Compact Stars

in the

QCD Phase Diagram

Effective description of QCD at finite density

Roberto Casalbuoni

University of Florence and INFN

http://alphateo.fi.infn.it/∼casalbuo/copen.ps

Copenhagen, August 15-18, 2001

Summary

- Introduction
- Field theory at the Fermi surface
- The gap equation
- Effective theory for CFL phase
- Weak coupling calculations
- Conclusions

Introduction

Ideas about **Color Superconductivity** (CS) back to B. Barrois, NP **B129** (1977),390; S. Frautschi, Erice 1978; D. Bailin and A. Love, Phys. Report **107** (1984) 325. Only recently it has been realized that CS methods are very powerful to analyze in a rigorous fashion the high density and zero temperature region of QCD phase space (for a complete review see: K. Rajagopal and F. Wilczek, hep-ph/0011333)

- Naive expectation at very high density: asymptotic freedom ⇛ Fermi sphere of almost free quarks

- **BCS proved the instability of the Fermi surface in presence of an attractive arbitrarily weak interaction.** The previous picture changes to a **coherent state of particle-hole pairs**, the **Cooper pairs**

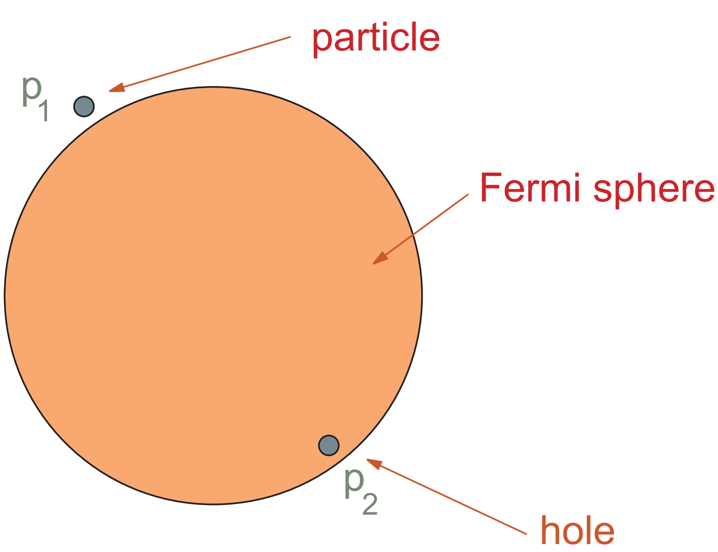
- The dominant interaction in QCD (gluon exchange) is attractive. **A diquark condensation is expected**

Field theory at the Fermi surface

The BCS theory is conveniently described in terms of a gas of almost free electrons (Landau). Main idea

quasiparticle ≈ dressed electrons

Quasiparticles are the excitations obtained by adding particles above the Fermi surface or removing particles from inside (holes).



Since we will consider particles and holes close to the Fermi sphere we can use a non-relativistic description (Polchinski, TASI 1992, `hep-th/9210046`)

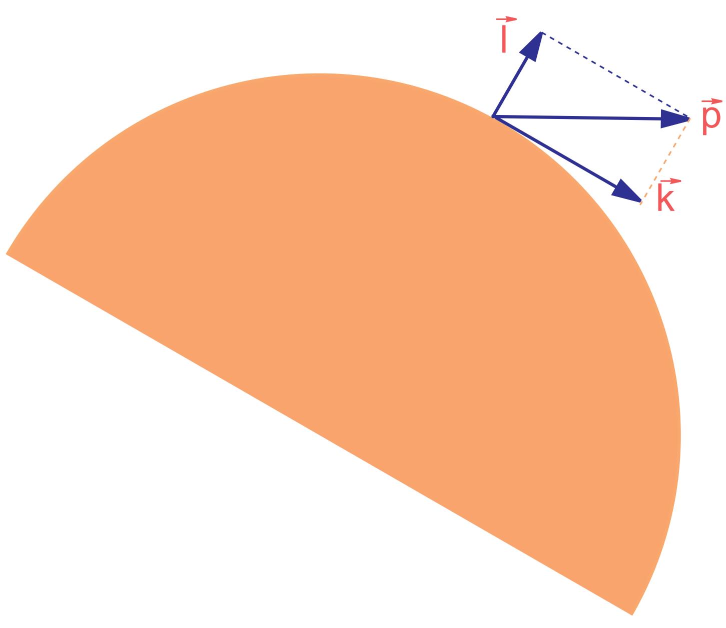
$$\int dt\, d^3\vec{p}\{i\psi_\sigma^\dagger(\vec{p})\partial_t\psi_\sigma(\vec{p}) - (\epsilon(\vec{p}) - \epsilon_F)\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})\}$$

($\sigma =$ spin index). The ground state is given by

states with $\epsilon(\vec{p}) < \epsilon_F$	filled
states with $\epsilon(\vec{p}) > \epsilon_F$	empty

The interestinq question is about the behaviour of the fields when scaling down the energies by a factor $s < 1$, that is toward ϵ_F. In order to realize the scaling momenta have to scale toward the Fermi surface, that is

when $E \to sE$, then $\vec{l} \to s\vec{l}, \quad \vec{k} \to \vec{k}$



Expanding around ϵ_F for small l

$$\epsilon(\vec{p}) - \epsilon_F = \left|\frac{\partial\epsilon(\vec{p})}{\vec{p}}\right|_{l=0} l + \mathcal{O}(l^2) \equiv v_F(\vec{k})l + \cdots$$

Under the scaling

$$dt \to s^{-1}dt, \quad d\vec{k} \to d\vec{k}, \quad d\vec{l} \to s\vec{l}$$
$$\partial_t \to s\partial_t, \quad l \to sl$$

we see that in the action

$$\int dt\, d^2\vec{k}\, d\vec{l}\{i\psi_\sigma^\dagger(\vec{p})\partial_t\psi_\sigma(\vec{p}) - lv_F(\vec{k})\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})\}$$

each term scales as s times the scaling of $\psi_\sigma^\dagger\psi_\sigma$

$$\psi_\sigma \approx s^{-1/2}\psi_\sigma$$

We can list the terms compatible with the symmetries of the problem

- Quadratic terms

$$\int dt\, d^2\vec{k}\, d\vec{l}\, \mu(\vec{k})\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})$$

scale as $s^{-1+1-2\times1/2} = s^{-1}$. This behaves as a mass term and it is relevant. But it can go into the definition of $\epsilon(\vec{p})$ producing at most a modification of the Fermi surface.

Adding one more time derivative or a term proportional to $|\vec{l}|$ makes the bilinear operators marginal as the ones already included. More time derivatives or $|\vec{l}|$ factors make the operator irrelevant.

- Quartic terms

$$\int dt\, d^2\vec{k}_1\, d\vec{l}_1\, d^2\vec{k}_2\, d\vec{l}_2\, d^2\vec{k}_3\, d\vec{l}_3\, d^2\vec{k}_4\, d\vec{l}_4$$
$$V(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)\psi_\sigma(\vec{p}_1)^\dagger\psi_\sigma(\vec{p}_3)\psi_{\sigma'}(\vec{p}_2)^\dagger\psi_{\sigma'}(\vec{p}_4)$$
$$\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$

scales as $s^{-1+4-4\times1/2} = s$ times the scaling of the delta-function. Generally one can neglect the longitudinal momenta inside the delta function getting

$$\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \approx \delta^3(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)$$

In this case the term is irrelevant. However consider the scattering $\vec{p}_1 + \vec{p}_2 \to \vec{p}_3 + \vec{p}_4$. Expanding

