{bR}|0\rangle \cong \Delta\varepsilon^{\alpha\beta C}\varepsilon_{abC}$$

$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \Rightarrow SU(3)_{c+L+R}$$

★ $m_u, m_d << \mu << m_s$ : 2SC phase

$$\langle 0|\psi^{\alpha}_{aL}\psi^{\beta}_{bL}|0\rangle = \Delta\varepsilon^{\alpha\beta 3}\varepsilon_{ab}$$

$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \Rightarrow SU(2)_c \otimes SU(2)_L \otimes SU(2)_R$$

In this situation strange quark decouples. But what happens in the intermediate region of μ? The interesting region is for **(see later)**

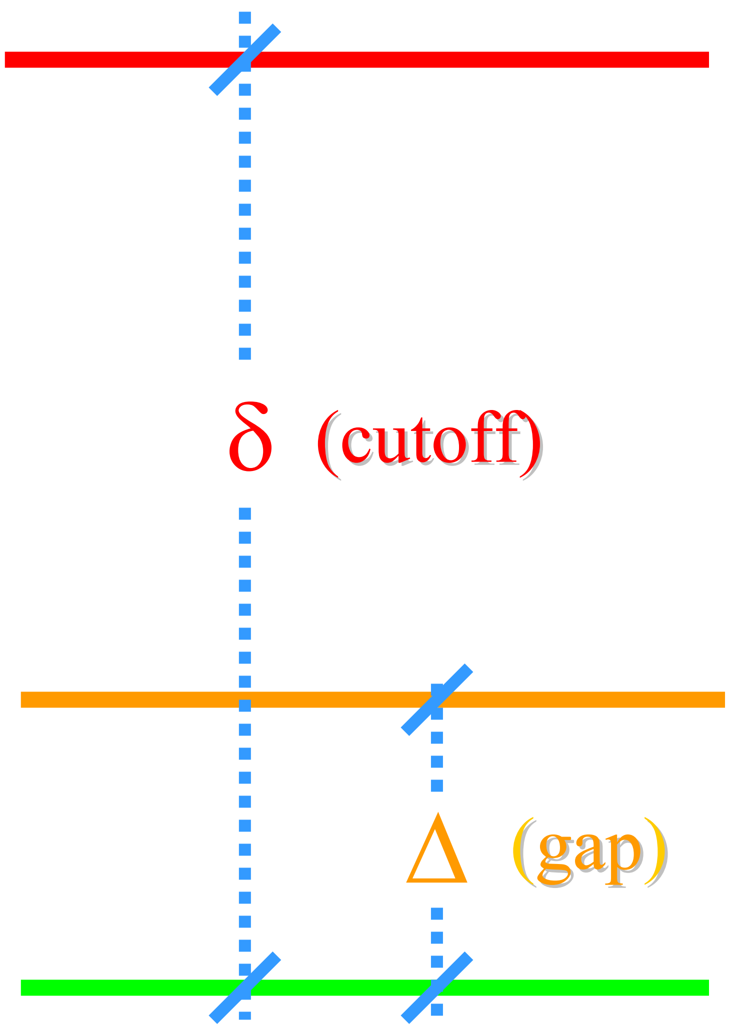
$$\mu \sim m_s^2/\Delta$$

**Possible new inhomogeneous phase of QCD**

# LOFF phase

# Effective theory of Color Superconductivity

# Relevant scales in CS




$p_F$

Fermi momentum defined by

$$E(\vec{p}_F) = \mu$$

The cutoff is of order $\omega_D$ in superconductivity and $> \Lambda_{QCD}$ in QCD

$$\Delta << \delta << p_F$$

# Hierarchies of effective lagrangians

Microscopic description → $L_{QCD}$

$p - p_F \gg \delta$

$p_F + \delta$

Quasi-particles (dressed fermions as electrons in metals). Decoupling of antiparticles (Hong 2000) → $L_{HDET}$

$\delta \gg p - p_F \gg \Delta$

$\Delta \ll \delta \ll p_F$

$p_F + \Delta$

Decoupling of gapped quasi-particles. Only light modes as Goldstones, etc. (R.C. & Gatto; Hong, Rho & Zahed 1999) → $L_{Gold}$

$p - p_F \ll \Delta$

$\Delta$

$p_F$

# Physics near the Fermi surface

$$(\Delta << \delta << p_F)$$

Relevant terms in the effective description

(see: Polchinski, TASI 1992, also Hong 2000; Beane, Bedaque & Savage 2000, also R.C., Gatto & Nardulli 2001)

$$S_R = \int \frac{d^3p}{(2\pi)^3}\, dt \left[ i\psi^{+}\partial_t\psi - (E(\vec{p}) - \mu)\psi^{+}\psi \right]$$

4-fermi attractive interaction is marginal (relevant at 1-loop)

$$(\vec{p}_1 = -\vec{p}_2,\ \vec{p}_3 = -\vec{p}_4)$$

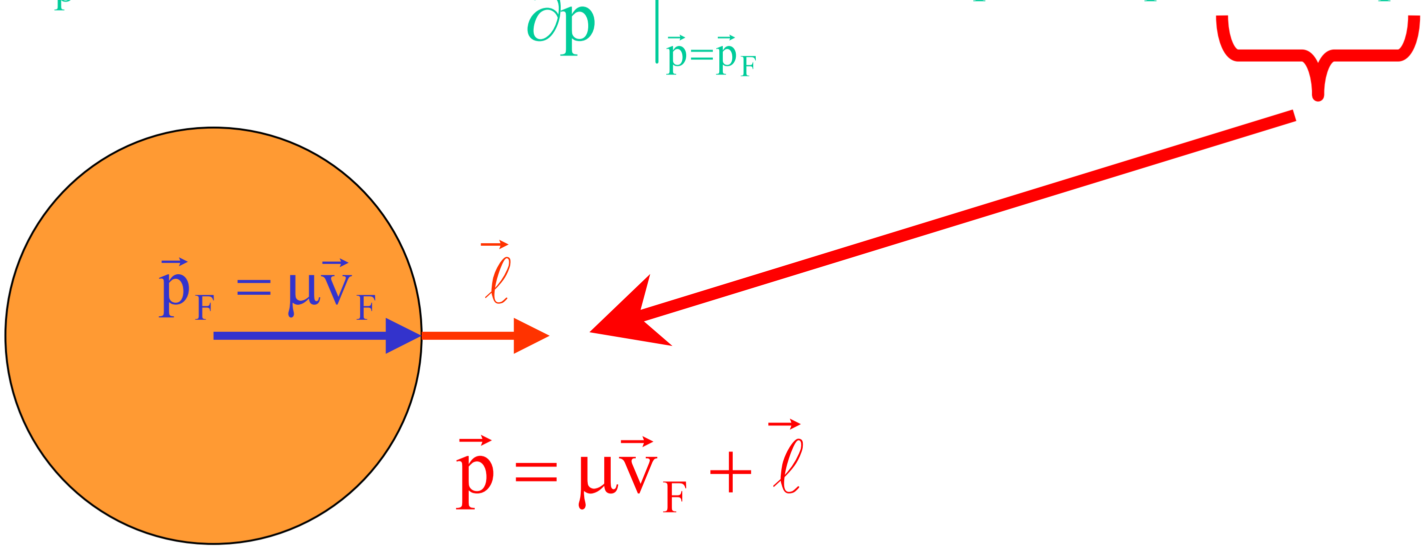
$$S_M = \frac{G}{2}\int \prod_{k=1}^{4} \frac{d^3p_k}{(2\pi)^3}\, dt\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\psi^{+}(\vec{p}_1)\psi(\vec{p}_3)\psi^{+}(\vec{p}_2)\psi(\vec{p}_4)$$

