The Standard Model of Electroweak Interactions

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Chapter 1

Introduction

1.1 Fermi theory of β-decay

The weak interactions were discovered by Bequerel, back in 1896, when he found accidentally that a nucleus (A, Z), where A is the atomic weight and Z the atomic number, may decay into a different nucleus plus a β ray (that is plus an electron)

$$(A, Z) \to (A, Z + 1) + e^- \tag{1.1}$$

It is important to notice that in a two-body decay of the type $A \to B + e^-$, the energy, E, of the final electron is fixed. In fact, from the conservation of the four-momentum, $p_B = p_A - p$, where p is the electron momentum, and going in the rest frame of the decaying nucleus, we have

$$m_B^2 = m_A^2 + m_e^2 - 2m_A E \tag{1.2}$$

from which

$$E = \frac{m_A^2 - m_B^2 + m_e^2}{2m_A} \tag{1.3}$$

This means that if one looks at the energy distribution of the emitted electrons, the spectrum should consist of a single line located at the energy (1.3). The experimental situation was quite confused up to the beginning of the thirties, when it was definitely shown that the spectrum was a continuous one. This was rather disturbing. In fact the only known particles, at that time, were the electron and the proton. Niels Bohr was ready to abandon the law of conservation of energy. But Pauli was much less radical and suggested that another particle was emitted along the electron [1]. Of course it had to be electrically neutral. As it is well known, the name proposed by Pauli for the new particle was **neutron**, but later Fermi suggested rather to use **neutrino**. In fact, in the mean time, the neutron was discovered by Chadwick (1932), and it was clear from the experiments that the neutrino had a very small mass. By then, the β decay of a nucleus was interpreted as the decay of a neutron, n, inside the nucleus, into a proton, p, plus a electron-neutrino pair

$$n \to p + e^- + \bar{\nu}_e \tag{1.4}$$

In this equation, we have anticipated two later ideas, that is the conservation of the lepton number, and the existence of various types of neutrinos. Also, one has to assign a spin 1/2 to the neutrino. After this, the real beginning of the weak interaction theory starts in 1934 with Fermi description of the β-decay in terms of quantum field theory [2]. Fermi assumed that the emission of a electron-neutrino pair was analogous to the electromagnetic emission of a photon. The relevance of the Fermi theory was that, apart describing successfully the β-decay, it was the first field theory in which the processes were described in terms of creation and annihilation of particles. Also, the work of Fermi made clear that it was not necessary to assume that the electron was inside the nucleus before the decay process. The electron neutrino pair is created by the weak interaction. As we said, the Fermi theory tried to copy the main features of the electromagnetism, just by substituting to the photon the electron neutrino pair. Since the electromagnetic interaction of a charged particle is given by

$$H_{\rm int} = e\int d^3\vec{x}\, A_\mu\bar{\psi}\gamma^\mu\psi \tag{1.5}$$

where ψ is the field describing the charged particle, and A_μ the electromagnetic field, the Fermi ansatz for $n \to p + e^- + \bar{\nu}_e$ was

$$H_W = G_F\int d^3\vec{x}\,(\bar{\psi}_e\gamma_\mu\psi_\nu)(\bar{\psi}_p\gamma^\mu\psi_n) \tag{1.6}$$

Of course this was not the most general choice one could do, because in principle one could substitute [3]

$$(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma_\mu\psi) \to \sum_i(\bar{\psi}\Gamma_i\psi)(\bar{\psi}\Gamma_i\psi) \tag{1.7}$$

where Γ_i is a generic 4×4 matrix. These matrices can be expressed in terms of the 16 independent Dirac matrices. These are given in Table 1.1 together with the related Dirac bilinear, their tensor character, and the non-relativistic limit, expressed in terms of two-component spinors, ϕ.

Γ_i	$\bar{\psi}\Gamma_i\psi$	tensor character	non-relativistic limit
$\mathbf{1}$	$\bar{\psi}\psi$	S (scalar)	$\phi^\dagger\phi$
γ^μ	$\bar{\psi}\gamma^\mu\psi$	V (vector)	$\phi^\dagger\phi$
$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$	$\bar{\psi}\sigma_{\mu\nu}\psi$	T (antisymmetric tensor)	$\phi^\dagger\vec{\sigma}\phi$
$\gamma^\mu\gamma_5$	$\bar{\psi}\gamma^\mu\gamma_5\psi$	A (axial-vector)	$\phi^\dagger\vec{\sigma}\phi$
$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$	$\bar{\psi}\gamma_5\psi$	P (pseudoscalar)	0

Table 1.1 -*The Dirac covariant bilinear and their non-relativistic limit.*

Therefore the most general interaction hamiltonian (parity-conserving) describing the nuclear β-decay $A \to B + e^- + \bar{\nu}_e$ can be written as

$$H_W = G \int d^3\vec{x} \sum_{i=S,V,T,A,P} c_i(\bar{\psi}_B \Gamma_i \psi_A)(\bar{\psi}_e \Gamma_i \psi_\nu) \tag{1.8}$$

where ψ_A and ψ_B are the nucleon (proton and/or neutron) fields within the nucleus. One can get a rough estimate for the nuclear β-decay, by neglecting the Coulomb interaction of the electron. This means that the outgoing leptons can be described by plane waves

$$\psi_\ell(\vec{x}) \approx \frac{1}{\sqrt{V}} e^{i\vec{p}\cdot\vec{x}} u(\vec{p}) \tag{1.9}$$

with the spinor $u(\vec{p})$ solution of the Dirac equation. Since the nucleon wave functions vanish outside the nuclear volume, we can re-write the weak hamiltonian in the form

$$H_W \approx G \sum_{i=S,V,T,A,P} c_i(\bar{u}_e(\vec{p})\Gamma_i v_\nu(\vec{q})) \int_{V_N} e^{i(\vec{p}-\vec{q})\cdot\vec{x}} \bar{\psi}_B(\vec{x})\Gamma_i \psi_A(\vec{x})\, d^3\vec{x} \tag{1.10}$$

where $\vec{p}$ and $\vec{q}$ are the momenta of the electron and of the neutrino respectively, and V_N is approximately the nuclear volume. The maximum energies available for the neutrino and/or for the electron are of the order of the mass difference between the two nuclei, which means a few MeV (for instance, for a free neutron the mass difference is about 1.29 MeV), or an equivalent length, λ, of order of 10^{-10} cm. The typical nuclear radius, R_N, is of the order $10^{-12} \div 10^{-13}$ cm., therefore the argument of the exponential in the previous integral is roughly of the order

$$(\vec{p}-\vec{q})\cdot\vec{x} \approx \frac{R_N}{\lambda} \approx 10^{-2} \div 10^{-3} \tag{1.11}$$

We see that it is possible to make two important simplifications:

- Treat the nucleon spinors in the non-relativistic approximation.
- Replace the exponent coming from the lepton fields by 1.

Then, the relevant matrix element for the transition is

$$\langle f|H_W|i\rangle \equiv \langle N_B, e^-, \bar{\nu}_e|H_W|N_A\rangle = G \int d^3\vec{x} \sum_{i=S,V,T,A,P} c_i \langle N_B|\bar{\psi}_B \Gamma_i \psi_A|N_A\rangle \bar{u}_e \Gamma_i v_\nu \tag{1.12}$$

If the matrix element of the **current** $\bar{\psi}_B \Gamma_i \psi_A$ between the nucleus states is different from zero, we speak about **allowed transitions**. Otherwise one has to expand the exponential to the next order, and one speaks of **forbidden transitions**. From our previous estimate we deduce that the the typical rate for a first order forbidden transition is reduced by a factor of about $10^{-4} \div 10^{-6}$ with respect to the rate of an allowed one. Then, for an allowed transition the matrix element becomes independent of the lepton momenta and the transition rate is proportional to

$$w_{fi} \approx \int d^3\vec{p}d^3\vec{q}\,\delta(E_0 - E - E_\nu)|\langle f|H_W|i\rangle|^2 \tag{1.13}$$

where $E_0 = m_A - m_B$, E the electron energy, and we have neglected the recoil of the nucleus. By using $d^3\vec{q} = |\vec{q}|E_\nu dE_\nu d\Omega_\nu$, we get

$$w_{fi} \approx \int p^2 dp E_\nu \sqrt{E_\nu^2 - m_\nu^2}\,|\langle f|H_W|i\rangle|^2 \tag{1.14}$$

Then, if we consider the Kurie plot of the energy spectrum, that is

$$\frac{1}{p}\sqrt{\frac{dw_{fi}}{dp}} \approx (E_0 - E)\left(1 - \left(\frac{m_\nu}{E_0 - E}\right)^2\right)^{1/4} \tag{1.15}$$

we see that it is a linear function of the electron energy, when $m_\nu = 0$. The deviation of the Kurie plot from linearity represents a powerful method to investigate the possibility of massive neutrinos (see Fig. 1.1).

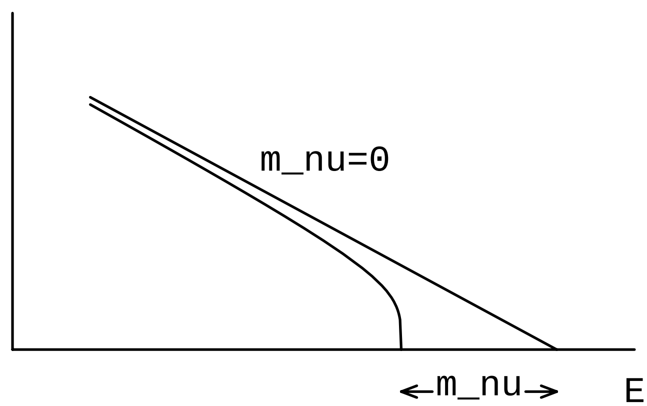

Fig. 1.1 -*Kurie plot showing the distribution* $(1/p)\sqrt{dw_{fi}/dp}$ *as a function of the electron energy* E.

We can perform a different type of expansion of the exponential, that is an expansion in spherical harmonics. Each term in the expansion corresponds to a given orbital momentum ℓ of the electron neutrino pair with respect to the final nucleus. But for allowed transitions $\ell = 0$. Then, the angular momentum conservation requires

$$\vec{J}_i = \vec{J}_f + \vec{S} \tag{1.16}$$

with $\vec{J}$ the nuclear spin, and $\vec{S}$ the spin. The possible values of S are $S = 0, 1$, giving the possibilities outlined in Table 1.2. Nuclear transitions are denoted **Fermi** or **Gamow-Teller** according to the lepton pair being in a singlet or in a triplet state. Both kind of transitions have been observed. Examples are

- **pure Fermi transition**

$$O_8^{14} \to N_7^{14} + e^+ + \nu_e \qquad (J_i = J_f = 0) \tag{1.17}$$

- **pure Gamow-Teller transition**

$$He_2^6 \to Li_3^6 + e^- + \bar{\nu}_e \qquad (J_i = 0,\ J_f = 1;\ \Delta J = 1) \tag{1.18}$$

- **mixture of Fermi and Gamow-Teller**

$$H_1^3 \to He_2^3 + e^- + \bar{\nu}_e \qquad (J_i = J_f = 1/2;\ \Delta J = 0) \tag{1.19}$$

Transition	S	ΔJ	Couplings
Fermi	0	0	S, V
Gamow-Teller	1	0,1 ($0 \to 0$ forbidden)	T, A

Table 1.2 -*The possible allowed transitions.*

It is also possible to show that if all the coefficients c_i are present in the weak hamiltonian, then the interference terms $c_S c_V$ and $c_A c_T$ give rise to contributions with a behaviour $1/E$ which destroys the linear behaviour of the Kurie plot. Therefore, according to the type of transition we have the following possibilities

- Fermi

$$c_S c_V = 0 \Longrightarrow S \text{ or } V \tag{1.20}$$

- Gamow-Teller

$$c_A c_T = 0 \Longrightarrow T \text{ or } A \tag{1.21}$$

In principle one could discriminate among the various cases, but the experiments are very difficult (one should look at the angular correlation between the electron and the neutrino). In practice only in 1957 the couplings were definitely determined to be V and A.

Another important process to mention in relation to the β-decay is the **muon-decay**. The muon (μ) was discovered in the late thirties by Anderson in cosmic rays [4]. At first it was thought as the quantum of nuclear interactions (the Yukawa

π-meson), but later it was shown that μ's interact weakly with matter. The muon decay is into electron and neutrinos

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \tag{1.22}$$

where we have already introduced the distinction between electron and muon neutrinos. This decay is very similar to the neutron decay with the pair (n, p) replaced by the pair (μ^-, ν_μ), and in fact it can be described by a very similar hamiltonian interaction. It turns out that the global effective coupling constant G has practically the same value that was found for the nuclear β-decay, and for the μ-capture process $\mu^- + p \to \nu_\mu + n$. This lead to the hypothesis of a **universal Fermi interaction**. The value of G obtained from the μ-decay is today one of the input parameters in the Standard Model (SM) ($G = 1.166389(22) \cdot 10^{-5}\ GeV^{-2}$).