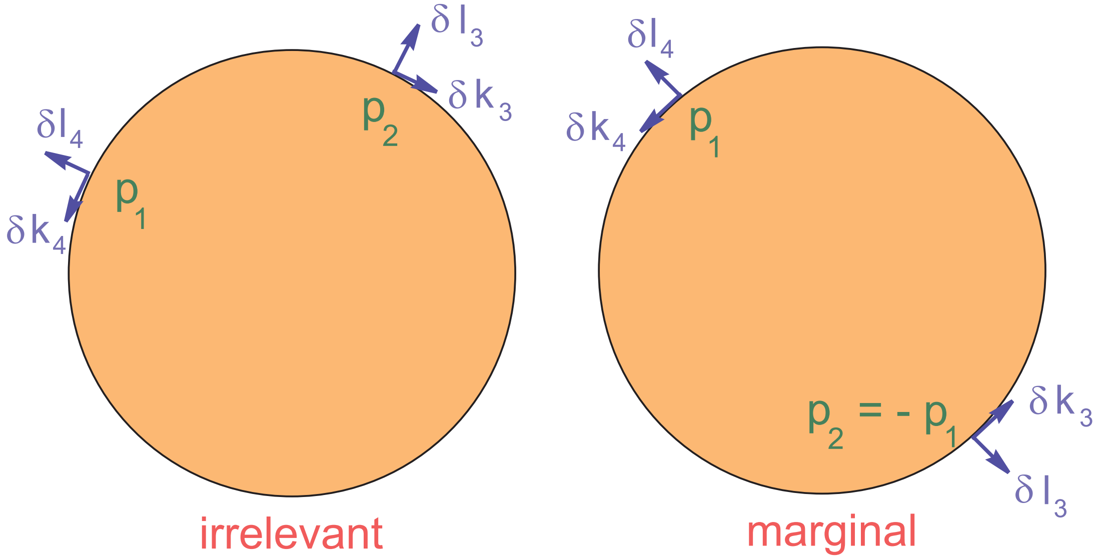
$$\vec{p}_3 = \vec{p}_1 + \delta\vec{k}_3 + \delta\vec{l}_3,\ \ \vec{p}_4 = \vec{p}_2 + \delta\vec{k}_4 + \delta\vec{l}_4$$

we get for the delta function

$$\delta^3(\delta\vec{k}_3 + \delta\vec{k}_4 + \delta\vec{l}_3 + \delta\vec{l}_4)$$

For arbitrary $\vec{p}_1$ and $\vec{p}_2$ the transverse momenta $\vec{\delta} k_3$ and $\vec{\delta} k_4$ span all the space. However for $\vec{p}_1 = -\vec{p}_2$ the delta function factorizes

$$\delta^2(\delta\vec{k}_3 + \delta\vec{k}_4)\delta(\delta\vec{l}_3 + \delta\vec{l}_4)$$

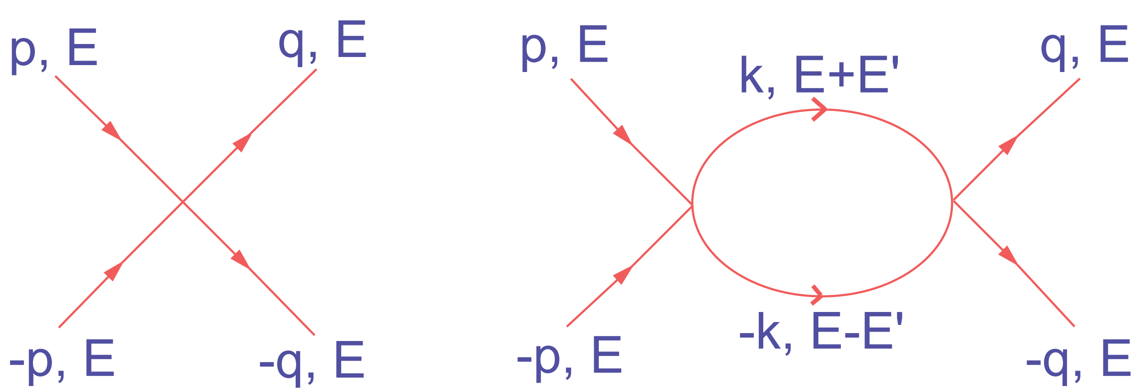


In this case the delta function scales as s^{-1} and the interaction is marginal. Notice that the previous arguments holds for any number of space dimensions. The exceptions are one-dimensional problems where quartic interactions are always marginal.

- Higher interactions

All interactions with a higher number of fermion fields are irrelevant. For instance, with 6 fermi fields we get a scaling factor $s^{-1+6-6\times 1/2} = s^2$ times the scaling of the delta-function. For N fermi fields we get $s^{-1+N-N\times 1/2} = s^{N/2-1}$ again times the scaling of the delta function.

The previous analysis shows that the excitations around the Fermi surface are essentially free, BUT one has to check the quantum corrections to the marginal operators



Assuming $V(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \equiv V$ as a constant one gets for the four-fermi coupling at one loop

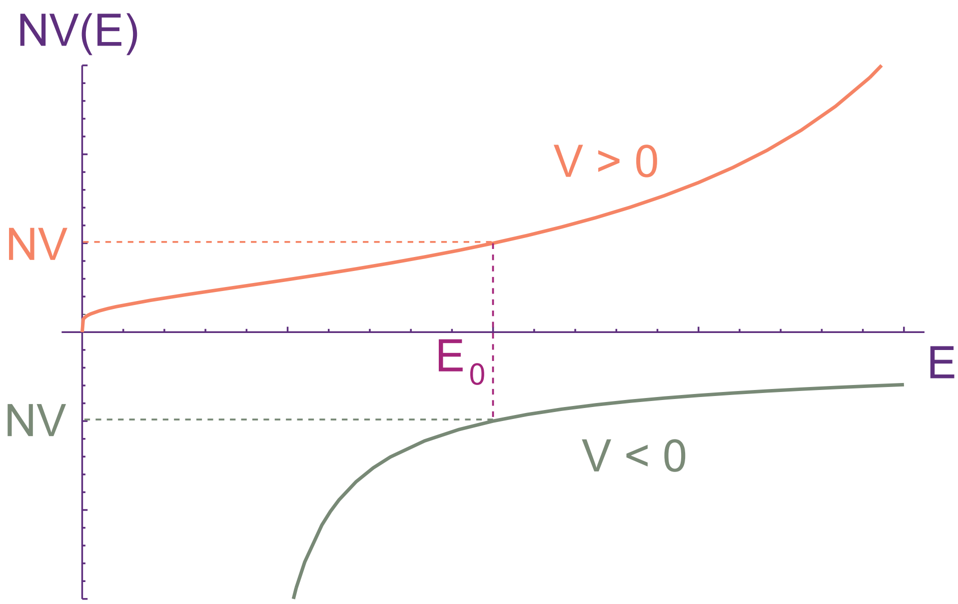
$$V(E) = V - NV^2 \log(E_0/E) + \mathcal{O}(V^3)$$

where E_0 is an upper cutoff and

$$N = \int \frac{d^2\vec{k}}{(2\pi)^3} \frac{1}{v_F(\vec{k})}$$

is the density of states at the Fermi energy. By using RG equations one gets

$$V(E) = \frac{V}{1 - NV \log(E/E_0)}$$



According to the sign of $V(E_0) = V$ we have

$V > 0$ repulsive $\to V(E)$ weaker for $E \to 0$
$V < 0$ attractive $\to V(E)$ stronger for $E \to 0$

An attractive four-fermi interaction, no matters how weak it is at some scale E_0 becomes stronger scaling toward the Fermi surface. The one-loop approximation does not hold any more. Higher orders are important and a BCS condensate $\langle \psi(\vec{p})\psi(-\vec{p}) \rangle$ is formed. **This is the physical origin of superconductivity**.