# $S_M$ gives rise di-fermion condensation producing a Majorana mass term. Work in the Nambu-Gorkov basis:

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi\ (\vec{p}) \\ C\psi^*(-\vec{p}) \end{pmatrix}$$

## Near the Fermi surface

$$\xi_{\vec{p}} \equiv E(\vec{p}) - \mu \approx \left. \frac{\partial E(\vec{p})}{\partial \vec{p}} \right|_{\vec{p}=\vec{p}_F} \cdot (\vec{p} - \vec{p}_F) \equiv \vec{v}_F \cdot (\vec{p} - \vec{p}_F)$$



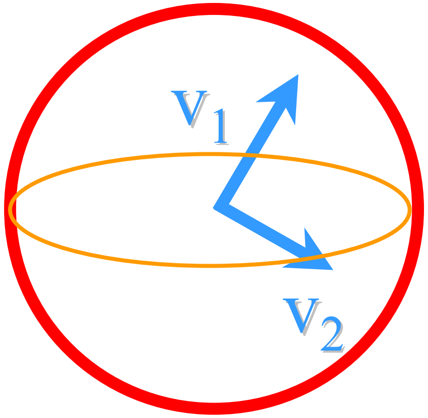

$$\vec{p} = \mu \vec{v}_F + \vec{\ell}$$

$$S^{-1} = \begin{bmatrix} E - \xi_{\vec{p}} & -\Delta \\ -\Delta^* & E + \xi_{\vec{p}} \end{bmatrix}$$

$$S = \frac{1}{E^2 - \xi_{\vec{p}} - |\Delta|^2} \begin{bmatrix} E + \xi_{\vec{p}} & \Delta \\ \Delta^* & E - \xi_{\vec{p}} \end{bmatrix}$$

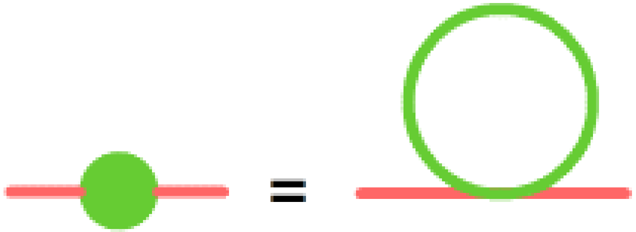
Dispersion relation $\varepsilon(\vec{p}) = \pm\sqrt{\xi_{\vec{p}}^2 + |\Delta|^2}$

At fixed $v_F$ only energy and momentum along $v_F$ are relevant




**Infinite copies of 2-d physics**

# Gap equation



$$1 = G\int \frac{d^4p}{(2\pi)^4}\frac{1}{p_4{}^2 + |\vec{p}|^2 + \Delta_{BCS}^2}$$

$$1 = GT\int \frac{d^3p}{(2\pi)^3}\sum_{n=-\infty}^{+\infty}\frac{1}{((2n+1)\pi T)^2 + \varepsilon(\vec{p},\Delta)^2}$$

$$1 = \frac{G}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1 - n_u - n_d}{\varepsilon(\vec{p},\Delta)}$$

$$n_u = n_d = \frac{1}{e^{\varepsilon(\vec{p},\Delta)/T} + 1}$$

For $T \to 0$

$$1 = \frac{G}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\xi^2(\vec{p}) + \Delta_{BCS}^2}}$$

At weak coupling

$$1 \cong \frac{G}{2\pi^2}\frac{p_F^2}{v_F}\log\frac{2\delta}{\Delta_{BCS}}$$

$(\delta = \text{cutoff})$

$$\Delta_{BCS} \approx 2\delta e^{-\frac{2}{G\rho}}$$

$$\rho = \frac{p_F^2}{\pi^2 v_F}$$

density of states

With G fixed by $\chi$SB at T = 0, requiring

$M_{const} \sim 400$ MeV

and for typical values of $\mu \sim 400 - 500$ MeV one gets

$$\Delta \approx 10 \div 100 \text{ MeV}$$

Evaluation from QCD first principles at asymptotic $\mu$

(Son 1999)

$$\Delta \approx b \mu g_s^5 e^{-\frac{3\pi^2}{\sqrt{2} g_s}}$$

Notice the behavior exp(-c/g) and not exp(-c/g$^2$) as one would expect from four-fermi interaction

For $\mu \sim 400$ MeV one finds again $\Delta \approx 10 \div 100$ MeV

# The inhomogeneous phase (LOFF)

In many different situations the “would be” pairing fermions belong to Fermi surfaces with different radii:

- Quarks with different masses
- Requiring electrical neutrality and/or weak equilibrium

Consider 2 fermions with $m_1 = M$, $m_2 = 0$ at the same chemical potential $\mu$. The Fermi momenta are

$$p_{F1} = \sqrt{\mu^2 - M^2}$$

$$p_{F2} = \mu$$

To form a BCS condensate one needs common momenta of the pair $p_F^{comm}$

$$p_F^{comm} = \mu - \frac{M^2}{4\mu}$$

Grand potential at $T = 0$ for a single fermion

$$\Omega = 2\int_0^{p_F} \frac{d^3p}{(2\pi)^3}\left(\varepsilon(\vec{p}) - \mu\right)$$

$$\Delta\Omega \approx 2\sum_{i=1}^{2}\mu^2 (p_F^{comm} - p_{Fi})(\varepsilon_i(p_F^{comm}) - \mu) \approx M^4$$

Pairing energy $\approx -\mu^2\Delta^2$

Pairing possible if $\frac{M^2}{\mu} \leq \Delta$

The problem may be simulated using massless fermions with different chemical potentials (Alford, Bowers & Rajagopal 2000)

Analogous problem studied by Larkin & Ovchinnikov, Fulde & Ferrel 1964. Proposal of a new way of pairing. LOFF phase

❖ LOFF: ferromagnetic alloy with paramagnetic impurities.