1.2 Parity non-conservation in weak interactions

Since 1956 it was a common belief that parity was conserved in any interaction. But at that time particle physicists had to deal with the famous $\theta - \tau$ puzzle [5]. This had to do with existence of two mesons identified by the decays

$$\theta^+ \to \pi^+\pi^0, \qquad \tau^+ \to \pi^+ + \pi^+ + \pi^- \tag{1.23}$$

From the point of view of mass, spin and life-time the two particles looked as the same particle (see Table 1.3)

	θ^+	τ^+
$M(MeV)$	966.7 ± 2.0	966.3 ± 2.1
Life-time$\times 10^8$ sec.	1.21 ± 0.02	1.19 ± 0.05
J^P	0^+	0^-

Table 1.3 -*The elements of the $\theta - \tau$ puzzle.*

However, the two different ways of decay were pointing at a different parity property of the two particles. In 1956 Lee and Yang [6] made a review of the experimental proofs of parity conservation and drew the conclusion that there was no evidence of parity conservation in weak interactions. They also proposed new experiments in order to check the parity conservation law. These were based on the fact that only the measure of **pseudoscalar quantities** could give informations about the conservation of parity. For instance, in the β-decay of a polarized nucleus, one could look for the pseudoscalar quantity $\vec{J} \cdot \vec{p}$, with $\vec{J}$ the spin of the nucleus and $\vec{p}$ the momentum of the outgoing electron. The dependence of the transition rate from such a quantity would signal a violation of parity. This suggestion was taken up immediately, the same year, by Wu and collaborators who analyzed the decay [7]

$$Co^{60}(J^P = 5^+) \to Ni^{*60}(J^P = 4^+) + e^- + \bar{\nu}_e \tag{1.24}$$

This is a pure Gamow-Teller transition ($\Delta J = S = 1$). The Co^{60} nuclei are polarized by a magnetic field, and the polarization of the Ni^{*60} was determined by the anysotropy of the two γ's in the decay $Ni^* \to Ni(J^P = 0^+) + 2\gamma$. The experiment showed a big asymmetry in the momentum distribution of the outgoing electrons. This asymmetry changed sign inverting the magnetic field. This was a very clear evidence of parity violation. The effect was that the electrons were emitted preferentially in the opposite direction to the nuclear spin. In a very schematic way this can be represented as follows

$$Co \Uparrow \ (J = 5) \longrightarrow Ni \ \Uparrow (J = 4) + e^- \downarrow\Uparrow + \bar{\nu}_e \uparrow\Uparrow$$

Here $\Uparrow$ denotes the spin, whereas $\uparrow$ denotes the momentum direction. Neglecting the nucleus recoil, the momentum of the electron and of the neutrino must be opposite. Then, the outgoing electron must be mostly **left-handed**, and, as a consequence, the antineutrino must be right-handed. An independent confirmation came the same year by Garwin, Lederman and Weinrich [8] from the analysis of $\pi^+ \to \mu^+ + \nu_\mu$ followed by $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$. It is interesting from an historical point of view to know that Cox and coworkers, in 1928, observed polarized electrons in Radium decay [9]!

To take into account parity violation, one has to modify the weak hamiltonian of eq. (1.8), inserting, for each scalar term, a corresponding pseudoscalar contribution. For instance, in the case of the nuclear β-decay one writes

$$H_W = \frac{G_F}{\sqrt{2}} \sum_{i=S,V,T,A,P} \int d^3\vec{x}\, (\bar{\psi}_p(x)\Gamma_i\psi_n(x))(\bar{\psi}_e(x)\Gamma_i(c_i + c_i'\gamma_5)\psi_\nu(x)) \tag{1.25}$$

The $\sqrt{2}$ is introduced in order to have the same normalization as in the case of the Fermi interaction, and we have introduced the standard notation, G_F, for the Fermi constant. The experiments show that there is a maximal violation of parity, that is $|c_i| = |c_i'|$, therefore

$$H_W = \frac{G_F}{\sqrt{2}} \sum_{i=S,V,T,A,P} c_i \int d^3\vec{x}\, (\bar{\psi}_p\Gamma_i\psi_n)(\bar{\psi}_e\Gamma_i(1 \pm \gamma_5)\psi_{\nu_e}) \tag{1.26}$$

For a massless neutrino, the matrices $1 \pm \gamma_5$ project out definite helicity states. In fact, the helicity projectors are

$$\Sigma_\pm = \frac{1}{2}\left(1 \pm \frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}\right) \tag{1.27}$$

where

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{1.28}$$

where $\vec{\sigma}$ are the Pauli matrices. The Dirac equation for a massless particle is

$$\not{p}\, u(p) = 0 \longrightarrow (E\gamma^0 - \vec{p}\cdot\vec{\gamma})u(p) = 0 \tag{1.29}$$

or

$$E\, u(p) = \vec{p}\cdot\vec{\alpha}\, u(p) = 0 \tag{1.30}$$

where $\vec{\alpha} = \gamma^0\vec{\gamma}$. In the case of a particle, $E = |\vec{p}|$, and therefore

$$\frac{\vec{p}\cdot\vec{\alpha}}{|\vec{p}|}u(p) = u(p) \tag{1.31}$$

In the basis we have chosen for the Dirac matrices we have

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.32}$$

It follows

$$\vec{\alpha} = \gamma_5\vec{\Sigma} \tag{1.33}$$

Then, on the spinors describing a particle, we get

$$\frac{\vec{p}\cdot\vec{\alpha}}{|\vec{p}|}u(p) = u(p) \longrightarrow \frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|}u(p) = \gamma_5\, u(p) \tag{1.34}$$

or

$$\gamma_5 = \frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|} \tag{1.35}$$

We see that the chirality, γ_5, and the helicity operators coincide for massless particles. In the case of antiparticles we get

$$\gamma_5 = -\frac{\vec{p}\cdot\vec{\Sigma}}{|\vec{p}|} \tag{1.36}$$

The same result applies to any high energy particle (that is with $E \gg m$).

Coming back to our problem, since we know already that in weak interactions only the left-handed electron plays a role, and noticing that

$$\bar{\psi}_L \equiv \overline{(\psi_L)} = \psi^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0 = \bar{\psi}\frac{1+\gamma_5}{2} \tag{1.37}$$

we can argue that the neutrino component taking part in weak interactions must be left-handed or right-handed, according to the interaction being of type V, A, or S, T, P. This follows from

$$\Gamma_i(1 \pm \gamma_5) = (1 \pm \gamma_5)\Gamma_i, \quad (i = S,\ T,\ P) \tag{1.38}$$

and

$$\Gamma_i(1 \pm \gamma_5) = (1 \mp \gamma_5)\Gamma_i, \quad (i = V,\ A) \tag{1.39}$$

and the use of these equations in the expression

$$\bar{\psi}_e \Gamma_i (1 \pm \gamma_5)\psi_\nu \tag{1.40}$$

In conclusion, the interaction is dictated by the neutrino helicity. An analysis made on the Fermi-type ($\Delta J = 0$) reaction

$$A^{35} \longrightarrow Cl^{35} + e^+ + \nu_e \tag{1.41}$$

did show that the positron and the neutrino were emitted preferentially along the same direction, implying that the neutrino is left-handed (the electron being left-handed in weak interactions, the positron is right-handed). In this way we get rid of three of the coefficients c_i appearing in H_W, obtaining

$$H_W = \frac{G_F}{\sqrt{2}} \int d^3\vec{x}\, (\bar{\psi}_p \gamma^\mu (c_V + c_A\gamma_5)\psi_n)(\bar{\psi}_e \gamma_\mu (1 - \gamma_5)\psi_{\nu_e}) \tag{1.42}$$

The coefficient c_V can be absorbed into G_F, that is we will take

$$c_V = 1 \tag{1.43}$$

By looking at the interference between Fermi and Gamow-Teller amplitudes, which gives rise to some anysotropy in the angular distribution of $\vec{p}_e$ with respect to the neutron polarization in $n \to p + e^- + \bar{\nu}_e$, one gets

$$c_A = -1.25 \pm 0.009 \tag{1.44}$$

It should be also noted that the absence of a right-handed neutrino (or a left-handed antineutrino) implies that also the charge conjugation C is violated. In fact, under C a left-handed neutrino goes into a left-handed antineutrino which does not exist. However the CP operation is a symmetry, since the CP-conjugated of a left-handed neutrino is a right-handed antineutrino.

1.3 Current-current interaction

The form $V - A$ of the weak interactions was generalized by Feynman and Gell-Mann and independently by Sudarshan and Marshak in 1958 through the **current-current hypothesis** [10]. This consists in writing the weak hamiltonian in the form

$$H_W = \frac{G_F}{\sqrt{2}} \int d^3\vec{x}\, J^\mu(x) J^\dagger_\mu(x) \tag{1.45}$$

where the weak current J^μ (called the charged current) is the sum of two pieces, one coming from the leptons, and the other coming from the hadrons. That is

$$J^\mu = J^\mu_\ell + J^\mu_h \tag{1.46}$$

where

$$J^\mu_\ell = \bar{e}\gamma^\mu(1-\gamma_5)\nu_e + \bar{\mu}\gamma^\mu(1-\gamma_5)\nu_\mu + \cdots \tag{1.47}$$

and

$$J^\mu_h = \bar{p}\gamma^\mu(c_V + c_A\gamma_5)n + \cdots \tag{1.48}$$

In the leptonic part we have assumed the two neutrinos hypothesis. That is the existence of two different type of neutrinos [11]. Our expression implies also the conservation of two different leptonic numbers, the electron number

$$N_e = n_{e^-} + n_{\nu_e} - n_{e^+} - n_{\bar{\nu}_e} \tag{1.49}$$

and the muon number

$$N_\mu = n_{\mu^-} + n_{\nu_\mu} - n_{\mu^+} - n_{\bar{\nu}_\mu} \tag{1.50}$$

In fact, the leptonic current, and consequently H_W, are invariant under the phase transformations generated by the previous operators. This hypothesis has further support by the non observation of processes as

$$\begin{aligned} \mu^\pm \to e^\pm + \gamma, \quad &(BR \leq 4.9\times 10^{-11}) \\ \mu^\pm \to e^\pm + e^+ + e^-, \quad &(BR \leq 1.0\times 10^{-12}) \\ \mu^\pm \to e^\pm + 2\gamma, \quad &(BR \leq 7.2\times 10^{-11}) \end{aligned} \tag{1.51}$$

The dots in the leptonic current stands for a further contribution coming from the discovery (1975) of a third charged lepton, the τ [12]. It is natural, and consistent with the experiments to associate a third neutrino to it, ν_τ, and a third conserved leptonic number. The dots in the hadronic current stand for contributions from strange mesons as

$$\bar{\Lambda}\gamma^\mu(c_V + c_A\gamma_5)p \tag{1.52}$$

as well as from mesons.

On the basis of the current-current interaction, we can divide up the weak processes in three categories:

- **Leptonic processes** originating from $J^\ell_\mu J^{\mu\dagger}_\ell$, as, for example, the μ-decay

$$\mu^- \to e^- + \nu_e + \bar{\nu}_\mu, \quad \mu^+ \to e^+ + \bar{\nu}_e + \nu_\mu \tag{1.53}$$

  or the ν ($\bar{\nu}$) elastic scattering

$$\nu_e + e^- \to \nu_e + e^-, \quad \bar{\nu}_e + e^+ \to \bar{\nu}_e + e^+ \tag{1.54}$$

- **Semi-leptonic processes**, originating form $J^\mu_\ell J^{h\,\dagger}_\mu + h.c.$. These processes can be further classified according to the variation of strangeness induced by the hadronic current:

  $\Delta S = 0$ **transitions**

$$\begin{array}{rl} \beta-\text{nuclear decay} & n \to p + e^- + \bar{\nu}_e \\ \text{e}-\text{capture} & e^- + p \to n + \nu_e \\ \mu-\text{capture} & \mu^- + p \to n + \nu_\mu \\ \text{neutrino reactions} & \nu_\mu + p \to \nu_\mu + p \\ \pi-\text{decay} & \pi^+ \to \mu^+ + \nu_\mu \\ \text{strange particle decay} & \Sigma^+ \to \Lambda^0 + e^+ + \nu_e \end{array} \tag{1.55}$$

  $\Delta S = 1$ **transitions**

$$\begin{array}{rl} \text{hyperon decay} & \Lambda^0 \to p + e^- + \bar{\nu}_e \\ \text{K}-\text{decay} & K^+ \to \mu^+ + \nu_\mu \\ \text{neutrino reactions} & \bar{\nu}_\mu + p \to \mu^+ + \Lambda^0(\Sigma^0) \end{array} \tag{1.56}$$

- **Non-leptonic processes**, originating from $J^h_\mu J^{\mu\,\dagger}_h$. Again, we may classify these processes according to the strangeness variation.

  $\Delta S = 0$ **transitions**

$$\text{parity violation in nuclei} \qquad n + p \to n + p \tag{1.57}$$

  $\Delta S = 1$ **transitions**

$$K-(\text{decay}) \qquad K \to 2\pi,\ K \to 3\pi \tag{1.58}$$

Although isospin is violated in weak interactions, it is possible to establish selection rules by using the conservation of the electric charge, Q, and of the baryonic number, B, together with the relation

$$Q = I_3 + \frac{1}{2}(B + S) \tag{1.59}$$

where I_3 is the third component of isospin. From this relation we get

$$\Delta Q = \Delta I_3 + \frac{1}{2}\Delta S \tag{1.60}$$

In the case of semi-leptonic reactions the selection rule applies directly to the hadronic current. Then for $\Delta S = 0$ one gets $\Delta Q = \Delta I_3$. Since $|\Delta Q| = 1$, it follows $|\Delta I_3| = 1$, implying $\Delta I = 1, 2\cdots$. In practice, transitions with $\Delta I > 1$ have not been observed, therefore one has the selection rule $\Delta I = 1$, $\Delta I_3 = \pm 1$. In the case $|\Delta S| = 1$, since $|\Delta S| = |\Delta Q| = 1$, one has two possibilities:

- $\Delta S = \Delta Q \longrightarrow |\Delta I_3| = \frac{1}{2} \longrightarrow \Delta I = \frac{1}{2}, \frac{3}{2}, \cdots$
- $\Delta S = -\Delta Q \longrightarrow |\Delta I_3| = \frac{3}{2} \longrightarrow \Delta I = \frac{3}{2}, \frac{5}{2}, \cdots$

We shall see, that at the quark level, the $|\Delta S| = 1$ transitions are induced by the quark process $s \to u$, implying $\Delta S = \Delta Q = 1$, $\Delta I_3 = 1/2$, $\Delta I = 1/2$. In practice, the $\Delta S = \Delta Q$ or the stronger $\Delta I = 1/2$ rules were advocated on the basis of the experimental facts. In the case of non-leptonic decays, since $\Delta Q = 0$ (we have only the H_W invariance to exploit), one gets $\Delta I_3 = 1/2$ and $\Delta I = 1/2, 3/2, \cdots$, in the case $|\Delta S| = 1$. Also in this circumstance, in order to explain the smallness of some rate, the rule $\Delta I = 1/2$ was advocated, but one needs a dynamical hypothesis in order to justify it.