The gap

The instability due to marginal four-fermi interactions is physically avoided through the formation of condensates. This can be studied by various methods: energy considerations, auxiliary fields, CJT analysis, etc.. We describe here the first method using a toy model with only two Fermi oscillators (Hubbard toy-model) and hamiltonian

$$H = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) + G a_1^\dagger a_2^\dagger a_1 a_2$$

We introduce a trial wave-function

$$|\Psi\rangle_{trial} = \left[\cos\theta + \sin\theta\, a_1^\dagger a_2^\dagger\right]|0\rangle$$

from which

$$\Gamma \equiv {}_{trial}\langle\Psi|a_1 a_2|\Psi\rangle_{trial} = -\sin\theta\cos\theta$$

We can write $H = H_0 + H_{\rm res}$ with

$$H_0 = \epsilon(a_1^\dagger a_1 + a_2^\dagger a_2) - G\Gamma(a_1 a_2 - a_1^\dagger a_2^\dagger)$$

and

$$H_{\rm res} = G(a_1^\dagger a_2^\dagger + \Gamma)(a_1 a_2 - \Gamma) + G\Gamma^2$$

We get

$$\langle\Psi|H|\Psi\rangle_{trial} = 2\epsilon\sin^2\theta - G\Gamma^2$$

Minimizing the average of H

$$2\epsilon\sin 2\theta + 2G\Gamma\cos 2\theta = 0 \longrightarrow \tan 2\theta = -\frac{G\Gamma}{\epsilon}$$

and from Γ

$$\Gamma = -\frac{1}{2}\sin 2\theta = \frac{1}{2}\frac{G\Gamma}{\sqrt{\epsilon^2 + G^2\Gamma^2}}$$

or

$$\boxed{1 = \frac{1}{2}\frac{G}{\sqrt{\epsilon^2 + \Delta^2}}}$$

with

$$\Delta = G\Gamma$$

This is the gap equation. The state $|\Psi\rangle_{\rm trial}$ is nothing but the fundamental state in the broken phase corresponding to the formation of the condensate Γ.

Neglecting H_{res} (*mean-field approximation*) we can look for the Bogoliubov transformation such to make H_0 canonical

$$A_1 = a_1 \cos\theta - a_2^\dagger \sin\theta$$
$$A_2 = a_1^\dagger \sin\theta + a_2 \cos\theta$$

one finds

$$A_{1,2}|\Psi\rangle_{\text{trial}} = 0$$

$$H_0 = (\epsilon - \sqrt{\epsilon^2 + \Delta^2}) + \sqrt{\epsilon^2 + \Delta^2}(A_1^\dagger A_1 + A_2^\dagger A_2)$$

showing that the energy of the quasiparticles (created by $A_i^\dagger$) is given by

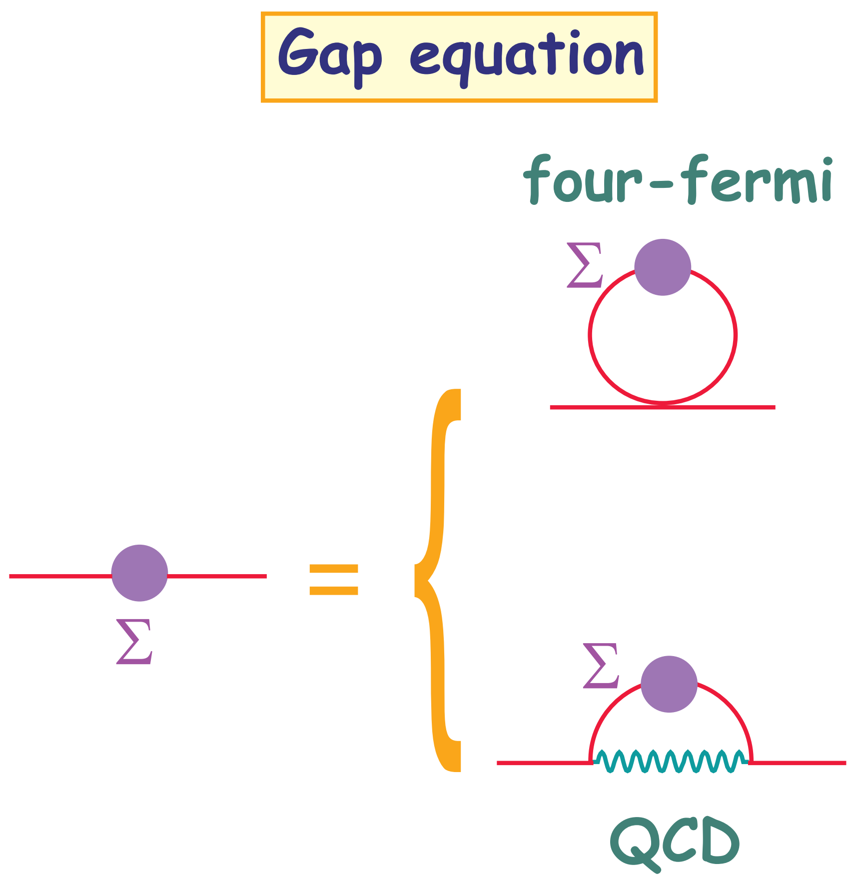
$$\boxed{E = \sqrt{\epsilon^2 + \Delta^2}}$$

At asymptotically high density one can calculate the gap from first principles since QCD is weakly coupled (Son, 1999; Schäfer and Wilczek, 1999; Pisarski and Rischke, 2000; Hong, 2000; Hong, Miransky, Shovkovy and Wijewardhana, 2000; Brown, Liu and Ren, 2000; Hsu and Schwetz, 2000; Schäfer, 2000; Shovkovy and Wijewardhana, 2000). However the physically interesting density regime for neutron star or heavy ions is up to a few times the nuclear density, $\mu \lesssim 500\ MeV$, whereas the actual calculations are unlikely to be extrapolated below $10^8\ MeV$. Alternatively one can use phenomenological interactions known to capture the essential features of QCD in the regime of interest. For two flavors one can use the instanton vertex producing a four-fermi interactions, or with more flavors to work in the one-gluon exchange approximation.

The corresponding gap equations are illustrated in figure. The results agree within a factor of about two. The gap equation has the form

$$\Sigma(k) = -\frac{1}{(2\pi)^4}\int d^4q G^{-1}(q) V(k-q)$$

with $G^{-1}(q)$ the full propagator and $V(p)$ the vertex function, momentum independent in the four-fermi case, whereas for QCD includes the gluon propagator



Since one expects a diquark condensate it is convenient to introduce the Nambu-Gorkov fields

$$\Psi = (\psi, \bar{\psi}^T)$$

The wave operator is then postulated of the form

$$D(q) = D(q)_{\text{free}} + \Sigma = \begin{pmatrix} \not{q} + \mu\gamma_0 & \gamma_0\Delta\gamma_0 \\ \Delta & (\not{q} - \mu\gamma_0)^T \end{pmatrix}$$

equivalent to introduce in the lagrangian a mass gap term of the type $\psi^T C \Delta \psi$. For a four-fermi interaction Δ is a matrix in color-flavor-spin space. In the gluon-exchange case there is also a momentum dependence. The typical gap equation for a four-fermi interaction (neglecting antiparticle contribution) is

$$\Delta = 4G \int_0^\Lambda \frac{d^4p}{(2\pi)^4} \Big(\frac{\Delta}{p_0^2 + (|\vec{p}| - \mu)^2 + \Delta^2} \Big)$$

When the interaction strength, $G \to 0$, and $\Delta^2 \ll \mu\Lambda$

$$\Delta \propto \Delta\mu^2 G \log(\mu/\Delta) \to \Delta \propto \mu \exp(-c/(G\mu^2))$$

Replacing the effective four-fermi interaction with one-gluon exchange one expects

$$\Delta \propto \mu \exp(-c/g^2)$$

But taking into account the gluon propagator the gap equation is of the type

$$\Delta \propto g^2 \int d\epsilon \frac{\Delta}{\sqrt{\epsilon^2 + \Delta^2}} d\theta \frac{\mu^2}{\theta\mu^2 + \Delta^2}$$

The θ (angle between the internal and external momentum) term comes from the gluon propagator which, due to the spontaneous breaking of color, picks up a **Meissner mass** $\propto \Delta$ (Casalbuoni, Gatto, Nardulli, 2001). This gives a double-log contribution (Son, 1999; Rajagopal and Wilczek, 2001)

$$\Delta \propto \Delta g^2 (\log(\mu/\Delta))^2 \rightarrow \boxed{\Delta \propto \mu \exp(-c/g)}$$

The gap at large μ is much larger in QCD than in the case of a point-like interaction

If the coupling g is evaluated at μ and assume $1/g^2 \approx \log \mu$ the exponential gives a very weak suppression and for $\mu \to \infty$ we have $\Delta \to \infty$ and $\Delta/\mu \to 0$.