❖ The impurities produce a constant exchange field acting upon the electron spins giving rise to an effective difference in the chemical potentials of the opposite spins.

❖ Very difficult experimentally but claims of observations in heavy fermion superconductors (Gloos & al 1993) and in quasi-two dimensional layered organic superconductors (Nam & al. 1999, Manalo & Klein 2000)

$\mu_1 \neq \mu_2$ or paramagnetic impurities ($\delta\mu \sim H$) give rise to an energy additive term

$$H_I = -\delta\mu\sigma_3$$

Gap equation

$$1 = G\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p_4 + i\delta\mu)^2 + |\vec{p}|^2 + \Delta^2}$$

Solution as for BCS $\Delta = \Delta_{BCS}$, up to (for $T = 0$)

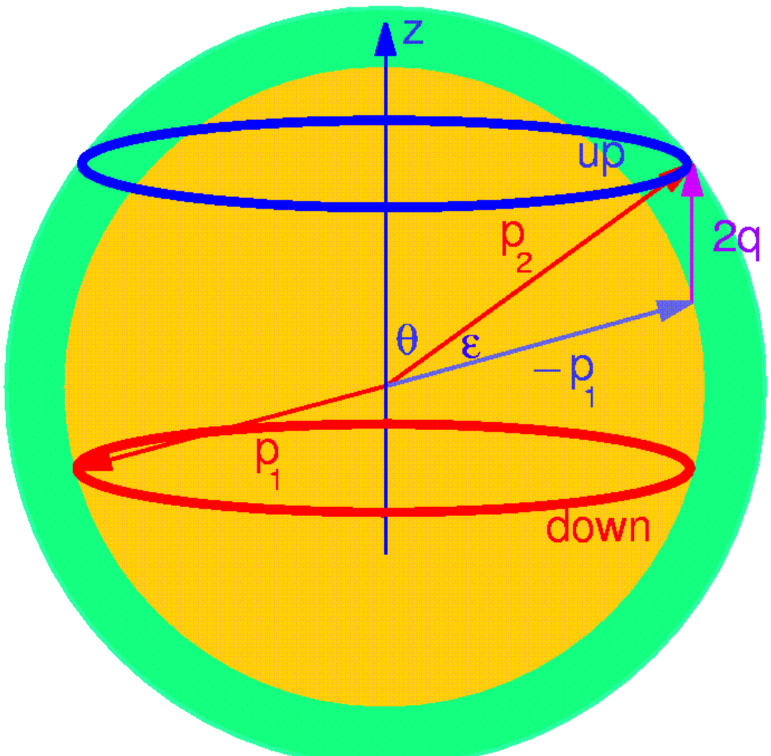
$$\delta\mu_1 = \frac{\Delta_{BCS}}{\sqrt{2}} \approx 0.707\Delta_{BCS}$$

$$\Omega_{BCS} - \Omega_{normal} = -\frac{\rho}{4}(\Delta^2{}_{BCS} - 2\delta\mu^2)$$

# According LOFF, close to first order line, possible condensation with **non zero total momentum**

$$\vec{p}_1 = \vec{k} + \vec{q} \qquad \vec{p}_2 = -\vec{k} + \vec{q} \rightarrow \langle \psi(x)\psi(x) \rangle = \Delta e^{2i\vec{q}\cdot\vec{x}}$$

More generally $\longrightarrow \langle \psi(x)\psi(x) \rangle = \Delta \sum_m c_m e^{2i\vec{q}_m\cdot\vec{x}}$




$$\vec{p}_1 + \vec{p}_2 = 2\vec{q}$$

$|\vec{q}|$ fixed variationally

$\vec{q}/|\vec{q}|$ chosen spontaneously

## Simple plane wave: energy shift

$$E(\vec{p})-\mu \to E(\pm\vec{k}+\vec{q})-\mu\mp\delta\mu \approx \xi \mp \bar{\mu}$$

$$\bar{\mu}=\delta\mu-\vec{v}_F\cdot\vec{q}$$

Gap equation:

$$1=\frac{g}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1-n_u-n_d}{\varepsilon(\vec{p},\Delta)}$$

$$n_u \neq n_d \quad \longrightarrow \quad n_{u,d}=\frac{1}{e^{(\varepsilon(\vec{p},\Delta)\pm\bar{\mu})/T}+1}$$

For $T\to 0$

blocking region $\varepsilon<|\bar{\mu}|$

$$1=\frac{g}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\varepsilon(\vec{p},\Delta)}(1-\theta(-\varepsilon-\bar{\mu})-\theta(-\varepsilon+\bar{\mu}))$$

The blocking region reduces the gap:

$$\Delta_{LOFF} << \Delta_{BCS}$$

Possibility of a crystalline structure (Larkin & Ovchinnikov 1964, Bowers & Rajagopal 2002)

$$\langle \psi(x)\psi(x) \rangle = \sum_{|\vec{q}_i|=1.2\delta\mu} \Delta_{\vec{q}_i} e^{2i\vec{q}_i \cdot \vec{x}}$$

$|\vec{q}_i|=1.2\delta\mu$ → see later

The $q_i$'s define the crystal pointing at its vertices.

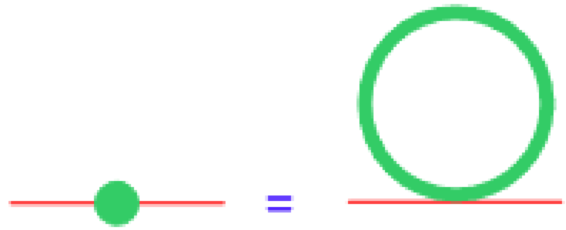
The LOFF phase is studied via a Ginzburg-Landau expansion of the grand potential

$$\Omega = \alpha\Delta^2 + \frac{\beta}{2}\Delta^4 + \frac{\gamma}{3}\Delta^6 + \cdots$$

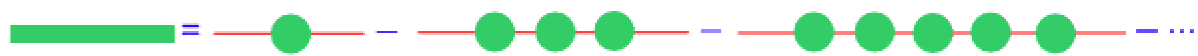
(for regular crystalline structures all the $\Delta_q$ are equal)

The coefficients can be determined microscopically for the different structures (Bowers and Rajagopal (2002))