We have assumed that the hadronic current is made up by pieces with $\Delta S = 0$ (as $\bar{p}\gamma^\mu(c_V + c_A\gamma_5)n$) and $\Delta S = 1$ (as $\bar{\Lambda}\gamma^\mu(c_V + c_A\gamma_5)p$) having the same strength. But the rates of the processes with $\Delta S = 0$ turn out to be bigger of about a factor 20 with respect to the $\Delta S = 1$ transitions. Cabibbo proposed to maintain the universality by assuming J_h^μ as a normalized combination of the previous two pieces [13], that is

$$J_h^\mu = J_{\Delta S=0}^\mu \cos\theta + J_{\Delta S=1}^\mu \sin\theta \tag{1.61}$$

with $\sin\theta = 0.21 \pm 0.03$. In particular, this implies a small difference between the Fermi constant measured in a leptonic (as μ-decay) or in a semi-leptonic (as β-nuclear decay) process

$$G_F^\beta = G_F^\mu cos\theta \tag{1.62}$$

This explains a small difference that was in the data. In conclusion we write

$$H_W = \frac{G_F}{\sqrt{2}} \int d^3\vec{x}\, J^\mu J_\mu^\dagger \tag{1.63}$$

with

$$J^\mu = J_\ell^\mu + J_{\Delta S=0}^\mu \cos\theta + J_{\Delta S=1}^\mu \sin\theta \tag{1.64}$$

We will be more explicit about the strangeness violating current when we will formulate the SM for quarks.

1.4 Problems of the Fermi theory

We will consider here only the leptonic part of the weak hamiltonian

$$H_W^\ell = \frac{G_F}{\sqrt{2}} \int d^3\vec{x}\, J_\ell^\mu J_\mu^{\ell\,\dagger} \tag{1.65}$$

with

$$J_\mu^\ell = \sum_{i=e,\mu,\tau} \bar{\ell}_i\gamma_\mu(1-\gamma_5)\nu_\ell \tag{1.66}$$

The main problem of this theory is its non-renormalizability due to the four-fermi interaction. We will recall here some elementary facts about renormalization in field theory. First of all we will review the idea of renormalization in a theory like QED. In general, when evaluating Feynman diagrams beyond the tree level, one encounters divergences due to the bad ultraviolet behaviour of the theory. For instance, in 1-loop perturbative QED the diagrams in Fig. 1.2 are divergent ones. However, one

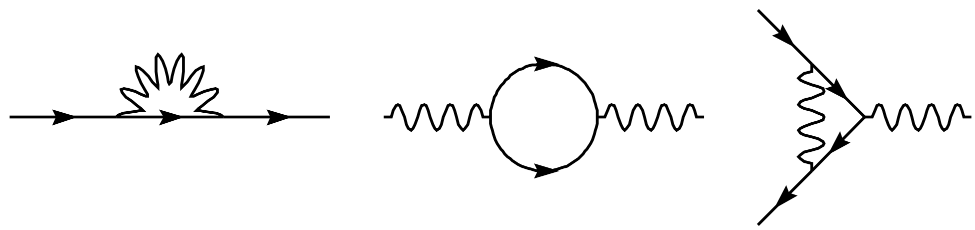

Fig. 1.2 - *Feynman diagrams in QED.*

can show that these divergences can be absorbed into the definition of the coupling constant of the theory, that is the electron mass, m, and the electric charge, e, and into a renormalization of the fields. The idea is that these parameters are arbitrary and one has to fix them by evaluating some observable quantity. Therefore what we are going to measure are not the original parameters appearing in the lagrangian, called the bare ones, $\bar{g}_i$. But rather an expression of the form

$$\bar{g}_i + \delta g_i \equiv g_i \tag{1.67}$$

where δg_i is the contribution from the divergent diagrams. This means that the bare parameters are not measurable quantities, and we have to require that only the previous combinations (renormalized parameters) are finite. Of course, the theory must satisfy special conditions in order that all the divergences can be absorbed into a redefinition of the couplings. When these conditions are satisfied we say that our field theory is renormalizable. One can show that a field theory is renormalizable if the coupling constants g_i (at this level we can avoid the distinction between bare and renormalized couplings) have positive dimension in mass, that is

$$[g_i] \geq 0, \qquad \text{for any } i \tag{1.68}$$

How do we count the dimension of a coupling constant? In our system of units, $\hbar = c = 1$, all the physical quantities have dimension of a mass (or energy, or length^{-1}) to some power. Let us start analyzing the dimension of a field operator. Noticing that an action is dimensionless (in our units), the lagrangian density

$$S = \int d^4x\, \mathcal{L} \tag{1.69}$$

has mass dimension 4

$$[\mathcal{L}] = 4 \tag{1.70}$$

Then, looking at the kinetic term for the various fields

$$\begin{aligned} \mathcal{L}_{\rm Dirac} &= \bar\psi i\partial\!\!\!/ \psi + \cdots \\ \mathcal{L}_{\rm KG} &= \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \cdots \\ \mathcal{L}_{\rm photon} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{1.71}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.72}$$

we see that

$$[\psi] = \frac{3}{2}, \qquad [\phi] = [A_\mu] = 1 \tag{1.73}$$

Knowing the field dimension it is an easy matter to evaluate the dimension of a coupling constant. For instance, in the lagrangian density of the Fermi theory, the coupling constant G_F multiplies an operator of the structure $(\bar\psi\psi)^2$. Since this operator has dimension 6, it follows

$$[G_F] = -2 \tag{1.74}$$

Therefore the Fermi theory is not renormalizable. On the contrary, QED lagrangian density is

$$\mathcal{L}_{QED} = \bar\psi(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.75}$$

and we see that

$$[m] = 1. \qquad [e] = 0 \tag{1.76}$$

implying that theory is renormalizable. The reason why the Fermi theory is not renormalizable in the sense we have explained before is the following. The diagrams in Fig. 1.3 are all divergent. The divergence coming from the first diagram can be absorbed into the coefficient of $(\bar\psi\psi)^2$, but to compensate the divergences coming from the other two diagrams terms of the type $(\bar\psi\psi)^3$ and $(\bar\psi\psi)^4$ should be required inside the lagrangian. In general, an infinite number of terms would be necessary in order to absorb all the divergences. This is why one generally does not take into consideration non-renormalizable theories, they lack predictivity. However one could ask if it would be possible to use a non-renormalizable theory as an effective theory. This means that one would regard the theory as a model in which all the corrections to the couplings are already included in the form one starts with, but restricting to a finite number the, a priori infinite, number of terms in the lagrangian.

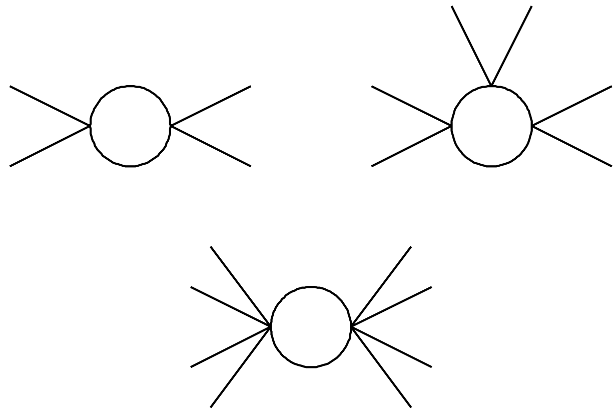

Fig. 1.3 -*Feynman diagrams for the Fermi theory.*

This can be generally done, but only up to some characteristic energy, which, for instance, can be determined by looking at the unitarity property of the theory. The unitarity requires that the scattering amplitudes are limited. However the bad high-energy behaviour of the non-renormalizable theory leads to increasing amplitudes, and therefore to a violation of unitarity. Consider, for instance, the amplitude for a given process in a theory characterized by a single coupling constant g. Necessarily the invariant amplitude coming from that particular lagrangian term will behave, for high energies where we can neglect all the mass scales, as

$$A \approx g E^n \tag{1.77}$$

where E is the characteristic energy of the process. If $[g] = d$, it follows

$$n = -d \tag{1.78}$$

since the probability amplitude is dimensionless. However from unitarity A must be bounded, and therefore the theory satisfies unitarity only if

$$n \leq 0 \tag{1.79}$$

or

$$d \geq 0 \tag{1.80}$$

We see that renormalizabilty and unitarity of a field theory are strictly related. In the case of the Fermi theory, this argument can be made more quantitative, considering, for instance, $\nu_\mu + e \to \nu_e + \mu$. One finds that the differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{\pi^2} E^2 = |f(\theta)|^2 \tag{1.81}$$

where $f(\theta)$ is the scattering amplitude. This can be expanded in partial waves

$$f(\theta) = \frac{1}{E} \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2}\right) M_\ell P_\ell(\cos\theta) \tag{1.82}$$

Unitarity requires

$$|M_\ell| \leq 1 \qquad \text{for any } \ell \tag{1.83}$$

Since in this case, $\ell = 0$, it follows

$$f(\theta) = \frac{1}{2E} M_0 \tag{1.84}$$

and

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E^2}|M_0|^2 = \frac{G_F^2}{\pi^2} E^2 \tag{1.85}$$

from which

$$\frac{4G_F^2 E^4}{\pi^2} \leq 1 \tag{1.86}$$

We see that we can use the Fermi theory as an effective theory (and it does work very well) up to energies not exceeding the threshold E_0 given by

$$E_0 = \sqrt{\frac{\pi}{2G_F}} \approx 367\ GeV \tag{1.87}$$

Chapter 2

Toward the construction of the Standard Model

2.1 Intermediate vector bosons

The analogy between QED and the current-current interaction can be pushed further if one assumes that weak interactions are mediated by a spin 1 field W_μ. In QED, since the photon is massless, a four-fermi amplitude goes like α/q^2, giving rise to a long range force. But weak interactions are short range, so we would rather need to exchange a massive particle. Assuming that the charged weak current couples to such a field

$$\mathcal{L} = \frac{g}{2\sqrt{2}} \left(J_\mu W^\mu + h.c. \right) \tag{2.1}$$

a process like the neutron decay (see Fig. 2.1) has an amplitude given by (see later

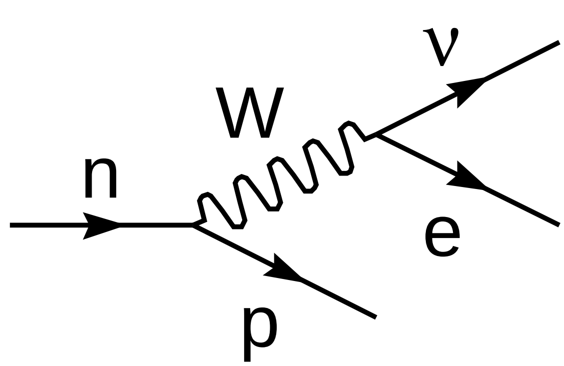

Fig. 2.1 - *The neutron decay mediated by the W vector boson.*

in this section)

$$\mathcal{M} = \langle p|J_\mu^h|n\rangle \frac{g^2}{8}\frac{1}{q^2 - M_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M_W^2}\right)\langle\bar{\nu}_e e^-|J_\nu^\ell|0\rangle \tag{2.2}$$

In the limit $q^2 \ll M_W^2$ this reproduces the Fermi expression after the identification

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}} \tag{2.3}$$

The idea of intermediate vector bosons is due to Schwinger [14] and extended to a neutral vector boson by Bludman [15] and Glashow [16]. In fact, the analogy with the Yang-Mills theory strongly suggests the existence of a neutral current. In fact, the Yang-Mills theory (see later) implies that the vector boson fields couple to conserved currents with associated charges generating the Lie algebra of the gauge group. Let us consider the $\nu_e - e$ contribution to the charged currents. The corresponding charges are

$$Q^{(-)} = \int d^3x\ J_0, \qquad Q^{(+)} = \int d^3x\ J_0^\dagger \tag{2.4}$$

with $J_0 = \psi_{\nu_e}^\dagger(1-\gamma_5)\psi_e$. The commutator of these two charges gives

$$[Q^{(+)}, Q^{(-)}] = 2Q^3 \tag{2.5}$$

with

$$Q^3 = \int d^3x\, J_0^3 = \int d^3x\ [\psi_e^\dagger(1-\gamma_5)\psi_e - \psi_\nu^\dagger(1-\gamma_5)\psi_\nu] \tag{2.6}$$

The Yang-Mills theory says that there should be a further vector boson coupled to the current J_μ^3

$$J_\mu^3 = \bar{\psi}_e\gamma_\mu(1-\gamma_5)\psi_e - \bar{\psi}_\nu\gamma_\mu(1-\gamma_5)\psi_\nu \tag{2.7}$$