Color superconductivity is bounded to dominate physics at high density

The previous calculation shows that in 2-flavors QCD the following condensate is formed ($i, j = 1, 2$; $\alpha, \beta = 1, 2, 3$; $\psi^{i\alpha}_{L,R}$ Weyl spinors)

$$\boxed{\langle \psi^{i\alpha}_L(\vec{p}) C \psi^{j\beta}_L(-\vec{p}) \rangle = \frac{\Delta}{G} \epsilon^{\alpha\beta 3} \epsilon^{ij}}$$

The condensate originates the symmetry breaking (barring $U(1)$ factors)

$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R$$
$$\Downarrow$$
$$SU(2)_c \otimes SU(2)_L \otimes SU(2)_R$$

No global symmetry is broken and therefore no Goldstone bosons in the spectrum, but we get 5 massive and 3 massless gluons (more in the following). Numerically one takes a cut-off $\Lambda = 800\ MeV$, and fixing $M = 400\ MeV$ (constituent quark mass), one finds

$$\mu = 400\ MeV \;\Rightarrow\; \Delta = 106\ MeV$$
$$\mu = 500\ MeV \;\Rightarrow\; \Delta = 145\ MeV$$

A similar analysis for 3-flavors QCD leads to the condensate

$$\langle \psi^i_{\alpha L}(\vec{p}) C \psi^j_{\beta L}(-\vec{p}) \rangle = \frac{1}{G} P^{ij}_{\alpha\beta}$$

$$P^{ij}_{\alpha\beta} = \frac{1}{3}(\Delta_8 + \frac{1}{8}\Delta_1)\delta^i_\alpha \delta^j_\beta + \frac{1}{8}\Delta_1 \delta^i_\beta \delta^j_\alpha$$

This originates the breaking

$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R$$
$$\Downarrow$$
$$SU(3)_{c+L+R}$$

There are 8 global broken symmetries giving 8 Goldstone bosons and 8 massive gluons. We have now two gap parameters corresponding to the singlet and octet quark excitations. In color we have the symmetric $\mathbf{6}$ and the anti-symmetric $\bar{\mathbf{3}}$ channels. For $\Delta_1 = -2\Delta_8$ only the $\bar{\mathbf{3}}$ survives (see next). For $\Lambda = 800\ MeV$, $M = 400\ MeV$ and $\mu = 400\ MeV$ one gets

$$\Delta_8 = 80\ MeV, \quad \Delta_1 = -176\ MeV$$

very close to $\Delta_1 = -2\Delta_8$

For $\Delta_1 = -2\Delta_8$

$$P^{ij}_{\alpha\beta} = -\frac{1}{2}\Delta_1\left(\delta^i_\alpha\delta^j_\beta - \delta^i_\beta\delta^j_\alpha\right) = -\frac{1}{2}\Delta_1\epsilon^{ijA}\epsilon_{\alpha\beta A}$$

showing that the condensate corresponds to the antisymmetric channel both in color and in flavor (due to the statistics), that is, it belongs to the $(\bar{3}, \bar{3})$ representation of $SU(3)_c \otimes SU(3)_{L(R)}$

Effective Theory for the CFL phase

In 3 massless flavors QCD at high density the condensate can also be written as

$$\langle q^i_{\alpha L}(\vec{p}) C\, q^j_{\beta L}(-\vec{p})\rangle \approx \epsilon_{ijX}\epsilon^{\alpha\beta X} + \kappa(\delta^\alpha_i \delta^\beta_j + \delta^\beta_i \delta^\alpha_j)$$

$$\langle q^i_{\alpha L}(\vec{p}) C\, q^j_{\beta L}(-\vec{p})\rangle = -\langle q^i_{\alpha R}(\vec{p}) C\, q^j_{\beta R}(-\vec{p})\rangle$$

For $\kappa = 0$, $(\Delta_1 = -2\Delta_8)$ we have the channel $(\bar{3}, \bar{3})$. This produces the symmetry breaking

$$G = SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes U(1)_A$$
$$\Downarrow$$
$$H = SU(3)_{c+L+R} \otimes Z_2 \otimes Z_2$$

The $U(1)_A$ symmetry is restored at very high density for $N_f = 3$. The corresponding breaking originates asymptotically a massless Goldstone boson $(U(1)_A \to Z_2)$ (R. Rapp, T. Schäfer, E,V, Shuryak, Velkowsky, hep-ph/9904353; T. Schäfer, hep-ph/9909574; D.T. Son and M.A. Stephanov, Phys. Rev. **D61** (2000) 074012, hep-ph/9910941; *ibidem*, Erratum, **D62** (2000) 059902, hep-ph/0004095)

The condensates belong to the representation $(\bar{3},\bar{3})\oplus(6,6)$ of $SU(3)_c\otimes SU(3)_{L,R}$ To represent the Goldstone fields it is enough to consider one representation, say the $(\bar{3},\bar{3})$. We then introduce **Goldstone fields as the phases of the following condensates**

$$X^i_\alpha \approx \epsilon^{ijk}\epsilon_{\alpha\beta\gamma}\langle q^j_{\beta L} q^k_{\gamma L}\rangle^*$$

$$Y^i_\alpha \approx \epsilon^{ijk}\epsilon_{\alpha\beta\gamma}\langle q^j_{\beta R} q^k_{\gamma R}\rangle^*$$

With the notations

$$g_c \in SU(3)_c, \quad g_{L(R)} \in SU(3)_{L(R)}$$

$$\exp(i\alpha) \in U(1)_V, \quad \exp(i\beta) \in U(1)_A$$

we have

$$q_{L(R)} \in (\mathbf{3},\mathbf{3}) \text{ of } SU(3)_c \otimes SU(3)_{L(R)}$$

$$q_L \to \exp i(\alpha+\beta) q_L \text{ under } U(1)_V \otimes U(1)_A$$

$$q_R \to \exp i(\alpha-\beta) q_R \text{ under } U(1)_V \otimes U(1)_A$$

and under the total symmetry group

$$X \to g_c X g_L^T \exp(-2i\alpha - 2i\beta)$$

$$Y \to g_c Y g_R^T \exp(-2i\alpha + 2i\beta)$$

Being X and Y the phases of the condensates $(\bar{3}, \bar{3})$ we have

$$X, Y \in U(3)$$

The number of fields is

$$\#X + \#Y = (1+8) + (1+8) = 18$$

8 of these fields give mass to the gluons. **The physical NGB are 10** correponding to the breaking of the global symmetry

$$SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes U(1)_A$$
$$\Downarrow$$
$$SU(3)_{L+R} \otimes Z_2 \otimes Z_2$$