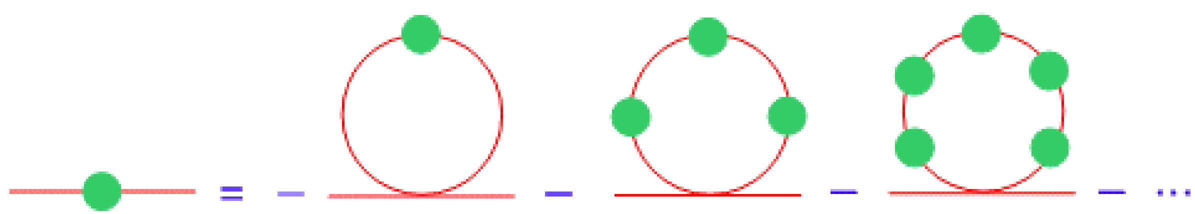
# Gap equation



# Propagator expansion



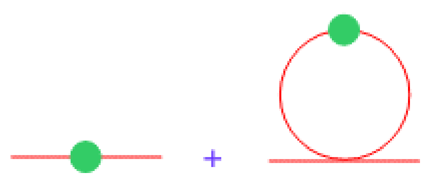
# Insert in the gap equation

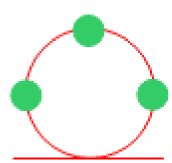


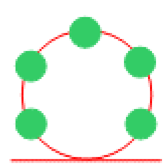
# We get the equation

$$\alpha\Delta + \beta\Delta^3 + \gamma\Delta^5 + \cdots = 0$$

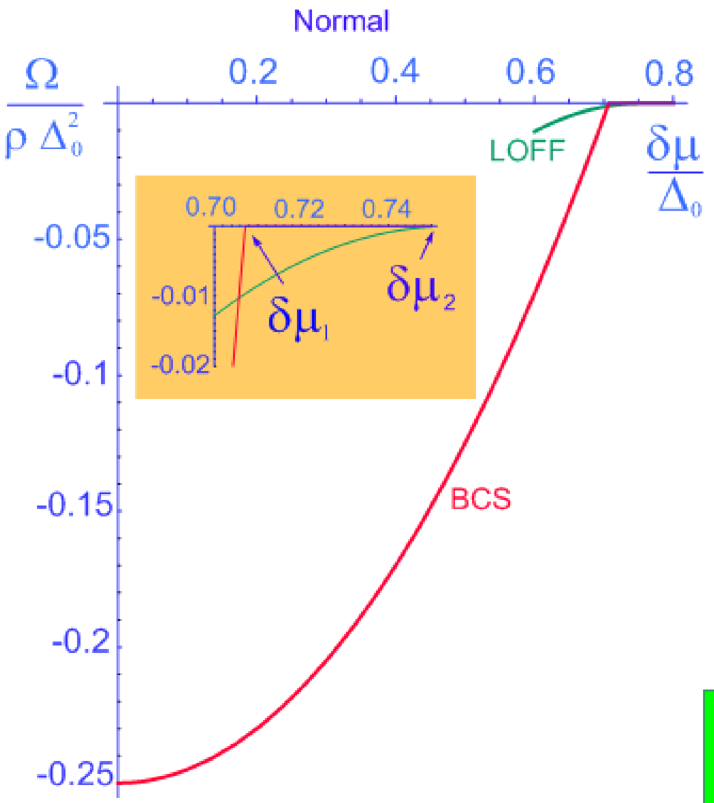
Which is the same as $$\frac{\partial\Omega}{\partial\Delta} = 0$$ with

$\alpha\Delta =$ 

$\beta\Delta^3 =$ 

$\gamma\Delta^5 =$ 

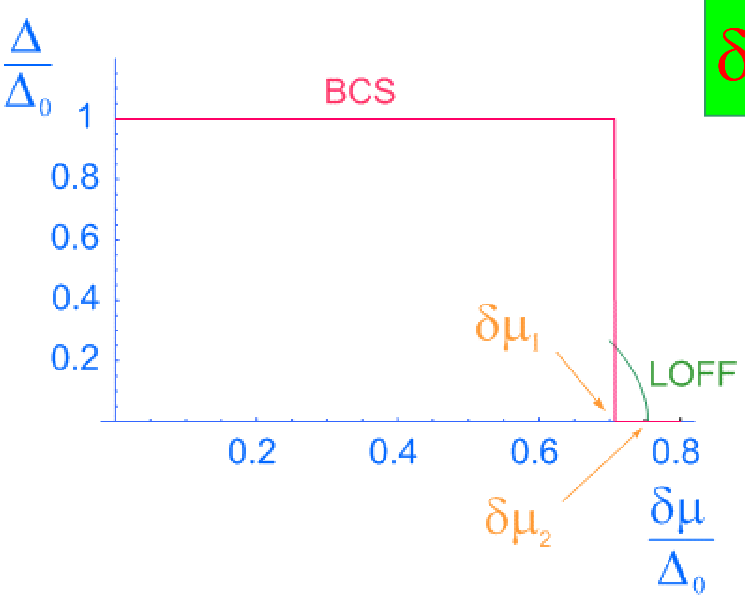
The first coefficient has universal structure, independent on the crystal. From its analysis one draws the following results




$$\Omega_{BCS} - \Omega_{normal} = -\frac{\rho}{4}(\Delta^2{}_{BCS} - 2\delta\mu^2)$$