This current has vanishing electric charge, and the corresponding vector boson should be neutral as well. Let us notice that this neutral vector boson cannot be identified with the photon, since the electromagnetic current is given by (considering again only the electron-neutrino system)

$$J_\mu^{\rm em} = \bar{\psi}_e\gamma_\mu\psi_e \tag{2.8}$$

In particular the new vector boson, call it Z, should couple to a neutrino pair. The existence of neutral weak currents was established at CERN in 1973, by the discovery of the elastic scattering $\nu_\mu + e^- \to \nu_\mu + e^-$. Since then, neutral-current interactions have been studied in many experiments. A distinctive feature is that the strangeness-changing processes are strongly suppressed

$$\frac{\Gamma(\Sigma^+ \to pe^+e^-)}{\Gamma(\Sigma^- \to ne^-\bar{\nu}_e)} < 1.3\times 10^{-2}, \qquad \frac{\Gamma(K_L^+ \to \mu^+\mu^-)}{\Gamma(K^+ \to \mu^+\nu_m u)} < 1.3\times 10^{-2} \tag{2.9}$$

A bonus of the introduction in the theory of the intermediate vector bosons is the improvement in the high-energy behaviour of the amplitudes. For instance, from eq. (2.2) it is not difficult to show that the differential cross-section for the process $\nu_\mu + e^- \to \mu^- + \nu_e$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 M_W^4}{4\pi^2 s}\left(\frac{s}{s - M_W^2}\right)^2 \tag{2.10}$$

To get this result one has to notice that the term proportional to $q^\mu q^\nu$ gives a contribution proportional to the lepton masses (and therefore negligible). We see from the previous expression how we get different behaviours for different energies

$$\begin{aligned} s \ll M_W^2 \quad &\to \quad \frac{d\sigma}{d\Omega} \approx \frac{G_F^2 s}{4\pi^2} \\ s \gg M_W^2 \quad &\to \quad \frac{d\sigma}{d\Omega} \approx \frac{G_F^2 M_W^4}{4\pi^2 s} \end{aligned} \tag{2.11}$$

For high energies the cross-section decreases avoiding the unitarity problem. However, this theory is not a renormalizable one. In fact the W-propagator goes to a constant at large energies. This bad behaviour comes from the term $q^\mu q^\nu$, which is harmless for the four-fermi interaction, but it is effective in reactions as $\bar{\nu}\nu \to W^+W^-$, or $e^+e^- \to W^+W^-$. The effect is to make the cross-section increase as E^2.

Summarizing what we have learned so far, we see that the necessary ingredients we need in order to build up a sensible theory of electroweak interactions are:

- Intermediate massive vector bosons, $W^\pm$, Z, and a massless photon, γ.
- Conservation of the various leptonic numbers.
- The $W^\pm$ field should couple only to left-handed doublets of fermions

  $$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \qquad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \tag{2.12}$$

  (we shall see later how things generalize to quarks).
- The Z field couples only diagonally in strangeness and in the other quantum numbers associated to new quarks. In general one says that Z should have only flavour-diagonal couplings.
- Renormalizability.

As we shall see, in order to satisfy all these requirements new ideas in particle physics have been necessary, namely

- gauge theories,

- spontaneous symmetry breaking,
- Higgs mechanism.

At the end of this section we collect some features about the field-theoretical description of a massive vector boson.

Massive vector bosons are described by a vector field V_μ which satisfies an equation of motion generalizing the Maxwell equation

$$(\partial^2 + m^2)V_\mu - \partial_\mu(\partial^\nu V_\nu) = 0 \tag{2.13}$$

The lagrangian giving rise to this equation of motion is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2V^2 \tag{2.14}$$

where

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{2.15}$$

If we take the four-divergence of the equation of motion we get

$$m^2\partial_\mu V^\mu = 0 \tag{2.16}$$

Therefore V_μ has zero four-divergence as a consequence of the equation of motion. Let us now count the number of degrees of freedom of the field V_μ. To this end let us consider the Fourier transform of $V_\mu(x)$

$$V_\mu(x) = \int d^4k e^{ikx} V_\mu(k) \tag{2.17}$$

Let us introduce four independent four-vectors,

$$k^\mu = (E, \vec{k}), \qquad \epsilon^{\mu(i)} = (0, \vec{n}_i), \quad i = 1, 2, \qquad \epsilon^{\mu(3)} = \frac{1}{m}\left(|\vec{k}|, \frac{E\vec{k}}{|\vec{k}|}\right) \tag{2.18}$$

with $\vec{k}\cdot\vec{n}_i = 0$, and $|\vec{n}_i|^2 = 1$. Then we can write $V_\mu(k)$ as

$$V_\mu(k) = \epsilon_\mu^{(\lambda)} a_\lambda(k) + k_\mu b(k) \tag{2.19}$$

By inserting this expression inside the equation of motion we get

$$(k^2 - m^2)(\epsilon_\mu^{(\lambda)} a_\lambda(k) + k_\mu b(k)) - k_\mu k^2 b(k) = 0 \tag{2.20}$$

that is

$$(k^2 - m^2)a_\lambda = 0, \qquad m^2 b = 0 \tag{2.21}$$

Therefore the field has three degrees of freedom corresponding to two transverse and to one longitudinal polarization. All these degrees of freedom satisfy a Klein-Gordon equation for a mass m. It is important to stress that the massive vector field has one extra-degree of freedom (the longitudinal), with respect to the massless case.

By introducing a fourth normalized four-vector

$$\epsilon_\mu^{(0)} = \frac{k_\mu}{m} \tag{2.22}$$

we get a basis in the four-dimensional Minkowski space, satisfying the completeness relation

$$\sum_{\lambda,\lambda'=0}^{3} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda')} g_{\lambda\lambda'} = g_{\mu\nu} \tag{2.23}$$

Therefore the sum over the physical polarizations is given by

$$\sum_{\lambda,\lambda'=1}^{3} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda')} g_{\lambda\lambda'} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \tag{2.24}$$

The propagator for the field $V_\mu(x)$ is defined in the usual way

$$\left[(\partial^2 + m^2) g_{\mu\nu} - \partial_\mu \partial_\nu\right] G^{\nu\lambda} = -g_\mu^\lambda \delta^4(x) \tag{2.25}$$

Its Fourier transform

$$G_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} G_{\mu\nu}(k) \tag{2.26}$$

satisfies

$$\left[(-k^2 + m^2) g_{\mu\nu} + k_\mu k_\nu\right] G^{\nu\lambda}(k) = -g_\mu^\lambda \tag{2.27}$$

and we get easily

$$G_{\mu\nu}(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{1}{k^2 - m^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\right] \tag{2.28}$$

The expression in parenthesis is nothing but the projector over the physical states of polarization.

2.2 QED as a gauge theory

Many field theories possess global symmetries. These are transformations leaving invariant the action of the system and are characterized by a certain number of parameters which are independent on the space-time point. As a prototype we can consider the free Dirac lagrangian

$$\mathcal{L}_0 = \bar\psi(x)[i\partial\!\!\!/ - m]\psi(x) \tag{2.29}$$

which is invariant under the global phase transformation

$$\psi(x) \to \psi'(x) = e^{iQ\alpha} \psi(x) \tag{2.30}$$

If one has more than one field, Q is a diagonal matrix having as eigenvalues the charges of the different fields measured in unit e. For instance, a term as $\bar{\psi}_2\psi_1\phi$, with ϕ a scalar field, is invariant by choosing $Q(\psi_1) = Q(\phi) = 1$, and $Q(\psi_2) = 2$. This is a so called **abelian symmetry** since

$$e^{i\alpha Q}e^{i\beta Q} = e^{i(\alpha+\beta)Q} = e^{i\beta Q}e^{i\alpha Q} \tag{2.31}$$

It is also referred to as a $U(1)$ symmetry. The physical meaning of this invariance lies in the possibility of assigning the phase to the fields in an arbitrary way, without changing the observable quantities. This way of thinking is in some sort of contradiction with causality, since it requires to assign the phase of the fields simultaneously at all space-time points. It looks more physical to require the possibility of assigning the phase in an arbitrary way at each space-time point. This invariance, formulated by Weyl in 1929 [17], was called gauge invariance. The free lagrangian (2.29) cannot be gauge invariant due to the derivative coming from the kinetic term. The idea is simply to generalize the derivative ∂_μ to a so called covariant derivative D_μ having the property that $D_\mu\psi$ transforms as ψ, that is

$$D_\mu\psi(x) \to [D_\mu\psi(x)]' = e^{iQ\alpha(x)}D_\mu\psi(x) \tag{2.32}$$

In this case the term

$$\bar{\psi}D_\mu\psi \tag{2.33}$$

will be invariant as the mass term under the local phase transformation. To construct the covariant derivative, we need to enlarge the field content of the theory, by introducing a vector field, the **gauge field** A_μ, in the following way

$$D_\mu = \partial_\mu - ieQA_\mu \tag{2.34}$$

The transformation law of A_μ is obtained from eq. (2.32)

$$\begin{aligned}
[(\partial_\mu - ieQA_\mu)\psi]' &= (\partial_\mu - ieQA'_\mu)\psi'(x) \\
&= (\partial_\mu - ieQA'_\mu)e^{iQ\alpha(x)}\psi \\
&= e^{iQ\alpha(x)}\left[\partial_\mu - ieQ(A'_\mu - \frac{1}{e}\partial_\mu\alpha)\right]\psi
\end{aligned} \tag{2.35}$$

from which

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha \tag{2.36}$$

The lagrangian

$$\mathcal{L}_\psi = \bar{\psi}[i\not{D} - m]\psi = \bar{\psi}[i\gamma^\mu(\partial_\mu - ieQA_\mu) - m)\psi = \mathcal{L}_0 + e\bar{\psi}Q\gamma^\mu\psi A_\mu \tag{2.37}$$

is then invariant under gauge transformations, or under the local group $U(1)$. In order to determine the kinetic term for the vector field A_μ we notice that eq. (2.32) implies that under a gauge transformation, the covariant derivative undergoes a unitary transformation

$$D_\mu \to D_\mu{}' = e^{iQ\alpha(x)} D_\mu e^{-iQ\alpha(x)} \tag{2.38}$$

Then, also the commutator of two covariant derivatives

$$[D_\mu, D_\nu] = [\partial_\mu - ieQA_\mu, \partial_\nu - ieQA_\nu] = -ieQF_{\mu\nu} \tag{2.39}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.40}$$

transforms in the same way

$$F_{\mu\nu} \to e^{iQ\alpha(x)} F_{\mu\nu} e^{-iQ\alpha(x)} = F_{\mu\nu} \tag{2.41}$$

The last equality follows from the commutativity of $F_{\mu\nu}$ with the phase factor. The complete lagrangian is then

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A = \bar{\psi}[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.42}$$

The gauge principle has automatically generated an interaction between the gauge field and the charged field. We notice also that gauge invariance prevents any mass term, $\frac{1}{2}M^2 A^\mu A_\mu$. Therefore, the photon field is massless.

2.3 Non-abelian gauge theories

The approach of the previous section can be easily extended to local non-abelian symmetries [18]. We will consider the case of N Dirac fields. The free lagrangian

$$\mathcal{L}_0 = \sum_{a=1}^{N} \bar{\psi}_a (i\partial\!\!\!/ - m)\psi_a \tag{2.43}$$

is invariant under the global transformation

$$\Psi(x) \to \Psi'(x) = A\Psi(x) \tag{2.44}$$

where A is a unitary $N \times N$ matrix, and we have denoted by Ψ the column vector with components ψ_a. In a more general situation the actual symmetry could be a subgroup of $U(N)$. For instance, when the masses are not all equal. So we will consider here the gauging of a subgroup G of $U(N)$. The fields $\psi_a(x)$ will belong, in general, to some reducible representation of G. Denoting by U the generic element of G, we will write the corresponding matrix U_{ab} acting upon the fields ψ_a as

$$U = e^{i\alpha_A T^A}, \qquad U \in G \tag{2.45}$$

where T^A denote the generators of the Lie algebra associated to G, Lie(G), in the fermion representation. The generators T^A satisfy the algebra

$$[T^A, T^B] = i f_C^{AB} T^C \tag{2.46}$$

where f_C^{AB} are the structure constants of Lie(G). For instance, if $G = SU(2)$, and we take the fermions in the fundamental representation,

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{2.47}$$

we have

$$T^A = \frac{\sigma^A}{2}, \qquad A = 1, 2, 3 \tag{2.48}$$

where σ^A are the Pauli matrices In the general case the T^A's are $N \times N$ hermitian matrices that we will choose normalized in such a way that

$$Tr(T^A T^B) = \frac{1}{2}\delta^{AB} \tag{2.49}$$

To make local the transformation (2.45), means to promote the parameters α_A to space-time functions

$$\alpha_A \to \alpha_A(x) \tag{2.50}$$

Notice, that now the group does not need to be abelian, and therefore, in general

$$e^{i\alpha_A T^A} e^{i\beta_A T^A} \neq e^{i\beta_A T^A} e^{i\alpha_A T^A} \tag{2.51}$$

Let us recall that for each global symmetry (and consequently the following will be valid also for the local case) there is a an associated four current with vanishing divergence

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \delta\phi^i \tag{2.52}$$

where $\delta\phi^i$ are the infinitesimal transformations undergone by the fields, and a conserved charge given by

$$Q = \int d^3x \; J^0 = \int d^3x \; \Pi_i \delta\phi^i, \qquad \Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} \tag{2.53}$$