It is convenient to separate the $U(1)$ factors defining

$$X = \hat{X} \exp(2i\phi + 2i\theta), \quad Y = \hat{Y} \exp(2i\phi - 2i\theta)$$

with $\hat{X}, \hat{Y} \in SU(3)$. We introduce also

$$\det(X) = \exp(6i\phi + 6i\theta)$$

$$\det(Y) = \exp(6i\phi - 6i\theta)$$

The transformation properties are

$$\hat{X} \to g_c \hat{X} g_L^T, \quad \hat{Y} \to g_c \hat{Y} g_R^T$$

$$\phi \to \phi - \alpha, \quad \theta \to \theta - \beta$$

The breaking of the global symmetry can be also described by gauge invariant order parameters given by

$$\Sigma_j^i = (\hat{Y}_\alpha^j)^* \hat{X}_\alpha^i \to \Sigma = \hat{Y}^\dagger \hat{X}$$

$$d_X = \text{det}(X), \qquad d_Y = \text{det}(Y)$$

These are the 8+2 NGB's corresponding to the breaking of the global symmetry. Notice that

$$\Sigma \to g_R^* \Sigma \, g_L^T$$

Σ^{T} transforms as the usual chiral field

The effective lagrangian

Invariant terms can be built starting from the currents (ignoring for a while the local color symmetry)

$$J^\mu_X = \hat{X}\partial^\mu\hat{X}^\dagger, J^\mu_Y = \hat{Y}\partial^\mu\hat{Y}^\dagger$$

$$J^\mu_\phi = U\partial^\mu U^\dagger, J^\mu_\theta = V\partial^\mu V^\dagger$$

with

$$\hat{X} = e^{i\tilde{\Pi}^a_X T_a}, \hat{Y} = e^{i\tilde{\Pi}^a_Y T_a}, U = e^{i\phi/f^V_T}, V = e^{i\theta/f^A_T}$$

and $T_a \in$ Lie SU(3). The transformation properties under the total symmetry G are

$$J^\mu_{X,Y} \to g_c J^\mu_X g_c^\dagger, \qquad J^\mu_{\phi,\theta} \to J^\mu_{\phi,\theta}$$

The most general invariant lagrangian under G the space rotation group $O(3)$ and Parity ($X \leftrightarrow Y$, $U \leftrightarrow U$, $V \leftrightarrow V^\dagger$) is (R. C. and R. Gatto, Phys. Lett. **B464** (1999) 111)

$$\mathcal{L} = -\frac{F_T^2}{4} Tr[(J_X^0 - J_Y^0)^2] - \alpha_T \frac{F_T^2}{4} Tr[(J_X^0 + J_Y^0)^2]$$

$$-\frac{{f_T^V}^2}{2}(J_\phi^0)^2 - \frac{{f_T^A}^2}{2}(J_\theta^0)^2$$

$$+\frac{F_S^2}{4} Tr[(\vec{J}_X - \vec{J}_Y)^2] + \alpha_S \frac{F_S^2}{4} Tr[(\vec{J}_X + \vec{J}_Y)^2]$$

$$+\frac{{f_S^V}^2}{2}(\vec{J}_\phi)^2 + \frac{{f_S^A}^2}{2}(\vec{J}_\theta)^2$$

With the definition

$$\Pi_X = \frac{\sqrt{\alpha_T} F_T}{2}(\tilde{\Pi}_X + \tilde{\Pi}_Y), \;\; \Pi_Y = \frac{F_T}{2}(\tilde{\Pi}_X - \tilde{\Pi}_Y)$$

and using $Tr[T_a T_b] = \delta_{ab}/2$ we get the properly normalized kinetic term for the 18 Goldstone bosons

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\dot{\Pi}_X^a)^2 + \frac{1}{2}(\dot{\Pi}_Y^a)^2 + \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\theta})^2$$

$$-\frac{v_X^2}{2}|\vec{\nabla}\Pi_X^a|^2 - \frac{v_Y^2}{2}|\vec{\nabla}\Pi_Y^a|^2 - \frac{v_\phi^2}{2}|\vec{\nabla}\phi|^2 - \frac{v_\theta^2}{2}|\vec{\nabla}\theta|^2$$

$$v_X^2 = \frac{\alpha_S}{\alpha_T}\frac{F_S^2}{F_T^2},\ v_Y^2 = \frac{F_S^2}{F_T^2},\ v_\phi^2 = \frac{{f_S^V}^2}{{f_T^V}^2},\ v_\theta^2 = \frac{{f_S^A}^2}{{f_T^A}^2}$$

Let us consider now the local color invariance. To keep this into account use covariant derivatives (g_μ are the gluon fields)

$$\partial_\mu \hat{X} \to D_\mu \hat{X} = \partial_\mu \hat{X} - g_\mu \hat{X}$$

$$\partial_\mu \hat{Y} \to D_\mu \hat{Y} = \partial_\mu \hat{Y} - g_\mu \hat{Y}$$

$$g_\mu = i g_s g_\mu^a T^a/2 \in \text{Lie } SU(3)_c$$

$$J_X^\mu \to J_X^\mu = \hat{X}\partial^\mu \hat{X}^\dagger + g^\mu,\ J_Y^\mu \to J_Y^\mu = \hat{Y}\partial^\mu \hat{Y}^\dagger + g^\mu$$

We obtain the invariant lagrangian

$$\mathcal{L} = -\frac{F_T^2}{4} Tr[(X\partial^0 X^\dagger - Y\partial^0 Y^\dagger)^2]$$

$$-\alpha_T \frac{F_T^2}{4} Tr[(X\partial^0 X^\dagger + Y\partial^0 Y^\dagger + 2\, g^0)^2]$$

$$-\frac{{f_T^V}^2}{2}(J_\phi^0)^2 - \frac{{f_T^A}^2}{2}(J_\theta^0)^2$$

$+$ spatial terms and kinetic part for g^μ

Using gauge invariance, choose $\hat{X} = \hat{Y}^\dagger$ (unitary gauge), where

$$\tilde{\Pi}_X = -\tilde{\Pi}_Y, \quad \text{or} \quad \Pi_X = 0, \quad \Pi_Y = F_T \tilde{\Pi}_X$$

8 Goldstone bosons disappear to give mass to the 8 gluons. The gluons g_0^a and g_i^a acquire Debye and Meissner masses respectively

$$m_D^2 = \alpha_T\, g_s^2\, F_T^2, \quad m_M^2 = v_X^2\, \alpha_T\, g_s^2\, F_T^2$$

The true mass of the gluons is not given by the previous expressions since, in general, the **gluon kinetic term appearing in the effective lagrangian is renormalized by the in-medium interactions** (see later)

The effective lagrangian is supposed to be a valid description below the gap Δ. Since the gluons (as to be seen later on) acquire a mass of order Δ, when $E \ll \Delta$ the gluons decouple and they can be expressed as

$$g_\mu = -\frac{1}{2}(\hat{X}\partial_\mu\hat{X}^\dagger + \hat{Y}\partial_\mu\hat{Y}^\dagger)$$

The lagrangian becomes

$$\mathcal{L} = -\frac{F_T^2}{4}Tr[(\hat{X}\partial^0\hat{X}^\dagger - \hat{Y}\partial^0\hat{Y}^\dagger)^2]$$

$$-\frac{{f_T^V}^2}{2}(J_\phi^0)^2 - \frac{{f_T^A}^2}{2}(J_\theta^0)^2 + \text{spatial terms}$$

This can be easily expressed in terms of the chiral field Σ

$$\mathcal{L} = \frac{F_T^2}{4}\left(Tr\,[\dot{\Sigma}\dot{\Sigma}^\dagger] - v_Y^2\,Tr[\vec{\nabla}\Sigma\cdot\vec{\nabla}\Sigma^\dagger]\right)$$

$$-\frac{{f_T^V}^2}{2}\left((J_\phi^0)^2 - v_\phi^2|\vec{J}_\phi|^2\right) - \frac{{f_T^A}^2}{2}\left((J_\theta^0)^2 - v_\theta^2|\vec{J}_\theta|^2\right)$$

Notice that the first term coincides with the chiral lagrangian except for the breaking of the Lorentz invariance

Perturbative calculations

Once taken into account the diquark condensation, it is possible to do perturbative calculations at very high density taking advantage of asymptotic freedom. We will follow the following steps

- We go from $\mathcal{L}_{QCD}$ at high density to an effective theory describing gapped fermionic excitations close to the Fermi surface.