$$\Omega_{LOFF} - \Omega_{normal} = -0.44\rho(\delta\mu - \delta\mu_2)^2$$

$$\Delta_{LOFF} \approx 1.15\sqrt{(\delta\mu_2 - \delta\mu)}$$




$$\delta\mu_1 = \Delta_{BCS}/\sqrt{2}$$

$$\delta\mu_2 \approx 0.754\Delta_{BCS}$$

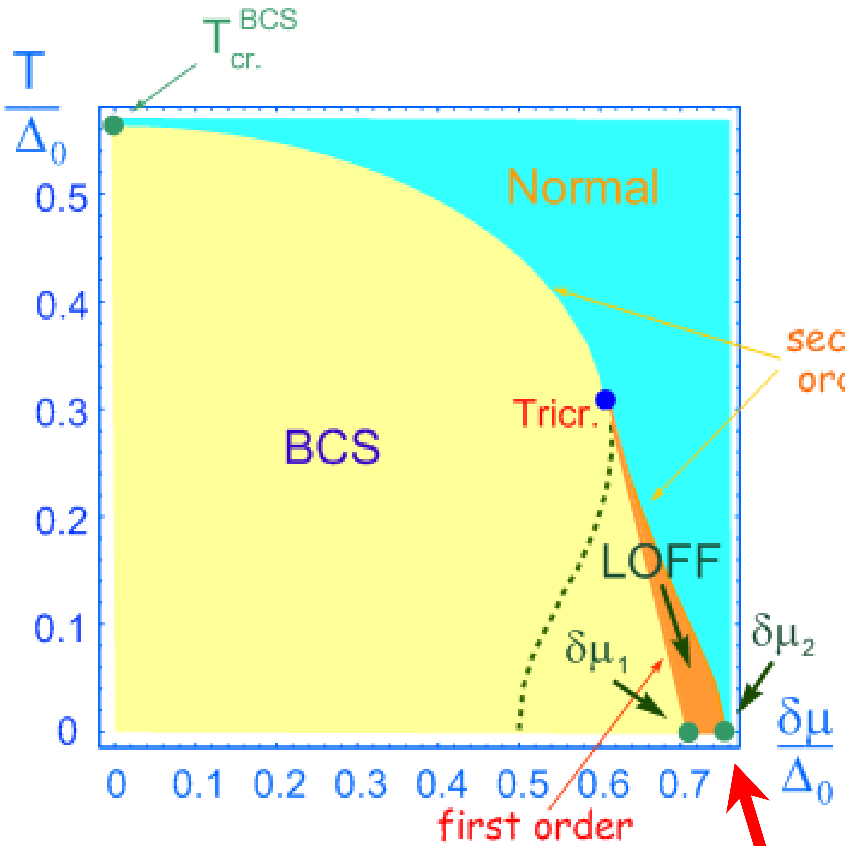
**Small window. Opens up in QCD?**

(Leibovich, Rajagopal & Shuster 2001;
Giannakis, Liu & Ren 2002)

# Results of Leibovich, Rajagopal & Shuster (2001)

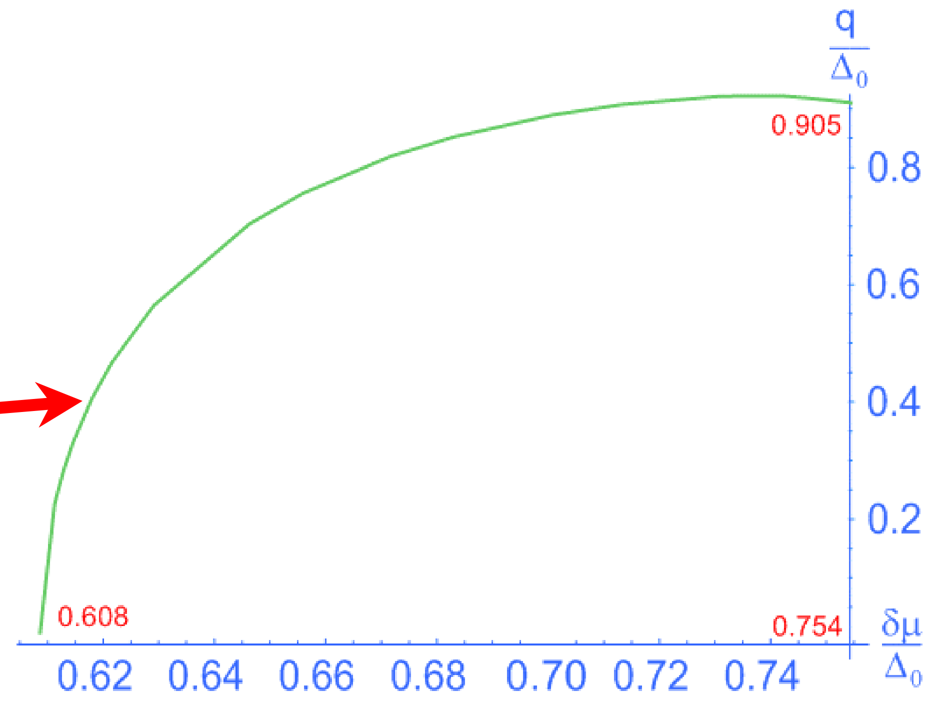
| $\mu$(MeV) | $\delta\mu_2/\Delta_{BCS}$ | $(\delta\mu_2 - \delta\mu_1)/\Delta_{BCS}$ |
|---|---|---|
| LOFF | 0.754 | 0.047 |
| 400 | 1.24 | 0.53 |
| 1000 | 3.63 | 2.92 |

# Single plane wave




## Critical line from

$$\frac{\partial\Omega}{\partial\Delta} = 0, \qquad \frac{\partial\Omega}{\partial q} = 0$$




Along the critical line

$$(\text{at } T = 0, \; q = 1.2\delta\mu_2)$$

| Structure | P | $\mathcal{G}$(Föppl) | $\bar{\beta}$ | $\bar{\gamma}$ | $\bar{\Omega}_{min}$ | $\delta\mu_*/\Delta_0$ |
|---|---|---|---|---|---|---|
| point | 1 | $C_{\infty v}(1)$ | 0.569 | 1.637 | 0 | 0.754 |
| antipodal pair | 2 | $D_{\infty h}(11)$ | 0.138 | 1.952 | 0 | 0.754 |
| triangle | 3 | $D_{3h}(3)$ | -1.976 | 1.687 | -0.452 | 0.872 |
| tetrahedron | 4 | $T_d(13)$ | -5.727 | 4.350 | -1.655 | 1.074 |
| square | 4 | $D_{4h}(4)$ | -10.350 | -1.538 | – | – |
| pentagon | 5 | $D_{5h}(5)$ | -13.004 | 8.386 | -5.211 | 1.607 |
| trigonal bipyramid | 5 | $D_{3h}(131)$ | -11.613 | 13.913 | -1.348 | 1.085 |
| square pyramid | 5 | $C_{4v}(14)$ | -22.014 | -70.442 | – | – |
| octahedron | 6 | $O_h(141)$ | -31.466 | 19.711 | -13.365 | 3.625 |
| trigonal prism | 6 | $D_{3h}(33)$ | -35.018 | -35.202 | – | – |
| hexagon | 6 | $D_{6h}(6)$ | 23.669 | 6009.225 | 0 | 0.754 |
| pentagonal bipyramid | 7 | $D_{5h}(151)$ | -29.158 | 54.822 | -1.375 | 1.143 |
| capped trigonal antiprism | 7 | $C_{3v}(13\bar{3})$ | -65.112 | -195.592 | – | – |
| cube | 8 | $O_h(44)$ | -110.757 | -459.242 | – | – |
| square antiprism | 8 | $D_{4d}(44)$ | -57.363 | -6.866 | – | – |
| hexagonal bipyramid | 8 | $D_{6h}(161)$ | -8.074 | 5595.528 | $-2.8 \times 10^{-6}$ | 0.755 |
| augmented trigonal prism | 9 | $D_{3h}(3\bar{3}\bar{3})$ | -69.857 | 129.259 | -3.401 | 1.656 |
| capped square prism | 9 | $C_{4v}(144)$ | -95.529 | 7771.152 | -0.0024 | 0.773 |
| capped square antiprism | 9 | $C_{4v}(14\bar{4})$ | -68.025 | 106.362 | -4.637 | 1.867 |
| bicapped square antiprism | 10 | $D_{4d}(14\bar{4}1)$ | -14.298 | 7318.885 | $-9.1 \times 10^{-6}$ | 0.755 |
| icosahedron | 12 | $I_h(15\bar{5}1)$ | 204.873 | 145076.754 | 0 | 0.754 |
| cuboctahedron | 12 | $O_h(4\bar{4}4)$ | -5.296 | 97086.514 | $-2.6 \times 10^{-9}$ | 0.754 |
| dodecahedron | 20 | $I_h(5555)$ | -527.357 | 114166.566 | -0.0019 | 0.772 |