The charge, multiplied by $-i$ generate the infinitesimal transformation on the fields

$$[\phi^j(\vec{x}, t), -iQ] = \int d^3y [\phi^j(\vec{x}, t), -i\Pi_i(\vec{y}, t)\delta\phi^i(\vec{y}, t)] = \delta\phi^j(\vec{x}, t) \tag{2.54}$$

where we have used the canonical commutation relations for the fields, and we have assumed that the transformation is linear, implying

$$[\phi^i(\vec{x}, t), \delta\phi^j(\vec{y}, t)] = 0 \tag{2.55}$$

For the transformation in eq. (2.45) we have

$$\delta\phi^i = i\alpha_A (T^A)^i_j \phi^j \tag{2.56}$$

and the current

$$J^\mu = i\alpha_A \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} (T^A)^i_j \phi^j \tag{2.57}$$

Since the parameters α_A are arbitrary we can extract a number of currents equal to the number of the generators of the group

$$j^A_\mu = -i \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} (T^A)^i_j \phi^j \tag{2.58}$$

with the corresponding conserved charges

$$Q^A = -i \int d^3x \; \Pi_i (T^A)^i_j \phi^j \tag{2.59}$$

Since the charges generate the infinitesimal transformations of the fields it is obvious that they give a representation of the Lie algebra of the symmetry group. Explicitly we have

$$\begin{aligned} [Q^A, Q^B] &= -\int d^3x \; d^3y [\Pi_i(\vec{x},t)(T^A)^i_j \phi^j(\vec{x},t), \Pi_l(\vec{y},t)(T^B)^l_m \phi^m(\vec{y},t)] \\ &= -i \int d^3x \; \Pi_i [T^A, T^B]^i_m \phi^m = i f^{AB}_C Q^C \end{aligned} \tag{2.60}$$

In particular, in the Dirac case, we have

$$j^A_\mu = \bar{\Psi} T^A \gamma_\mu \Psi \tag{2.61}$$

Let us now proceed to the case of the local symmetry by defining again the concept of covariant derivative

$$D_\mu \Psi(x) \to [D_\mu \Psi(x)]' = U(x)[D_\mu \psi(x)] \tag{2.62}$$

We will put again

$$D_\mu = \partial_\mu - igB_\mu \tag{2.63}$$

where B_μ is a $N \times N$ matrix acting upon $\Psi(x)$. In components

$$D^\mu_{ab} = \delta_{ab}\partial^\mu - ig(B^\mu)_{ab} \tag{2.64}$$

The eq. (2.62) implies

$$\begin{aligned} D_\mu \Psi &\to (\partial_\mu - igB'_\mu) U(x) \Psi \\ &= U(x)\partial_\mu \Psi + U(x)[-U^{-1}(x) igB'_\mu U(x)]\Psi + (\partial_\mu U(x))\Psi \\ &= U(x)[\partial_\mu - U^{-1}(x) igB'_\mu U(x) + U^{-1}(x)\partial_\mu U(x)]\Psi \end{aligned} \tag{2.65}$$

therefore

$$-U^{-1}(x)igB'_\mu U(x) + U^{-1}(x)\partial_\mu U(x) = -igB_\mu \tag{2.66}$$

and

$$B'_\mu(x) = U(x)B_\mu(x)U^{-1}(x) - \frac{i}{g}(\partial_\mu U(x))U^{-1}(x) \tag{2.67}$$

For an infinitesimal transformation

$$U(x) \approx 1 + i\alpha_A(x)T^A \tag{2.68}$$

we get

$$\delta B_\mu(x) = i\alpha_A(x)[T^A, B_\mu(x)] + \frac{1}{g}(\partial_\mu \alpha_A(x))T^A \tag{2.69}$$

Since $B_\mu(x)$ acquires a term proportional to Lie$G$, the transformation law is consistent with B_μ linear in T^A, that is

$$(B^\mu)_{ab} \equiv A^\mu_A (T^A)_{ab} \tag{2.70}$$

The transformation law for A_μ becomes

$$\delta A^\mu_C = -f_C^{AB}\alpha_A A^\mu_B - \frac{1}{g}\partial^\mu \alpha_C \tag{2.71}$$

The difference with respect to the abelian case is that the field undergoes also a homogeneous transformation.

The kinetic term for the gauge fields is constructed as in the abelian case. In fact the quantity

$$[D_\mu, D_\nu]\Psi \equiv -igF_{\mu\nu}\Psi \tag{2.72}$$

in virtue of the eq. (2.62), transforms as Ψ under gauge transformations, that is

$$\begin{aligned} ([D_\mu, D_\nu]\Psi)' &= -igF'_{\mu\nu}\Psi' = -igF'_{\mu\nu}U(x)\Psi \\ &= U(x)([D_\mu, D_\nu]\Psi) = U(x)\left(-igF_{\mu\nu}\right)\Psi \end{aligned} \tag{2.73}$$

This time the tensor $F_{\mu\nu}$ is not invariant but transforms homogeneously, since it does not commute with the gauge transformation as in the abelian case

$$F'_{\mu\nu} = U(x)F_{\mu\nu}U^{-1}(x) \tag{2.74}$$

The invariant kinetic term will be assumed as

$$\mathcal{L}_A = -\frac{1}{2}Tr[F_{\mu\nu}F^{\mu\nu}] \tag{2.75}$$

Let us now evaluate $F_{\mu\nu}$

$$\begin{aligned} -igF_{\mu\nu} &= [D_\mu, D_\nu] = [\partial_\mu - igB_\mu, \partial_\nu - igB_\nu] \\ &= -ig(\partial_\mu B_\nu - \partial_\nu B_\mu) - g^2[B_\mu, B_\nu] \end{aligned} \tag{2.76}$$

or

$$F_{\mu\nu} = (\partial_\mu B_\nu - \partial_\nu B_\mu) - ig[B_\mu, B_\nu] \tag{2.77}$$

in components

$$F^{\mu\nu} = F_C^{\mu\nu} T^C \tag{2.78}$$

with

$$F_C^{\mu\nu} = \partial^\mu A_C^\nu - \partial^\nu A_C^\mu + g f_C^{AB} A_A^\mu A_B^\nu \tag{2.79}$$

The main feature of the non-abelian gauge theories is the bilinear term in the previous expression. Such a term comes because $f_C^{AB} \neq 0$, expressing the fact that Lie(G), and G are not abelian. The kinetic term for the gauge field, expressed in components, is given by

$$\mathcal{L}_A = -\frac{1}{4} \sum_A F_{\mu\nu A} F_A^{\mu\nu} \tag{2.80}$$

Therefore, whereas in the abelian case $\mathcal{L}_A$ is a free lagrangian (it contains only quadratic terms), now it contains interaction terms cubic and quartic in the fields. The physical motivation lies in the fact that the gauge fields couple to everything which transforms in a non trivial way under the gauge group. Therefore they couple also to themselves (remember the homogeneous piece of transformation). From the point of view of renormalizability we can notice that both the covariant derivative term and the kinetic lagrangian for the gauge fields have mass dimension equal to 4.

2.4 The Standard Model for the electroweak interactions

We have now the elements to start to formulate the Weinberg Salam model for the electroweak interactions [19]. As usual we will consider here only the leptonic sector restricted to the electron and to its neutrino (here $\nu \equiv \nu_e$)

$$J_\mu^{(+)} = \bar{\psi}_e \gamma_\mu (1 - \gamma_5) \psi_\nu, \qquad J_\mu^{(-)} = J_\mu^{(+)\dagger} = \bar{\psi}_\nu \gamma_\mu (1 - \gamma_5) \psi_e \tag{2.81}$$

To understand the symmetry hidden in this couplings it is convenient to introduce the following spinors with 2×4 components

$$L = \begin{pmatrix} (\psi_\nu)_L \\ (\psi_e)_L \end{pmatrix} = \frac{1 - \gamma_5}{2} \begin{pmatrix} \psi_\nu \\ \psi_e \end{pmatrix} \tag{2.82}$$

where we have introduced left-handed fields. The right-handed spinors, projected out with $(1 + \gamma_5)/2$, don't feel the weak interaction. This is, in fact, the physical meaning of the $V - A$ interaction. By using

$$\bar{L} = \left((\bar{\psi}_\nu)_L, (\bar{\psi}_e)_L \right) = \left(\bar{\psi}_\nu, \bar{\psi}_e \right) \frac{1 + \gamma_5}{2} \tag{2.83}$$

we can write the weak currents in the following way

$$\begin{aligned} J_\mu^{(+)} &= 2\bar{\psi}_e \frac{1+\gamma_5}{2}\gamma_\mu \frac{1-\gamma_5}{2}\psi_\nu = 2\left((\bar{\psi}_\nu)_L, (\bar{\psi}_e)_L\right)\gamma_\mu \begin{pmatrix} 0 \\ (\psi_\nu)_L \end{pmatrix} \\ &= 2\left((\bar{\psi}_\nu)_L, (\bar{\psi}_e)_L\right)\gamma_\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\psi_\nu)_L \\ (\psi_e)_L \end{pmatrix} = 2\bar{L}\gamma_\mu \tau_- L \end{aligned} \tag{2.84}$$

where we have defined

$$\tau_\pm = \frac{\tau_1 \pm i\tau_2}{2} \tag{2.85}$$

the τ_i's being the Pauli matrices. Analogously

$$J_\mu^{(-)} = 2\bar{L}\gamma_\mu \tau_+ L \tag{2.86}$$

The interaction between these currents and the intermediate vector bosons is then given by

$$\mathcal{L}_{\rm int} = \frac{g}{2\sqrt{2}}\left[J_\mu^{(+)} W^{-\mu} + J_\mu^{(-)} W^{+\mu}\right] = \frac{g}{\sqrt{2}}\bar{L}\gamma_\mu(\tau_- W^{-\mu} + \tau_+ W^{+\mu})L \tag{2.87}$$

Introducing real fields in place of $W_\pm$

$$W^\pm = \frac{W_1 \mp iW_2}{\sqrt{2}} \tag{2.88}$$

we get

$$\mathcal{L}_{\rm int} = \frac{g}{2}\bar{L}\gamma_\mu(\tau_1 W_1^\mu + \tau_2 W_2^\mu)L \tag{2.89}$$

Let us now try to write the electromagnetic interaction within the same formalism

$$\begin{aligned} \mathcal{L}_{\rm em} = e\bar{\psi}_e Q \gamma_\mu \psi_e A^\mu &= e\bar{\psi}_e Q \left(\frac{1+\gamma_5}{2}\gamma_\mu \frac{1-\gamma_5}{2} + \frac{1-\gamma_5}{2}\gamma_\mu \frac{1+\gamma_5}{2}\right)\psi_e A^\mu \\ &= e[(\bar{\psi}_e)_L Q\gamma_\mu(\psi_e)_L + (\bar{\psi}_e)_R Q\gamma_\mu(\psi_e)_R]A^\mu \end{aligned} \tag{2.90}$$

where

$$(\psi_e)_R = \frac{1+\gamma_5}{2}\psi_e, \qquad (\bar{\psi}_e)_R = \bar{\psi}_e \frac{1-\gamma_5}{2} \tag{2.91}$$

where $Q = -1$ is the electric charge of the electron, in unit of the electric charge of the proton, e. It is also convenient to denote the right-handed components of the electron as

$$R = (\psi_e)_R \tag{2.92}$$

We can write

$$(\bar{\psi}_e)_L \gamma_\mu (\psi_e)_L = \bar{L}\gamma_\mu \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} L = \bar{L}\gamma_\mu \frac{1-\tau_3}{2} L \tag{2.93}$$

and

$$(\bar{\psi}_e)_R \gamma_\mu (\psi_e)_R = \bar{R}\gamma_\mu R \tag{2.94}$$

from which, using $Q = -1$,

$$\mathcal{L}_{\rm em} = e\left(-\frac{1}{2}\bar{L}\gamma_\mu L - \bar{R}\gamma_\mu R + \bar{L}\gamma_\mu \frac{\tau_3}{2} L\right) A^\mu = e j_\mu^{\rm em} A^\mu \tag{2.95}$$

From this equation we can write

$$j_\mu^{\rm em} = j_\mu^3 + \frac{1}{2} j_\mu^Y \tag{2.96}$$

with

$$j_\mu^3 = \bar{L}\gamma_\mu \frac{\tau_3}{2} L \tag{2.97}$$

and

$$j_\mu^Y = -\bar{L}\gamma_\mu L - 2\bar{R}\gamma_\mu R \tag{2.98}$$

These equations show that the electromagnetic current is a mixture of j_μ^3, which is a partner of the charged weak currents, and of j_μ^Y, which is another neutral current. The following notations unify the charged currents and j_μ^3

$$j_\mu^i = \bar{L}\gamma_\mu \frac{\tau_i}{2} L \tag{2.99}$$

These currents are of the form (2.61), and their associated charges span the Lie algebra of $SU(2)$,

$$[Q^i, Q^j] = i\epsilon_{ijk} Q^k \tag{2.100}$$

The charge Q^Y associated to the current j_μ^Y commutes with Q^i, as it can be easily verified noticing that the right-handed and left-handed fields commute with each other. Therefore the algebra of the charges is

$$[Q^i, Q^j] = i\epsilon_{ijk} Q^k, \qquad [Q^i, Q^Y] = 0 \tag{2.101}$$

This is the Lie algebra of $SU(2) \otimes U(1)$, since the Q^i's generate a group $SU(2)$, whereas Q^Y generates a group $U(1)$, with the two groups commuting among themselves. To build up a gauge theory from these elements we have to start with an initial lagrangian possessing an $SU(2) \otimes U(1)$ global invariance. This will produce 4 gauge vector bosons, which is the number we just need, see Section 2.1. By evaluating the commutators of the charges with the fields we can read immediately the quantum numbers of the various particles. We have

$$[L_c, Q^i] = \int d^3x \; [L_c, L_a^\dagger \left(\frac{\tau_i}{2}\right)_{ab} L_b] = \left(\frac{\tau_i}{2}\right)_{cb} L_b \tag{2.102}$$

and

$$[R, Q^i] = 0 \tag{2.103}$$

$$[L_c, Q^Y] = -L_c, \qquad [R, Q^Y] = -2R \tag{2.104}$$

These relations show that L transforms as an $SU(2)$ spinor, or, as the representation $\underline{2}$, where we have identified the representation with its dimensionality. R belongs to the trivial representation of dimension $\underline{1}$. Putting everything together we have the following behaviour under the representations of $SU(2) \otimes U(1)$

$$L \in (\underline{2}, -1), \qquad R \in (\underline{1}, -2) \tag{2.105}$$

The relation (2.96) gives the following relation among the different charges

$$Q_{\rm em} = Q^3 + \frac{1}{2} Q^Y \tag{2.106}$$

As we have argued, if $SU(2) \otimes U(1)$ has to be the fundamental symmetry of electroweak interactions, we have to start with a lagrangian exhibiting this global symmetry. This is the free Dirac lagrangian for a massless electron and a left-handed neutrino

$$\mathcal{L}_0 = \bar{L} i \hat{\partial} L + \bar{R} i \hat{\partial} R \tag{2.107}$$