- We couple Goldstone and gluons in an invariant way to the fermions at the Fermi surface and evaluate the relevant $n-$point functions. This allows the determination of the couplings appearing in the effective lagrangian for NG bosons and gluons

We start describing the effective theory around the Fermi surface (the physics has been described by J. Polchinski, TASI 1992, hep-th/9210046, see also: D.K. Hong, Phys. Lett. **B473** (2000) 118, hep-ph/9812510 and Nucl. Phys. **B582** (2000) 451, hep-ph/9905523; S.R. Beane, P.F. Bedaque, M.J. Savage, Phys. Lett. **B483** (2000) 131, hep-ph/0002209)

We consider QCD at finite density, with a chemical potential μ ($a = 1, \cdots, 8$)

$$\mathcal{L}_{QCD} = \bar\psi i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \mu \bar\psi \gamma_0 \psi$$

For $\mu \gg \Lambda_{QCD}$ quarks are almost free . We have, ($\vec\alpha = \gamma_0 \vec\gamma$)

$$(\not{p} + \mu\gamma_0)\psi(p) = 0 \;\;\Rightarrow\;\; (p^0 + \mu)\psi = \vec\alpha \cdot \vec p\, \psi$$

The energy eigenvalues are

$$p^0 = E_\pm = -\mu \pm |\vec p|$$

with eigenstates $|\pm\rangle$. For momenta close to the Fermi momentum $|\vec p| \approx \mu$ only the states $|+\rangle$ close to the Fermi surface ($E_+ \approx 0$) can be excited. The states $|-\rangle$ with $E_- \approx -2\mu$ decouple at large μ. More formally write

$$p^\mu = \mu v^\mu + \ell^\mu, \quad v^\mu = (0, \vec v_F), \quad |\vec v_F| = 1$$

The Hamiltonian is ($\vec\alpha = \gamma^0 \vec\gamma$)

$$H = -\mu + \vec\alpha \cdot \vec p \;\;\to\;\; H = -\mu(1 - \vec\alpha \cdot \vec v_F) + \vec\alpha \cdot \vec\ell$$

Introducing the projectors

$$P_\pm = \frac{1 \pm \vec\alpha \cdot \vec v_F}{2}$$

and $|\pm\rangle = P_\pm \psi$, we get

$$H\,|+\rangle = \vec\alpha \cdot \vec\ell\,|+\rangle, \quad H|-\rangle = (-2\mu + \vec\alpha \cdot \vec\ell)\,|-\rangle$$

We decompose the fields with $P_\pm$ and integrate out all the modes with $|\vec{\ell}| > \delta$ $(\Delta < \delta \ll \mu)$

$$\psi(x) = \sum_{\vec{v}_F} e^{-i\mu v\cdot x}\left[\psi_+(x) + \psi_-(x)\right]$$

$$\psi_\pm(x) = e^{i\mu v\cdot x} P_\pm\, \psi(x) = \int_{|\vec{\ell}|<\delta} \frac{d^4\ell}{(2\pi)^4} e^{-i\ell\cdot x}\psi_\pm(\ell)$$

Substituting inside the lagrangian we get (**off-diagonal terms in the velocity are cancelled by the exponential oscillations for $\mu \to \infty$**)

$$\mathcal{L} = \sum_{\vec{v}_F}\Big[\psi_+^\dagger iV\cdot D\,\psi_+ + \psi_-^\dagger(2\mu + i\tilde{V}\cdot D)\psi_-$$

$$+ \,(\bar{\psi}_+ i\not{D}_\perp \psi_- + \text{h.c.})\Big]$$

$$V^\mu = (1, \vec{v}_F), \qquad \tilde{V}^\mu = (1, -\vec{v}_F)$$

$$\not{D}_\perp = D_\mu \gamma_\perp^\mu, \quad \gamma_\perp^\mu = P_\perp^{\mu\nu}\gamma_\nu$$

$$P_\perp^{\mu\nu} = (2g^{\mu\nu} - V^\mu\tilde{V}^\nu - \tilde{V}^\mu V^\nu)$$

Fields inside $\mathcal{L}$ are evaluated at the same Fermi velocity, or

Fermi velocity selection rule

For large chemical potential the field ψ_- decouples and it can be eliminated through its equation of motion. At the leading order

$$iV\cdot D\,\psi_+ = 0, \quad \psi_- = -\frac{i}{2\mu}\gamma_0 \not{D}_\perp \psi_+$$

For fixed $\vec{v}_F$ only energy and momentum along the Fermi velocity are relevant. Due to the velocity selection rule we have

infinite copies of 2-d physics

Eliminating the field ψ_-:

$$\mathcal{L} = \sum_{\vec{v}_F} \left[\psi_+^\dagger iV\cdot D\psi_+ - \frac{1}{2\mu + i\tilde{V}\cdot D} \psi_+^\dagger (\not{D}_\perp)^2 \psi_+ \right]$$

The $1/\mu$ term may contribute to one-loop diagrams giving rise to an extra μ factor (see later).

Couplings to Goldstone bosons

We have seen that the NG fields, $\hat{X}$ ($\hat{Y}$), transform under G as $q_L(q_R)$, for instance

$$q_L \to g_c\, q_L g_L^T, \qquad \hat{X} \to g_c \hat{X} g_L^T$$

There are two possible invariant couplings with the NGB's, and similar for $\hat{Y}$ and ψ_R, corresponding to the two channels $(\bar{3}, \bar{3})$ and $(6, 6)$

$$\gamma_1\, Tr\,[q_L^T \hat{X}^\dagger] C Tr[q_L \hat{X}^\dagger] + \gamma_2\, Tr\,[q_L^T C \hat{X}^\dagger q_L \hat{X}^\dagger]$$

$$+ \text{ h.c.}$$

Since in the fundamental state $\langle \hat{X} \rangle = \langle \hat{Y} \rangle = 1$, the two couplings reproduce the correct breaking of the symmetry in the CFL phase. For simplicity we will take

$$\gamma_1 = -\gamma_2 \propto \frac{\Delta}{2}$$

corresponding to a condensate in the representation $(\bar{3}, \bar{3})$

In this case the coupling can be written as

$$-\frac{\Delta}{2}\sum_{I=1,2,3} Tr\,[(q_L\hat{X}^\dagger)^T C\epsilon_I(q_L\hat{X}^\dagger)\epsilon_I]$$

with$(\epsilon_I)_{ab}=\epsilon_{Iab}$. It is convenient to define (λ_a, $a=1,\cdots,8$, are the Gell-Mann matrices, $\lambda_0=\sqrt{2/3}\ 1$, and $\Delta_a=\Delta$, $\Delta_9=-2\Delta$)

$$\hat{X}=1+(\hat{X}-1)\equiv 1+X_1$$

$$\psi_\pm=\frac{1}{\sqrt{2}}\sum_{A=1}^{9}\lambda_A\psi_\pm^A$$

In terms of the velocity decomposition we get the lagrangian (R.C., R. Gatto and G. Nardulli, Phys. Lett. **B498** (2001) 179, hep-ph/0010321)

$$\mathcal{L}=\sum_{\vec{v}_F}\frac{1}{2}\Big[\sum_{A=1}^{9}\Big(\psi_+^{A\dagger}iV\cdot D\psi_+^A+\psi_-^{A\dagger}i\tilde{V}\cdot D\psi_-^A -\Delta_A(\psi_-^{A^T}C\psi_+^A+\text{h.c.})\Big) -\Delta\sum_{I=1,3}\Big(Tr[(\psi_-X_1^\dagger)^TC\epsilon_I(\psi_+X_1^\dagger)\epsilon_I]+\text{h.c.}\Big)\Big]$$