General analysis

(Bowers and Rajagopal (2002))

Preferred structure:

face-centered cube

# Phonons

In the LOFF phase translations and rotations are broken

phonons

Phonon field through the phase of the condensate (R.C., Gatto, Mannarelli & Nardulli 2002):

$$\langle \psi(x)\psi(x) \rangle = \Delta e^{2i\vec{q}\cdot\vec{x}} \rightarrow \Delta e^{i\Phi(x)} \qquad \langle \Phi(x) \rangle = 2\vec{q}\cdot\vec{x}$$

Introduce:

$$\frac{1}{f}\phi(x) = \Phi(x) - 2\vec{q}\cdot\vec{x}$$

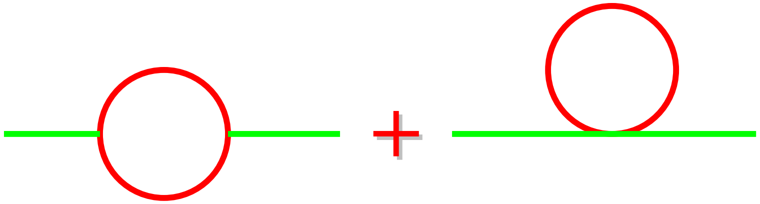
$$L_{phonon} = \left[ \frac{1}{2}\dot{\phi}^2 - v_\perp^2 \left( \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \right) - v_\parallel^2 \frac{\partial^2\phi}{\partial z^2} \right]$$

Coupling phonons to fermions (quasi-particles) trough the gap term

$$\Delta(x)\psi^T C\psi \rightarrow \Delta e^{i\Phi(x)}\psi^T C\psi$$

It is possible to evaluate the parameters of $L_{phonon}$

(R.C., Gatto, Mannarelli & Nardulli 2002)



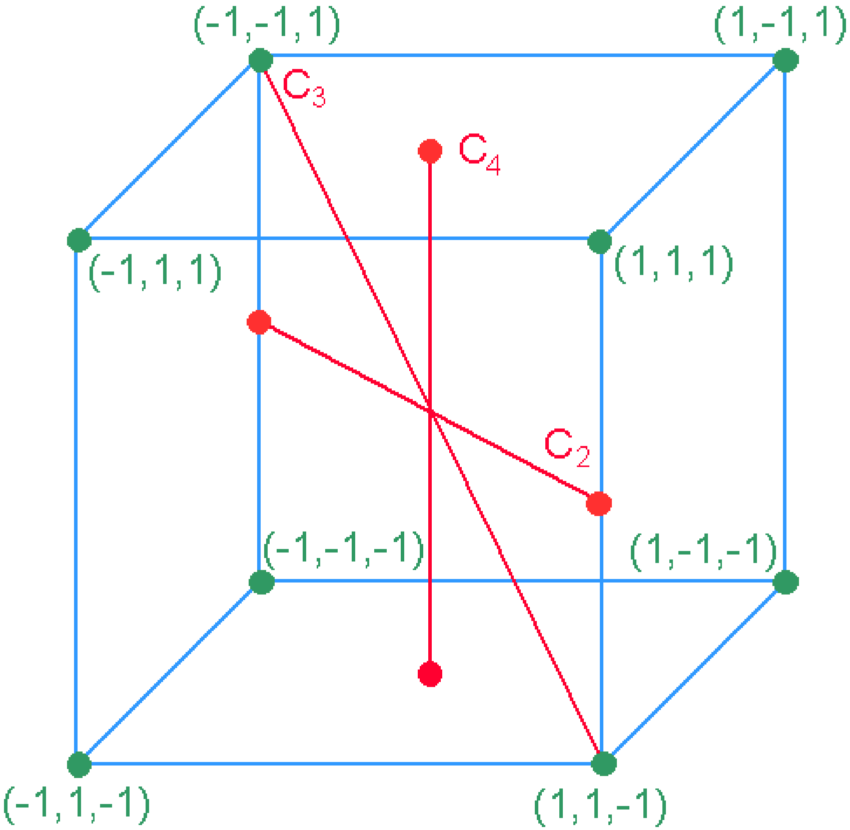
$$v_\perp^2 = \frac{1}{2}\left( 1 - \left( \frac{\delta\mu}{|\vec{q}|} \right)^2 \right) \approx 0.153 \qquad v_\parallel^2 = \left( \frac{\delta\mu}{|\vec{q}|} \right)^2 \approx 0.694$$

# Cubic structure

$$\Delta(x) = \Delta \sum_{k=1}^{8} e^{2i\vec{q}_k \cdot \vec{x}} = \Delta \sum_{i=1,2,3;\varepsilon_i=\pm} e^{2i|\vec{q}|\varepsilon_i x_i} \Rightarrow \Delta \sum_{i=1,2,3;\varepsilon_i=\pm} e^{i\varepsilon_i \Phi^{(i)}(x)}$$

$$\left\langle \Phi^{(i)}(x) \right\rangle = 2\,|\,\vec{q}\,|\,x_i$$

$$\frac{1}{f}\varphi^{(i)}(x) = \Phi^{(i)}(x) - 2\,|\,\vec{q}\,|\,x_i$$



# Using the symmetry group of the cube one gets:

$$L_{phonon} = \frac{1}{2}\sum_{i=1,2,3}\left(\frac{\partial\phi^{(i)}}{\partial t}\right)^2 - \frac{a}{2}\sum_{i=1,2,3}|\vec{\nabla}\phi^{(i)}|^2$$

$$-\frac{b}{2}\sum_{i=1,2,3}\left(\partial_i\phi^{(i)}\right)^2 - c\sum_{i<j=1,2,3}\left(\partial_i\phi^{(i)}\partial_j\phi^{(j)}\right)$$