Mass terms, mixing left- and right-handed fields, would destroy the global symmetry. For instance, a typical mass term gives

$$m\bar{\psi}\psi = m\psi^\dagger \left[\frac{1+\gamma_5}{2}\gamma_0 \frac{1-\gamma_5}{2} + \frac{1-\gamma_5}{2}\gamma_0 \frac{1+\gamma_5}{2} \right] \psi = m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) \tag{2.108}$$

We shall see later that the same mechanism giving mass to the vector bosons can be used to generate fermion masses. A lagrangian invariant under the local symmetry $SU(2) \otimes U(1)$ is obtained through the use of covariant derivatives

$$\mathcal{L} = \bar{L} i \gamma^\mu \left(\partial_\mu - ig \frac{\tau_i}{2} W^i_\mu + i \frac{g'}{2} Y_\mu \right) L + \bar{R} i \gamma^\mu \left(\partial_\mu + i g' Y^\mu \right) R \tag{2.109}$$

The interaction term is

$$\mathcal{L}_{\rm int} = g \bar{L} \gamma_\mu \left(\frac{\vec{\tau}}{2} \cdot \vec{W}^\mu \right) L - \frac{g'}{2} \bar{L} \gamma_\mu L Y^\mu - g' \bar{R} \gamma_\mu R Y^\mu \tag{2.110}$$

In this expression we recognize the charged interaction with $W^\pm$. We have also, as expected, two neutral vector bosons W^3 and Y. The photon must couple to the electromagnetic current which is a linear combination of j^3 and j^Y. The neutral vector bosons are coupled to these two currents, so we expect the photon field, A_μ, to be a linear combination of W^3_μ and Y_μ. It is convenient to introduce two orthogonal combination of these two fields, Z_μ, and A_μ. The mixing angle can be identified through the requirement that the current coupled to A_μ is exactly the electromagnetic current with coupling constant e. Let us consider the part of $\mathcal{L}_{\rm int}$ involving the neutral couplings

$$\mathcal{L}_{NC} = g j^3_\mu W_3^\mu + \frac{g'}{2} j^Y_\mu Y^\mu \tag{2.111}$$

By putting (θ is called the Weinberg angle)

$$\begin{aligned} W_3 &= Z\ \cos\theta + A\ \sin\theta \\ Y\ &= -Z\ \sin\theta + A\ \cos\theta \end{aligned} \tag{2.112}$$

we get

$$\mathcal{L}_{NC} = \left[g\sin\theta\ j^3_\mu + g'\cos\theta\frac{1}{2}j^Y_\mu\right]A^\mu + \left[g\cos\theta\ j^3_\mu - g'\sin\theta\frac{1}{2}j^Y_\mu\right]Z^\mu \tag{2.113}$$

The electromagnetic coupling is reproduced by requiring the two conditions

$$g\sin\theta = g'\cos\theta = e \tag{2.114}$$

These two relations allow us to express both the Weinberg angle and the electric charge e in terms of g and g'.

In the low-energy experiments performed before the LEP era (< 1990) the fundamental parameters used in the theory were e (or rather the fine structure constant α) and $\sin^2\theta$. We shall see in the following how the Weinberg angle can be eliminated in favor of the mass of the Z, which is now very well known. By using eq. (2.114), and $j^{\rm em}_\mu = j^3_\mu + \frac{1}{2}j^Y_\mu$, we can write

$$\mathcal{L}_{NC} = e j^{\rm em}_\mu A^\mu + [g\cos\theta j^3_\mu - g'\sin\theta(j^{\rm em}_\mu - j^3_\mu)]Z^\mu \tag{2.115}$$

Expressing g' through $g' = g\tan\theta$, the coefficient of Z_μ can be put in the form

$$[g\cos\theta + g'\sin\theta]j^3_\mu - g'\sin\theta\ j^{\rm em}_\mu = \frac{g}{\cos\theta}j^3_\mu - \frac{g}{\cos\theta}\sin^2\theta\ j^{\rm em}_\mu \tag{2.116}$$

from which

$$\mathcal{L}_{NC} = e j^{\rm em}_\mu A^\mu + \frac{g}{\cos\theta}[j^3_\mu - \sin^2\theta\ j^{\rm em}_\mu]Z^\mu \equiv e j^{\rm em}_\mu A^\mu + \frac{g}{\cos\theta}j^Z_\mu Z^\mu \tag{2.117}$$

where we have introduced the neutral current coupled to the Z

$$j^Z_\mu = j^3_\mu - \sin^2\theta\ j^{\rm em}_\mu \tag{2.118}$$

The value of $\sin^2\theta$ was evaluated initially from processes induced by neutral currents (as the scattering $\nu - e$) at low energy. The approximate value is

$$\sin^2\theta \approx 0.23 \tag{2.119}$$

This shows that both g and g' are of the same order of the electric charge. Notice that the eq. (2.114) gives the following relation among the electric charge and the couplings g and g'

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2} \tag{2.120}$$

2.5 Spontaneous symmetry breaking

We have seen that the charged couplings are neatly recovered within the context of a gauge theory. Also one has a renormalizable scheme. However we know that three of the four vector bosons we have introduced have to get mass. This requires that we break in some way the gauge invariance of the theory. However, doing that, one would destroy the renormalizability property. The only way out appears to be spontaneous symmetry breaking because, as we shall see this allows to get non-symmetric result out of an invariant formulation. This leaves us some hope to preserve renormalizability. The general framework of a spontaneous symmetry breaking theory is based on the following two points

- The theory is invariant under a symmetry group G.
- The fundamental state of the theory is degenerate and transforms in a non trivial way under the symmetry group.

Just as an example consider a scalar field described by a lagrangian invariant under parity

$$P: \quad \phi \to -\phi \tag{2.121}$$

The lagrangian will be of the type

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi^2) \tag{2.122}$$

If the vacuum state is non degenerate, barring a phase factor, we must have

$$P|0\rangle = |0\rangle \tag{2.123}$$

since P commutes with the hamiltonian. As a consequence $P|0\rangle$ and $|0\rangle$ having the same energy must coincide. It follows

$$\langle 0|\phi|0\rangle = \langle 0|P^{-1}P\phi P^{-1}P|0\rangle = \langle 0|P\phi P^{-1}|0\rangle = -\langle 0|\phi|0\rangle \tag{2.124}$$

from which

$$\langle 0|\phi|0\rangle = 0 \tag{2.125}$$

Things change if the fundamental state is degenerate. This would be the case in the example (2.122), if

$$V(\phi^2) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \tag{2.126}$$

with $\mu^2 < 0$. In fact, this potential has two minima located at

$$\phi = \pm v, \qquad v = \sqrt{-\frac{\mu^2}{\lambda}} \tag{2.127}$$

By denoting with $|R\rangle$ e $|L\rangle$ the two states corresponding to the classical configurations $\phi = \pm v$, we have

$$P|R\rangle = |L\rangle \neq |R\rangle \tag{2.128}$$

Therefore

$$\langle R|\phi|R\rangle = \langle R|P^{-1}P\phi P^{-1}P|R\rangle = -\langle L|\phi|L\rangle \tag{2.129}$$

which now does not imply that the expectation value of the field vanishes. From our point of view we will be rather interested in the case of continuous symmetries. So let us consider this case with two scalar fields, and a lagrangian with symmetry $O(2)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{2}\mu^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2 \tag{2.130}$$

where

$$\vec{\phi}\cdot\vec{\phi} = \phi_1^2 + \phi_2^2 \tag{2.131}$$

For $\mu^2 > 0$ there is a unique fundamental state (minimum of the potential) $\vec{\phi} = 0$, whereas for $\mu^2 < 0$ there are infinite degenerate states given by

$$|\vec{\phi}|^2 = \phi_1^2 + \phi_2^2 = v^2 \tag{2.132}$$

with v defined as in (2.127). By denoting with $R(\theta)$ the operator rotating the fields in the plane (ϕ_1, ϕ_2), in the non-degenerate case we have

$$R(\theta)|0\rangle = |0\rangle \tag{2.133}$$

and

$$\langle 0|\phi|0\rangle = \langle 0|R^{-1}R\phi R^{-1}R|0\rangle = \langle 0|\phi^\theta|0\rangle = 0 \tag{2.134}$$

since $\phi^\theta \neq \phi$. In the case $\mu^2 < 0$ (degenerate case), we have

$$R(\theta)|0\rangle = |\theta\rangle \tag{2.135}$$

where $|\theta\rangle$ is one of the infinitely many degenerate fundamental states lying on the circle $|\vec{\phi}|^2 = v^2$. Then

$$\langle 0|\phi_i|0\rangle = \langle 0|R^{-1}(\theta)R(\theta)\phi_i R^{-1}(\theta)R(\theta)|0\rangle = \langle\theta|\phi_i^\theta|\theta\rangle \tag{2.136}$$

with

$$\phi_i^\theta = R(\theta)\phi_i R^{-1}(\theta) \neq \phi_i \tag{2.137}$$

Again, the expectation value of the field (contrarily to the non-degenerate state) does not need to vanish. The situation can be described qualitatively saying that the existence of a degenerate fundamental state forces the system to choose one of these equivalent states, and consequently to break the symmetry. But the breaking is only at the level of the solutions, the lagrangian and the equations of motion preserve the symmetry. One can easily construct classical systems exhibiting spontaneous symmetry breaking. For instance, a classical particle in a double-well potential.

This system has parity invariance $x \to -x$, where x is the particle position. The equilibrium positions are around the minima positions, $\pm x_0$. If we put the particle close to x_0, it will perform oscillations around that point and the original symmetry is lost. A further example is given by a ferromagnet which has an hamiltonian invariant under rotations, but below the Curie temperature exhibits spontaneous magnetization, breaking in this way the symmetry. These situations are typical for the so called second order phase transitions. One can describe them through the Landau free-energy, which depends on two different kind of parameters:

- **Control parameters**, as μ^2 for the scalar field, and the temperature for the ferromagnet.
- **Order parameters**, as the expectation value of the scalar field or as the magnetization.

The system goes from one phase to another varying the control parameters, and the phase transition is characterized by the order parameters which assume different values in different phases. In the previous examples, the order parameters were zero in the symmetric phase and different from zero in the broken phase.

The situation is slightly more involved at the quantum level, since spontaneous symmetry breaking cannot happen in finite systems. This follows from the existence of the tunnel effect. Let us consider again a particle in a double-well potential, and recall that we have defined the fundamental states through the correspondence with the classical minima

$$\begin{aligned} x = x_0 &\to |R\rangle \\ x = -x_0 &\to |L\rangle \end{aligned} \tag{2.138}$$

But the tunnel effect gives rise to a transition between these two states and as a consequence it removes the degeneracy. In fact, due to the transition the hamiltonian acquires a non zero matrix element between the states $|R\rangle$ and $|L\rangle$. By denoting with $\underline{H}$ the matrix of the hamiltonian between these two states, we get

$$\underline{H} = \begin{bmatrix} \epsilon_0 & \epsilon_1 \\ \epsilon_1 & \epsilon_0 \end{bmatrix} \tag{2.139}$$

The eigenvalues of $\underline{H}$ are

$$(\epsilon_0 + \epsilon_1, \epsilon_0 - \epsilon_1) \tag{2.140}$$

We have no more degeneracy and the eigenstates are

$$|S\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) \tag{2.141}$$

with eigenvalue $E_S = \epsilon_0 + \epsilon_1$, and

$$|A\rangle = \frac{1}{\sqrt{2}}(|R\rangle - |L\rangle) \tag{2.142}$$

with eigenvalue $E_A = \epsilon_0 - \epsilon_1$. One can show that $\epsilon_1 < 0$ and therefore the fundamental state is the symmetric one, $|S\rangle$. This situation gives rise to the well known effect of quantum oscillations (for instance the $K^0 - \bar{K}^0$ transitions). We can express the states $|R\rangle$ and $|L\rangle$ in terms of the energy eigenstates

$$\begin{aligned} |R\rangle &= \tfrac{1}{\sqrt{2}}(|S\rangle + |A\rangle) \\ |L\rangle &= \tfrac{1}{\sqrt{2}}(|S\rangle - |A\rangle) \end{aligned} \tag{2.143}$$

Let us now prepare a state, at $t = 0$, by putting the particle in the right minimum. This is not an energy eigenstate and its time evolution is given by

$$|R,t\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_S t}|S\rangle + e^{-iE_A t}|A\rangle\right) = \frac{1}{\sqrt{2}}e^{-iE_S t}\left(|S\rangle + e^{-it\Delta E}|A\rangle\right) \tag{2.144}$$

with $\Delta E = E_A - E_S$. Therefore, for $t = \pi/\Delta E$ the state $|R\rangle$ transforms into the state $|L\rangle$. The state oscillates with a period given by

$$T = \frac{2\pi}{\Delta E} \tag{2.145}$$

In nature there are finite systems as sugar molecules, which seem to exhibit spontaneous symmetry breaking. In fact one observes right-handed and left-handed sugar molecules. The explanation is simply that the energy difference ΔE is so small that the oscillation period is of the order of $10^4 - 10^6$ years.