Goldstone and gap terms couple fields with opposite Fermi velocity (Cooper pairs). ψ_- **is obtained from** ψ_+ **sending** $\mathbf{v_F}\to-\mathbf{v_F}$

Formalism neater introducing Nambu-Gorkov fields

$$\chi = \begin{pmatrix} \psi_+ \\ C\psi_-^* \end{pmatrix}$$

We get the quadratic part of the lagrangian (plus analogous term for the right-handed fields)

$$\boxed{\mathcal{L}_0 = \int \frac{d\vec{v}_F}{8\pi}\, \frac{1}{2} \sum_{A=1}^{9} \chi^{A\dagger} \begin{bmatrix} iV\cdot D & \Delta^A \\ \Delta^A & i\tilde{V}\cdot D^* \end{bmatrix} \chi^A}$$

and the propagator

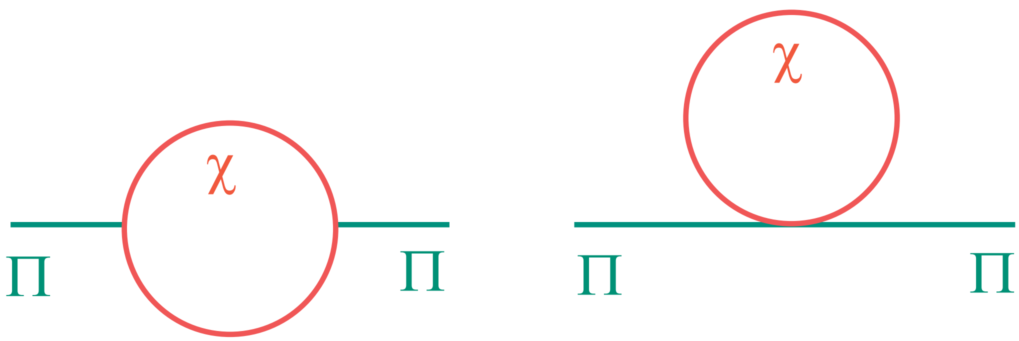
$$S_{AB}(p) = \frac{2\delta_{AB}}{(V\cdot p\,\tilde{V}\cdot p) - \Delta_A^2} \begin{bmatrix} \tilde{V}\cdot p & -\Delta_A \\ -\Delta_A & V\cdot p \end{bmatrix}$$

Expanding the Goldstone fields $\hat{X}$ and $\hat{Y}$ in the gauge $\hat{X} = \hat{Y}^\dagger$

$$\hat{X} = \exp i \left(\frac{\lambda_a \Pi^a}{2F} \right), \qquad a = 1, \cdots, 8$$

we get vertices $\Pi\chi\chi$ and $\Pi\Pi\chi\chi$. The Goldstone self-energy is given by the diagrams

Goldstone self-energy



Expanding to $\mathcal{O}(p^2)$ we get

$$i\mu^2 \frac{21 - 8\ln 2}{72\pi^2 F^2} \int \frac{d\vec{v}_F}{4\pi} \sum_{a=1}^{8} \Pi^a (V \cdot p)(\tilde{V} \cdot p) \Pi^a$$

from which

$$\mathcal{L}_{\text{eff}}^{\text{kin}} = \frac{\mu^2 (21 - 8\ln 2)}{72\pi^2 F^2} \sum_{a=1}^{8} \left(\dot{\Pi}^a \dot{\Pi}^a - \frac{1}{3} |\vec{\nabla} \Pi_a|^2 \right)$$

To get the proper normalization we must have

$$\boxed{F^2 = \frac{\mu^2(21 - 8\ln 2)}{36\pi^2}}$$

Comparing with the effective lagrangian we see that

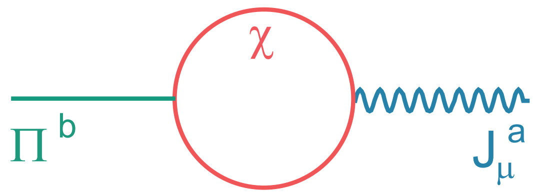
$$F_T = F, \quad F_S = \frac{F_T}{\sqrt{3}} \quad \Rightarrow \quad v_Y = \frac{1}{\sqrt{3}}$$

Therefore the pions satisfy the dispersion relation

$$(p^0)^2 - \frac{1}{3}|\vec{p}|^2 = 0 \quad \mapsto \quad p^0 = \pm\frac{1}{\sqrt{3}}|\vec{p}|$$

The same result has been obtained through the evaluation of the Debye and Meissner masses of the gluons (D. T. Son and M. A. Stephanov, Phys. Rev. **D61**, 074012 (2000), hep-ph/9910491; erratum, *ibid.* **D62**, 059902 (2000), hep-ph/0004095; M. Rho, A. Wirzba and I. Zahed, Phys. Lett. **B473**, 126 (2000), hep-ph/9910550; D. K. Hong, T. Lee and D. Min, Phys. Lett. **B477**, 137 (2000), hep-ph/9912531; C. Manuel and M. H. Tytgat, Phys. Lett. **B479**, 190 (2000), hep-ph/0001095; M. Rho, E. Shuryak, A. Wirzba and I. Zahed, Nucl. Phys. **A676**, 273 (2000), hep-ph/0001104; S. R. Beane, P. F. Bedaque and M. J. Savage, Phys. Lett. **B483**, 131 (2000), hep-ph/0002209; C. Manuel and M. Tytgat, hep-ph/0010274)

To the same result one can arrive through the evaluation of the diagram (R.C., R. Gatto and G. Nardulli, Phys. Lett. **B498** (2001) 179)



giving

$$\langle 0|J^a_\mu|\Pi^b\rangle = iF\delta_{ab}\tilde{p}_\mu, \quad \tilde{p}^\mu = \left(p^0, \frac{1}{3}\vec{p}\right)$$

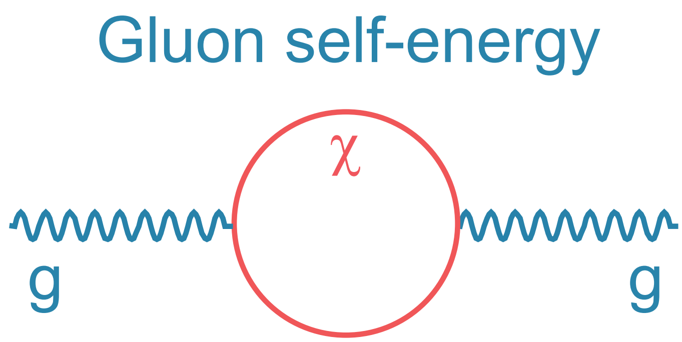
This shows the **current is conserved**, due to the dispersion relation for the pions

$$p\cdot\tilde{p} = (p^0)^2 - \frac{1}{3}|\vec{p}|^2 = 0$$

Similar calculations can be done for the NG fields ϕ and θ

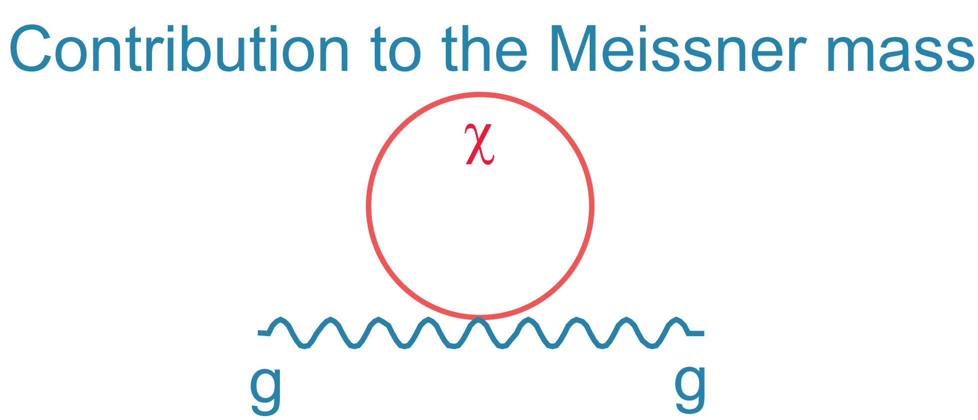
Couplings to the gluons

By the same techniques we can evaluate the gluon self-energy. The coupling of gluons to fermions is given by the covariant derivative. This gives rise to the diagram