Coupling phonons to fermions (quasi-particles) trough the gap term

$$\Delta(x)\psi^T C\psi \to \Delta \sum_{i=1,2,3;\varepsilon_i=\pm} e^{i\varepsilon_i\Phi^{(i)}(x)}\psi^T C\psi$$

we get for the coefficients

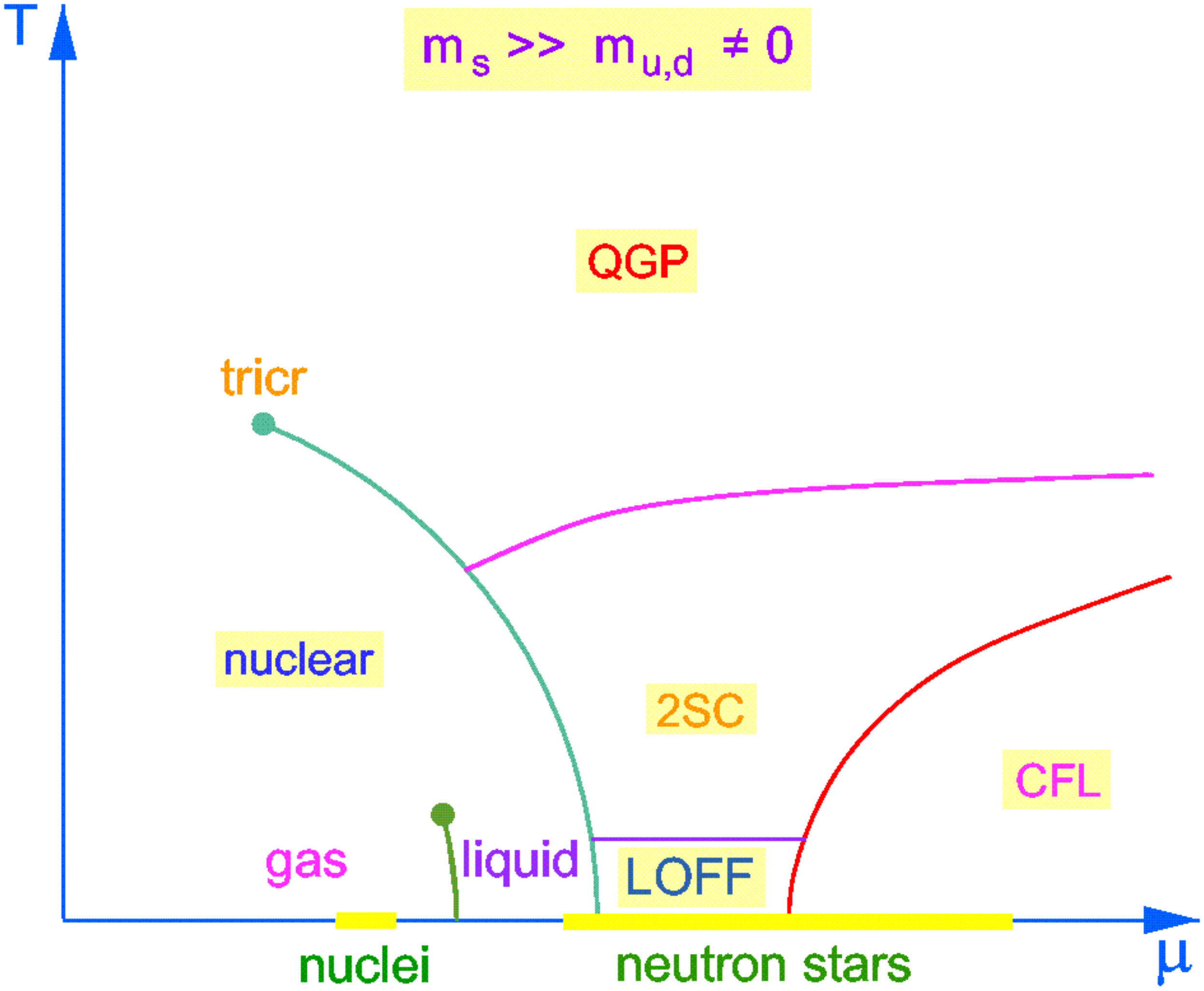
$$a = \frac{1}{12} \qquad b = 0 \qquad c = \frac{1}{12}\left(3\left(\frac{\delta\mu}{|\vec{q}\,|}\right)^2 - 1\right)$$

One can also evaluate the effective lagrangian for the gluons in the anisotropic medium. For the cube one finds

**Isotropic propagation**

This because the second order invariant for the cube and for the rotation group are the same!

# LOFF phase in CSO




Why the interest in the LOFF phase in QCD?

In neutron stars CS can be studied at T = 0

$$\frac{T_{ns}}{\Delta_{BCS}} \approx 10^{-6} \div 10^{-7} \qquad 20 \leq \Delta_{BCS}(\mathrm{MeV}) \leq 100$$

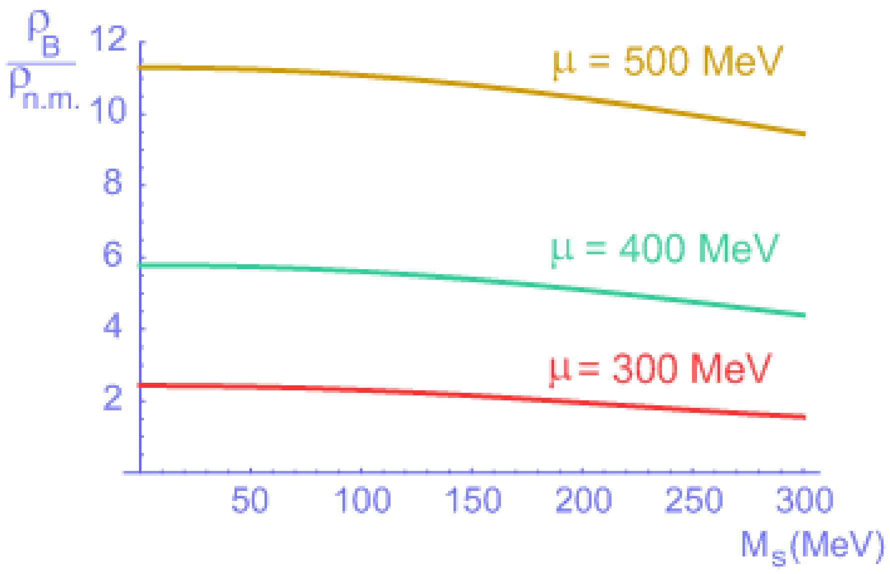
$$(1\mathrm{MeV} \approx 10^{10}\,\mathrm{K})$$