The splitting of the fundamental states decreases with the height of the potential between two minima, therefore, for infinite systems, the previous mechanism does not work, and we may have spontaneous symmetry breaking. In fact, coming back to the scalar field example, its expectation value on the vacuum must be a constant, as it follows from the translational invariance of the vacuum

$$\langle 0|\phi(x)|0\rangle = \langle 0|e^{iPx}\phi(0)e^{-iPx}|0\rangle = \langle 0|\phi(0)|0\rangle = v \tag{2.146}$$

and the height of the potential between the two minima becomes infinite in the limit of infinite volume

$$H(\phi = 0) - H(\phi = v) = -\int_V d^3x \left[\frac{\mu^2}{2}v^2 + \frac{\lambda}{4}v^4\right] = \frac{\mu^4}{4\lambda}\int_V d^3x = \frac{\mu^4}{4\lambda}V \tag{2.147}$$

2.6 The Goldstone theorem

From our point of view, the most interesting consequence of spontaneous symmetry breaking is the Goldstone theorem [20]. This theorem says that for any continuous symmetry spontaneously broken, there exists a massless particle (the Goldstone boson). The theorem holds rigorously in a local field theory, under the following hypotheses

- The spontaneous broken symmetry must be a continuous one.
- The theory must be manifestly covariant.
- The Hilbert space of the theory must have a definite positive norm.

We will analyze the theorem in the case of scalar field theory with symmetry $O(3)$. Let us consider the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\mu^2}{2}\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2 \tag{2.148}$$

An infinitesimal rotation along the direction $\vec{n}$ ($|\vec{n}|^2 = 1$) can be written as

$$\vec{\phi} \to \vec{\phi} + \theta\vec{\phi}\wedge\vec{n} \tag{2.149}$$

In fact a rotation leaves invariant the norm of a vector, therefore for an infinitesimal rotation we get

$$|\vec{\phi}|^2 \to |\vec{\phi} + \delta\vec{\phi}|^2 = |\vec{\phi}|^2 + 2\vec{\phi}\cdot\delta\vec{\phi} \tag{2.150}$$

from which

$$\vec{\phi}\cdot\delta\vec{\phi} = 0 \tag{2.151}$$

Since for a rotation around the unit vector $\vec{n}$, we must have $\delta\vec{\phi}$ orthogonal to $\vec{n}$, it follows

$$\delta\vec{\phi} = \theta\vec{\phi}\wedge\vec{n} \tag{2.152}$$

where θ is the infinitesimal rotation angle. For instance, if $\vec{n} = (0, 0, 1)$

$$\begin{aligned} \delta\phi_1 &= \theta\phi_2 \\ \delta\phi_2 &= -\theta\phi_1 \\ \delta\phi_3 &= 0 \end{aligned} \tag{2.153}$$

By defining $\vec{\eta} = \theta\vec{n}$, we can also write

$$\phi_l \to (\delta_{lm} + \epsilon_{alm}\eta_a)\phi_m = (1 + iT_a\eta_a)_{lm}\phi_m \tag{2.154}$$

with the infinitesimal generators defined by

$$(T_a)_{lm} = -i\epsilon_{alm} \tag{2.155}$$

The finite transformation will be

$$\phi_l \to \left(e^{i\theta\vec{T}\cdot\vec{n}}\right)_{lm}\phi_m \tag{2.156}$$

The conditions that V must satisfy in order to have a minimum are

$$\frac{\partial V}{\partial\phi_l} = \mu^2\phi_l + \lambda\phi_l|\vec{\phi}|^2 = 0 \tag{2.157}$$

with solutions

$$\phi_l = 0, \qquad |\vec{\phi}|^2 = v^2, \qquad v = \sqrt{\frac{-\mu^2}{\lambda}} \tag{2.158}$$

The character of the stationary points can be studied by evaluating the second derivatives

$$\frac{\partial^2 V}{\partial\phi_l \partial\phi_m} = \delta_{lm}(\mu^2 + \lambda|\vec{\phi}|^2) + 2\lambda\phi_l\phi_m \tag{2.159}$$

We have two possibilities

- $\mu^2 > 0$, we have only one real solution given by $\vec{\phi} = 0$, which is a minimum, since

$$\frac{\partial^2 V}{\partial\phi_l \partial\phi_m} = \delta_{lm}\mu^2 > 0 \tag{2.160}$$

- $\mu^2 < 0$, there are infinite solutions, among which $\vec{\phi} = 0$ is a maximum. The points of the sphere $|\vec{\phi}|^2 = v^2$ are degenerate minima. In fact, by choosing $\phi_l = v\delta_{l3}$ as a representative point, we get

$$\frac{\partial^2 V}{\partial\phi_l \partial\phi_m} = 2\lambda v^2 \delta_{l3}\delta_{m3} > 0 \tag{2.161}$$

Expanding the potential around this minimum we get

$$V(\vec{\phi}) \approx V\Big|_{\text{minimum}} + \frac{1}{2}\frac{\partial^2 V}{\partial\phi_l \partial\phi_m}\Big|_{\text{minimum}}(\phi_l - v\delta_{l3})(\phi_m - v\delta_{m3}) \tag{2.162}$$

If we are going to make a perturbative expansion, the right fields to be used are $\phi_l - \delta_{l3}$, and their mass is just given by the coefficient of the quadratic term

$$M^2_{lm} = \frac{\partial^2 V}{\partial\phi_l \partial\phi_m}\Big|_{\text{minimo}} = -2\mu^2\delta_{l3}\delta_{m3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\mu^2 \end{bmatrix} \tag{2.163}$$

Therefore the masses of the fields ϕ_1, ϕ_2, and $\chi = \phi_3 - v$, are given by

$$m^2_{\phi_1} = m^2_{\phi_2} = 0, \qquad m^2_\chi = -2\mu^2 \tag{2.164}$$

By defining

$$m^2 = -2\mu^2 \tag{2.165}$$

we can write the potential as a function of the new fields

$$V = -\frac{m^4}{16\lambda} + \frac{1}{2}m^2\chi^2 + \sqrt{\frac{m^2\lambda}{2}}\chi(\phi_1^2 + \phi_2^2 + \chi^2) + \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2 + \chi^2\right)^2 \tag{2.166}$$

In this form the original symmetry $O(3)$ is broken. However a residual symmetry $O(2)$ is left. In fact, V depends only from the combination $\phi_1^2 + \phi_2^2$, and it is invariant under rotations around the axis we have chosen as representative for the fundamental state, $(0,0,v)$. It must be stressed that this is not the most general potential invariant under $O(2)$, in fact this would depend on 6 coupling constants, whereas the one we got depends only on two parameters m and λ. Therefore spontaneous symmetry breaking puts heavy constraints on the dynamics of the system. We have also seen that we have two massless scalars and two continuous symmetry broken correspondingly to rotations around the axis 1 and 2. This can be seen also in terms of generators. The state that we have chosen as representative of the vacuum (the fundamental state) can be written, in field space, as

$$|0\rangle = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix} \tag{2.167}$$

The expression for the O(3) generators is (see eq. (2.155))

$$T_1 = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad T_2 = -i \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T_3 = -i \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.168}$$

and we easily verify that

$$T_1|0\rangle \neq 0, \qquad T_2|0\rangle \neq 0, \qquad T_3|0\rangle = 0 \tag{2.169}$$

The first two relations show that the vacuum is not invariant under rotations around the axis 1 and 2, whereas the third ensure the remaining $O(2)$ symmetry. The generators of $O(3)$ divide up naturally in the generators of the vacuum symmetry (here $O(2)$), and in the so called broken generators, each of them corresponding to a massless Goldstone boson. In general, if the original symmetry group G of the theory is spontaneously broken down to a subgroup H (which is the symmetry of the vacuum), the Goldstone bosons correspond to the generators of G which are left after subtracting the generators of H. Intuitively one can understand the origin of the massless particles noticing that the broken generators allow transitions from a possible vacuum to another. Since these states are degenerate the operation does not cost any energy. From the relativistic dispersion relation this implies that we must have massless particles. One can say that Goldstone bosons correspond to flat directions in the potential.

2.7 The Higgs mechanism

At first sight the spontaneous symmetry breaking mechanism does not seem to fulfill our hopes to solve the mass problem. On the contrary we get more massless particles, the Goldstone bosons. However, once one couples spontaneous symmetry breaking to a gauge symmetry, things change. In fact, if we look back at the hypothesis underlying a gauge theory, it turns out that Goldstone theorem does not hold in this context. The reason is that it is impossible to quantize a gauge theory in a way which is at the same time manifestly covariant and has a Hilbert space with positive definite metric. This is well known already for the electromagnetic field, where one has to choose the gauge before quantization. What happens is that, if one chooses a physical gauge, as the Coulomb gauge, in order to have a Hilbert space spanned by only the physical states, than the theory looses the manifest covariance. If one goes to a covariant gauge, as the Lorentz one, the theory is covariant but one has to work with a big Hilbert space, with non-definite positive metric, and where the physical states are extracted through a supplementary condition. The way in which the Goldstone theorem is evaded is that the Goldstone bosons disappear, and, at the same time, the gauge bosons corresponding to the broken symmetries acquire mass. This is the famous **Higgs mechanism** [21].

Let us start again with a scalar theory invariant under $O(2)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\mu^2}{2}\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2 \tag{2.170}$$

and let us analyze the spontaneous symmetry breaking mechanism. If $\mu^2 < 0$ the symmetry is broken and we can choose the vacuum as the state

$$\vec{\phi} = (v, 0), \qquad v = \sqrt{\frac{-\mu^2}{\lambda}} \tag{2.171}$$

After the translation $\phi_1 = \chi + v$, with $\langle 0|\chi|0\rangle = 0$, we get the potential $(m^2 = -2\mu^2)$

$$V = -\frac{m^4}{16\lambda} + \frac{1}{2}m^2\chi^2 + \sqrt{\frac{m^2\lambda}{2}}\chi(\phi_2^2 + \chi^2) + \frac{\lambda}{4}(\phi_2^2 + \chi^2)^2 \tag{2.172}$$

In this case the group $O(2)$ is completely broken (except for the discrete symmetry $\phi_2 \to -\phi_2$). The Goldstone field is ϕ_2 . This has a peculiar way of transforming under $O(2)$. In fact, the original fields transform as

$$\delta\phi_1 = -\alpha\phi_2, \qquad \delta\phi_2 = \alpha\phi_1 \tag{2.173}$$

from which

$$\delta\chi = -\alpha\phi_2, \qquad \delta\phi_2 = \alpha\chi + \alpha v \tag{2.174}$$

We see that the Goldstone field undergoes a rotation plus a translation, αv. This is the main reason for the Goldstone particle to be massless. In fact one can have invariance under translations of the field, only if the potential is flat in the corresponding direction. This is what happens when one moves in a way which is tangent to the surface of the degenerate vacua (in this case a circle). How do things change if our theory is gauge invariant? In that case we should have invariance under a transformation of the Goldstone field given by

$$\delta\phi_2(x) = \alpha(x)\chi(x) + \alpha(x)v \tag{2.175}$$

Since $\alpha(x)$ is an arbitrary function of the space-time point, it follows that we can choose it in such a way to make $\phi_2(x)$ vanish. In other words it must be possible to eliminate the Goldstone field from the theory. This is better seen by using polar coordinates for the fields, that is

$$\rho = \sqrt{\phi_1^2 + \phi_2^2}, \qquad \sin\theta = \frac{\phi_2}{\sqrt{\phi_1^2 + \phi_2^2}} \tag{2.176}$$

Under a finite rotation, the new fields transform as

$$\rho \to \rho, \qquad \theta \to \theta + \alpha \tag{2.177}$$

It should be also noticed that the two coordinate systems coincide when we are close to the vacuum, as when we are doing perturbation theory. In fact, in that case we can perform the following expansion

$$\rho = \sqrt{\phi_2^2 + \chi^2 + 2\chi v + v^2} \approx v + \chi, \qquad \theta \approx \frac{\phi_2}{v + \chi} \approx \frac{\phi_2}{v} \tag{2.178}$$

Again, if we make the theory invariant under a local transformation, we will have invariance under

$$\theta(x) \to \theta(x) + \alpha(x) \tag{2.179}$$

By choosing $\alpha(x) = -\theta(x)$ we can eliminate this last field from the theory. The only remaining degree of freedom in the scalar sector is $\rho(x)$.