We have also the tadpole contribution arising from the term

$$-\frac{1}{2\mu}\psi_+^\dagger(\not{D}_\perp)^2\psi_+$$



The loop integration gives an extra μ factor compensating the one in the denominator. From the constant part of the diagrams we get Debye and Meissner masses

$$m_D^2 = g_s^2 F^2 = \frac{\mu^2 g_s^2}{36\pi^2}(21 - 8\log 2)$$

$$m_M^2 = \frac{\mu^2 g_s^2}{108\pi^2}\left(-33 - 8\log 2 + \underbrace{54}_{\text{tadpole}}\right) = \frac{m_D^2}{3}$$

Comparison with the effective lagrangian shows

$$\alpha_T = \alpha_S = 1$$

The tadpole term is also essential in order to satisfy the Ward identity ($\Pi_{ab}^{\mu\nu}$ is the gluon self-energy)

$$p_\mu \Pi_{ab}^{\mu\nu} \propto \langle 0|J^\nu|\Pi^b\rangle = iF\delta_{ab}\tilde{p}^\nu$$

The Meissner and the Debye masses are not the physical masses of the gluons. This comes from the wave-function renormalization proportional to $\mu^2 g_s^2/\Delta^2$ making the **effective square masses proportional to Δ^2** rather than to $g_s^2\mu^2$($m_{\text{gluon}} \lesssim 2\Delta$). This changes also $g_s \to g_s\Delta/(g_s\mu) = \Delta/\mu$

Wave function renormalization of order $g_s^2\mu^2/\Delta^2$ for the gluons appears to be a rather general phenomenon. For instance, consider the 2SC phase. The low energy degrees of freedom are 3 gluons and the almost free quarks of color 3. The symmetries determining the effective lagrangian are: the gauge symmetry $SU(2)_c$ and rotation invariance (Lorentz is broken being at finite density). For the gluons one gets (Rischke, Son, Stephanov, 2000)

$$\mathcal{L}_{\rm eff} = \frac{\epsilon}{2}\vec{E}^a\cdot\vec{E}^a - \frac{1}{2\lambda}\vec{B}^a\cdot\vec{B}^a$$

with a propagation velocity for the gluons given by $v = 1/\sqrt{\epsilon\lambda}$. Values of ϵ and λ different from 1 originate from wave function renormalization. One finds

$$\epsilon = 1 + \frac{g_s^2\mu^2}{18\pi^2\Delta^2} \approx \frac{g_s^2\mu^2}{18\pi^2\Delta^2}, \quad \lambda = 1$$

The strong coupling constant gets modified

$$\alpha_s \ \rightarrow \ \alpha_s' = \frac{g_{\text{eff}}^2}{4\pi v} = \frac{g_s^2}{4\pi\sqrt{\epsilon}} = \frac{3}{2\sqrt{2}}\frac{g_s\Delta}{\mu}$$

due to the changes in the propagation velocity and in the Coulomb force

$$g_s^2/r \rightarrow g_s^2/(\epsilon r) \ \Rightarrow \ g_s^2 \rightarrow g_{\text{eff}}^2 = g_s^2/\epsilon$$

Similar results hold for the massive gluons of type 4, 5, 6 and 7 which acquire a mass of order Δ. Exceptions are the spatial components (but not the time one) of the gluon 8. In this case there is no wave function renormalization of the time derivative and the mass is of order $g_s\mu$ (R.C., R. Gatto, M. Mannarelli and G. Nardulli hep-ph/0107024). Also the em dielectric constant gets modified by the in-medium effects both in the CFL and in the 2SC phases (D.F. Litim and C. Manuel, hep-ph/0105165)

$$\tilde{\epsilon} = 1 + \frac{r}{18\pi^2}\frac{\tilde{e}^2\mu^2}{\Delta^2}$$

$\tilde{e}$ the in-medium rotated electric charge, and

$$r = 4 \text{ in CFL}, \quad r = 1 \text{ in 2SC}$$

Mass terms for the NGB's

The QCD mass terms have the form

$$\bar{\psi}_L M \psi_R \; + \; \text{c.c.}$$

They are invariant if we transform at the same time the fields and the mass matrix

$$\psi_L \to g_L \, e^{i(\alpha+\beta)} \psi_L, \quad \psi_R \to g_R \, e^{i(\alpha-\beta)} \psi_R$$

$$M \to e^{2i\beta} g_L M g_R^\dagger$$

with $e^{i\alpha} \in U(1)_V$, $e^{i\beta} \in U(1)_A$. It is convenient to introduce the field

$$\tilde{\Sigma} = Y^\dagger X = e^{4i\theta} \Sigma$$

transforming as

$$\tilde{\Sigma}^T \to e^{-4i\beta} g_L \tilde{\Sigma}^T g_R^\dagger$$

whereas ($d_X = \det(X)$, $d_Y = \det(Y)$)

$$d_X \to e^{-6i(\alpha+\beta)} d_X, \quad d_Y \to e^{-6i(\alpha-\beta)} d_Y$$

and

$$\det(M) \to e^{6i\beta} \det M$$

$U(1)_V$ invariance requires dependence on the combination $d_X d_{Y^\dagger} = \det(\tilde{\Sigma})$

$$\det(\tilde{\Sigma}) \to e^{-12i\beta} \det(\tilde{\Sigma})$$

Using the Cayley identity for 3×3 matrices it is not difficult to prove that at the lowest order in M there are only three invariant terms, quadratic in M. This follows from the Z_2 symmetry acting on the left-handed fermion fields, under which $M \to -M$. One finds (D.T. Son and M.A. Stephanov, Phys. Rev. **D61** (2000) 074012, hep-ph/9910491, (E) *ibid.* **D62** (2000) 059902, hep-ph/0004095)

$$\mathcal{L}_{masses} = -c\left[\text{det}(M)\, Tr[M^{-1}\tilde{\Sigma}^T] + \text{h.c.}\right]$$

$$-c'\left[\text{det}(\tilde{\Sigma})\,(Tr[M\Sigma^*])^2 + \text{h.c.}\right]$$

$$-c"\left[Tr[M\Sigma^*]\, Tr[M^\dagger\Sigma^T]\right]$$

In a weak coupling calculation the coefficient c' turns out to be very small compared with c, whereas $c"$ is zero at the leading order

In the weak coupling limit we can evaluate the coupling c:

$$c = \frac{3\Delta^2}{2\pi^2}$$

and the NGB's masses. For instance

$$m^2_{\pi^\pm} = \frac{2c}{F^2} m_s(m_u + m_d)$$

$$m^2_{K^\pm} = \frac{2c}{F^2} m_d(m_u + m_s)$$

showing that

$$\frac{m^2_{K^\pm}}{m^2_{\pi^\pm}} \approx \frac{m_d}{m_u + m_d}$$

In the CFL phase, kaons lighter than pions

Conclusions

- In high density QCD **color superconducting phases** are formed with different features according to m_s
  - $m_s = 0 \;\Rightarrow$ **CFL**
  - $m_s = \infty \Rightarrow$ **2SC**

- An **effective lagrangian** description for the CFL phase has been proposed (for 2SC see R.C., Z. Duan , F. Sannino, Phys. Rev. **D62** (2000) 094004)

- Asymptotic freedom allows for weak coupling calculations of the parameters of $\mathcal{L}_{\text{eff}}$. Use is made of a formalism describing **excitations close to the Fermi surface** simplifying calculations. Equivalent to

  ∞ **copies of 2-dim physics**