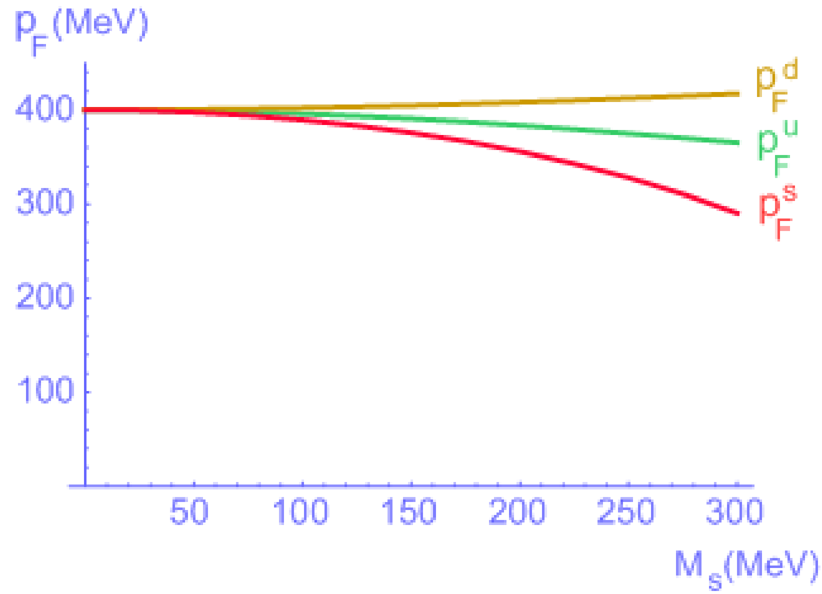
For LOFF state from $\delta p_F \sim 0.75\ \Delta_{BCS}$

$$14 \leq \delta\mu(\mathrm{MeV}) \leq 70$$

Orders of magnitude from a crude model: 3 free quarks

$$M_u = M_d = 0, \;\; M_s \neq 0$$




- $\rho_{n.m.}$ is the saturation nuclear density ~ $.15 \times 10^{15}$ g/cm$^3$
- At the core of the neutron star $\rho_B \sim 10^{15}$ g/cm$^3$

$$\frac{\rho_B}{\rho_{n.m.}} \approx 5 \div 6$$

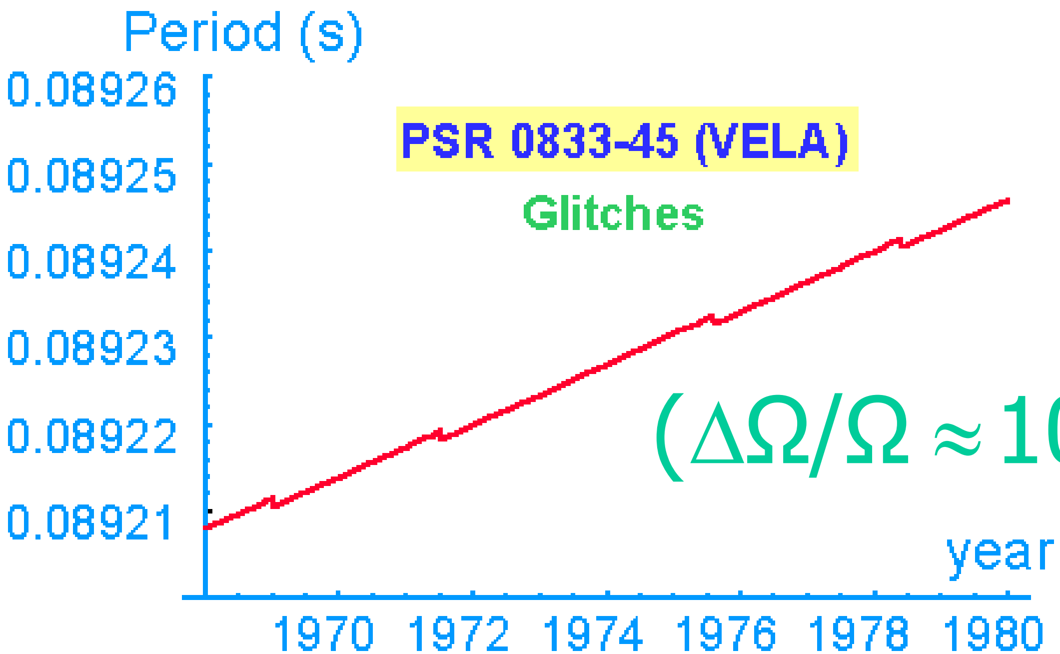
Choosing $\mu \sim 400$ MeV

| | |
|---|---|
| $M_s = 200$ | $\delta p_F = 25$ |
| $M_s = 300$ | $\delta p_F = 50$ |

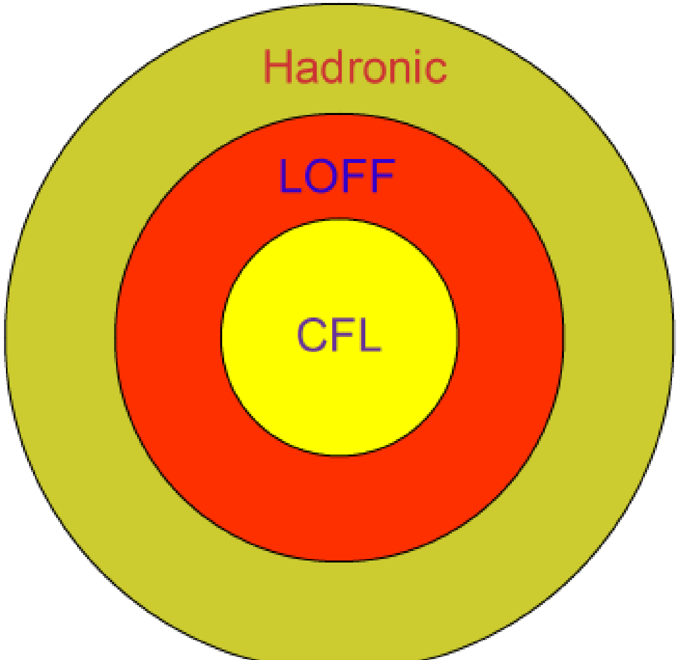
Right ballpark (14 - 70 MeV)

# Glitches: discontinuity in the period of the pulsars.

- Standard explanations require: metallic crust + superfluide inside (neutrons)
- LOFF region inside the star might provide a crystalline structure + superfluid CFL phase
- New possibilities for strange stars




$$(\Delta\Omega/\Omega \approx 10^{-6})$$



# Outlook

- **Theoretical problems**: Is the cube the optimal structure at T=0? Which is the size of the LOFF window?
- **Phenomenological problems**: Better discussion of the glitches (treatment of the vortex lines)
- **New possibilities**: Recent achieving of degenerate ultracold Fermi gases opens up new fascinating possibilities of reaching the onset of Cooper pairing of hyperfine doublets. Possibility of observing the **LOFF crystal?**