Let us study the gauging of this model. It is convenient to introduce complex variables

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{2.180}$$

The $O(2)$ transformations become phase transformations on ϕ

$$\phi \to e^{i\alpha}\phi \tag{2.181}$$

and the lagrangian (2.170) can be written as

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{2.182}$$

From Section 2.2 we know that we can promote a global symmetry to a local one by introducing the covariant derivative

$$\partial_\mu\phi \to (\partial_\mu - igA_\mu)\phi \tag{2.183}$$

From which

$$\mathcal{L} = (\partial_\mu + igA_\mu)\phi^\dagger(\partial^\mu - igA_\mu)\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.184}$$

In terms of the polar coordinates (ρ, θ) we have

$$\phi = \frac{1}{\sqrt{2}}\rho e^{i\theta}, \qquad \phi^\dagger = \frac{1}{\sqrt{2}}\rho e^{-i\theta} \tag{2.185}$$

By performing the following gauge transformation on the scalars

$$\phi \to \phi' = \phi e^{-i\theta} \tag{2.186}$$

and the corresponding transformation on the gauge fields

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{g}\partial_\mu \theta \tag{2.187}$$

the lagrangian will depend only on the fields ρ and A'_μ (we will put again $A'_\mu = A_\mu$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu - igA_\mu)\rho(\partial^\mu + igA_\mu)\rho - \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.188}$$

In this way the Goldstone boson disappears. We have now to translate the field ρ

$$\rho = \chi + v, \qquad \langle 0|\chi|0\rangle = 0 \tag{2.189}$$

and we see that this generates a bilinear term in A_μ, coming from the covariant derivative, given by

$$\frac{1}{2}g^2 v^2 A_\mu A^\mu \tag{2.190}$$

Therefore the gauge field acquires a mass

$$m_A^2 = g^2 v^2 \tag{2.191}$$

It is instructive to count the degrees of freedom before and after the gauge transformation. Before we had 4 degrees of freedom, two from the scalar fields and two from the gauge field. After the gauge transformation we have only one degree of freedom from the scalar sector, but three degrees of freedom from the gauge vector, because now it is a massive vector field. The result looks a little bit strange, but the reason why we may read clearly the number of degrees of freedom only after the gauge transformation is that before the lagrangian contains a mixing term

$$A_\mu \partial^\mu \theta \tag{2.192}$$

between the Goldstone field and the gauge vector which makes complicate to read the mass of the states. The previous gauge transformation realizes the purpose of making that term vanish. The gauge in which such a thing happens is called the **unitary gauge**.

We will consider now the further example of a symmetry $O(3)$, already discussed in the previous Section. The lagrangian invariant under local transformations is

$$\mathcal{L} = \frac{1}{2}(D_\mu)_{lm}\phi_m(D^\mu)_{ln}\phi_n - \frac{\mu^2}{2}\phi_l\phi_l - \frac{\lambda}{4}(\phi_l\phi_l)^2 \tag{2.193}$$

where

$$(D_\mu)_{lm} = \delta_{lm}\partial_\mu - ig(T_a)_{lm}W_\mu^a \tag{2.194}$$

and we recall also that $(T_a)_{lm} = -i\epsilon_{alm}$. In the case of broken symmetry ($\mu^2 < 0$), we choose again the vacuum along the direction 3, with v defined as in (2.171)

$$\phi_l = v\delta_{l3} \tag{2.195}$$

The mass term for the gauge field is given by

$$-\frac{1}{2}g^2v^2(T_a)_{l3}(T_b)_{l3}W_\mu^a W^{\mu a} \tag{2.196}$$

and using

$$(T_a)_{l3}(T_b)_{l3} = -\epsilon_{al3}\epsilon_{bl3} = -(\delta_{ab} - \delta_{a3}\delta_{b3}) \tag{2.197}$$

we get

$$(M_W)^2_{ab} = g^2v^2(\delta_{ab} - \delta_{a3}\delta_{b3}) = g^2v^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.198}$$

Therefore, the two fields W_μ^1 e W_μ^2, associated to the broken directions T_1 e T_2, acquire mass, whereas W_μ^3, associated to the unbroken symmetry $O(2)$, remains massless.

In general, if G is the global symmetry group of the lagrangian, H the subgroup of G leaving invariant the vacuum, and G_W the group of local (gauge) symmetries, $G_W \in G$, one can divide up the broken generators in two categories. In the first category fall the broken generators lying in G_W; they have associated massive vector bosons. In the second category fall the other broken generators; they have associated massless Goldstone bosons. The situation is represented in Fig. 2.2.

By using the $O(3)$ example we can show how to eliminate in general the Goldstone bosons. In fact we can define new fields ξ and χ as

$$\phi_l = \left(e^{iT_\alpha\xi_\alpha}\right)_{l3}(\chi + v) \tag{2.199}$$

where the index α takes the values 1 e 2, that is the sum is restricted to the broken directions. The other degree of freedom is in the other factor. The correspondence among the fields $\vec{\phi}$ and (ξ_1, ξ_2, χ) can be seen easily by expanding around the vacuum

$$\left(e^{iT_a\xi_a}\right)_{l3} \approx \delta_{l3} + i(T_1)_{l3}\xi_1 + i(T_2)_{l3}\xi_2 = \delta_{l3} + \epsilon_{1l3}\xi_1 + \epsilon_{2l3}\xi_2 = (-\xi_2, \xi_1, 1) \tag{2.200}$$

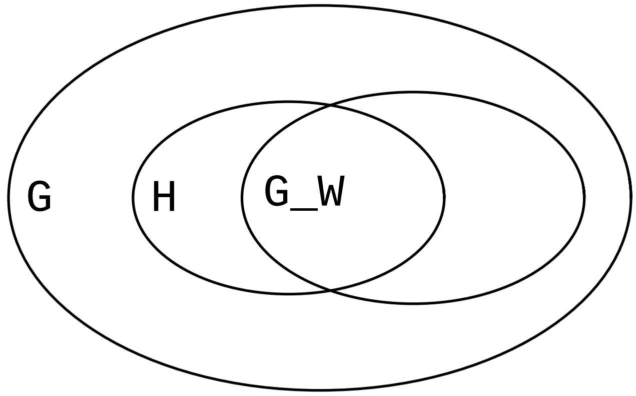

Fig. 2.2 -*This figure shows the various groups, G, the global symmetry of the lagrangian, $H \in G$, the symmetry of the vacuum, and G_W, the group of local symmetries. The broken generators in G_W correspond to massive vector bosons. The broken generators do not belonging to G_W correspond to massless Goldstone bosons.*

from which

$$\phi_l \approx (-\xi_2, \xi_1, 1)(\chi + v) \approx (-v\xi_2, v\xi_1, \chi + v) \tag{2.201}$$

showing that ξ_α are really the Goldstone fields. The unitary gauge is defined through the transformation

$$\phi_l \to \left(e^{-iT_\alpha \xi_a}\right)_{lm} \phi_m = \delta_{l3}(\chi + v) \tag{2.202}$$

$$W_\mu \to e^{-iT_\alpha \xi_a} W_\mu e^{iT_\alpha \xi_a} - \frac{i}{g}\left(\partial_\mu e^{-iT_\alpha \xi_a}\right) e^{+iT_\alpha \xi_a} \tag{2.203}$$

This transformation eliminates the Goldstone degrees of freedom and the resulting lagrangian depends on the field χ, on the two massive vector fields W_μ^1 and W_μ^2, and on the massless field W_μ^3.

2.8 The Higgs sector in the Standard Model

According to the discussion made in the previous Section, we need three broken symmetries in order to give mass to $W^\pm$ and Z. Since $SU(2) \otimes U(1)$ has four generators, we will be left with one unbroken symmetry, that we should identify with the group $U(1)$ of the electromagnetism ($U(1)_{\rm em}$), in such a way to have the corresponding gauge particle (the photon) massless. As we see from Fig. 2.2 the group $U(1)_{\rm em}$ must be a symmetry of the vacuum (saying that the vacuum should be electrically neutral). To realize this aim we have to introduce a set of scalar fields transforming in a convenient way under $SU(2) \otimes U(1)$. The simplest choice turns out to be a complex representation of $SU(2)$ of dimension 2 (Higgs doublet). As we already noticed the vacuum should be electrically neutral, so one of the components of the doublet must also be neutral. Assume that this component is the lower one, than we will write

$$\Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \tag{2.204}$$

To determine the weak hypercharge, Q^Y, of Φ, we use the relation

$$Q_{\text{em}} = Q^3 + \frac{1}{2}Q^Y \tag{2.205}$$

for the lower component of Φ. We get

$$Q^Y(\phi^0) = 2 \times \left[0 - \left(-\frac{1}{2}\right)\right] = 1 \tag{2.206}$$

Since Q^Y and Q^i commute, both the components of the doublet must have the same value of Q^Y, and using again eq. (2.205), we get

$$Q_{\text{em}}(\phi^+) = \frac{1}{2} + \frac{1}{2} = 1 \tag{2.207}$$

which justifies the notation ϕ^+ for the upper component of the doublet.

The most general lagrangian for Φ with the global symmetry $SU(2) \otimes U(1)$ and containing terms of dimension lower or equal to four (in order to have a renormalizable theory) is

$$\mathcal{L}_{\text{Higgs}} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2 \tag{2.208}$$

where $\lambda > 0$ and $\mu^2 < 0$. The potential has infinitely many minima on the surface

$$|\Phi|^2_{\text{minimum}} = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2} \tag{2.209}$$

with

$$v^2 = -\frac{\mu^2}{\lambda} \tag{2.210}$$

Let us choose the vacuum as the state

$$\langle 0|\Phi|0\rangle = \begin{bmatrix} 0 \\ v/\sqrt{2} \end{bmatrix} \tag{2.211}$$

By putting

$$\Phi = \begin{bmatrix} 0 \\ v/\sqrt{2} \end{bmatrix} + \begin{bmatrix} \phi^+ \\ (h + i\eta)/\sqrt{2} \end{bmatrix} \equiv \Phi_0 + \Phi' \tag{2.212}$$

with

$$\langle 0|\Phi'|0\rangle = 0 \tag{2.213}$$

we get

$$|\Phi|^2 = \frac{1}{2}v^2 + vh + |\phi^+|^2 + \frac{1}{2}(h^2 + \eta^2) \tag{2.214}$$

from which

$$V_{\text{Higgs}} = -\frac{1}{4}\frac{\mu^4}{\lambda} + \lambda v^2 h^2 + 2\lambda v h\left(|\phi^+|^2 + \frac{1}{2}(h^2+\eta^2)\right) + \lambda\left(|\phi^+|^2 + \frac{1}{2}(h^2+\eta^2)\right)^2 \tag{2.215}$$

From this expression we read immediately the particle masses ($\phi^- = (\phi^+)^\dagger$)

$$m_\eta^2 = m_{\phi^+}^2 = m_{\phi^-}^2 = 0 \tag{2.216}$$

and

$$m_h^2 = 2\lambda v^2 = -2\mu^2 \tag{2.217}$$

It is also convenient to express the parameters appearing in the original form of the potential in terms of m_h^2 e v ($\mu^2 = -2m_h^2$, $\lambda = 2m_h^2/v^2$). In this way we get

$$V_{\text{Higgs}} = -2m_h^2 + \frac{m_h^2}{2}h^2 + \frac{m_h^2}{v}h\left(|\phi^+|^2 + \frac{1}{2}(h^2+\eta^2)\right) + \frac{m_h^2}{2v^2}\left(|\phi^+|^2 + \frac{1}{2}(h^2+\eta^2)\right)^2 \tag{2.218}$$

Summarizing we have three massless Goldstone bosons $\phi^\pm$ and η, and a massive scalar h. This is called the Higgs field.

As usual, by now, we promote the global symmetry to a local one by introducing the covariant derivative. Recalling that $\Phi \in (\underline{2}, 1)$ of $SU(2)\otimes U(1)$, we have

$$D_\mu^H = \partial_\mu - i\frac{g}{2}\vec{\tau}\cdot\vec{W}_\mu - i\frac{g'}{2}Y_\mu \tag{2.219}$$

and

$$\mathcal{L}_{\text{Higgs}} = (D_\mu^H\Phi)^\dagger D^{H\mu}\Phi - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 \tag{2.220}$$

To study the mass generation it is convenient to write Φ in a way analogous to the one used in eq. (2.199)

$$\Phi = e^{i\vec{\xi}\cdot\vec{\tau}/v}\begin{bmatrix} 0 \\ (v+h)/\sqrt{2}\end{bmatrix} \tag{2.221}$$

According to our rules the exponential should contain the broken generators. In fact there are τ_1, and τ_2 which are broken. Furthermore it should contain the combination of 1 and τ_3 which is broken (remember that $(1+\tau_3)/2$ is the electric charge of the doublet Φ, and it is conserved). But, at any rate, τ_3 is a broken generator, so the previous expression gives a good representation of Φ around the vacuum. In fact, expanding around this state

$$\begin{aligned} e^{i\vec{\xi}\cdot\vec{\tau}/v}\begin{bmatrix} 0 \\ (v+h)/\sqrt{2}\end{bmatrix} &\approx \begin{bmatrix} 1+i\xi_3/v & i(\xi_1-i\xi_2)/v \\ i(\xi_1+i\xi_2)/v & 1-i\xi_3/v\end{bmatrix}\begin{bmatrix} 0 \\ (v+h)/\sqrt{2}\end{bmatrix} \\ &\approx \frac{1}{\sqrt{2}}\begin{bmatrix} i(\xi_1-i\xi_2) \\ (v+h-i\xi_3)\end{bmatrix} \end{aligned} \tag{2.222}$$

and introducing real components for Φ ($\phi^+ = (\phi_1 - i\phi_2)/\sqrt{2}$)

$$\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix} \phi_1 - i\phi_2 \\ v+h+i\eta\end{bmatrix} \tag{2.223}$$

we get the relation among the two sets of coordinates

$$(\phi_1, \phi_2, \eta, h) \approx (\xi_2, -\xi_1, -\xi_3, h) \tag{2.224}$$

By performing the gauge transformation

$$\Phi \to e^{-i\vec{\xi}\cdot\vec{\tau}/v}\Phi \tag{2.225}$$

$$\vec{W}_\mu \cdot \frac{\vec{\tau}}{2} \to e^{-i\vec{\xi}\cdot\vec{\tau}/v}\vec{W}_\mu \cdot \frac{\vec{\tau}}{2} e^{i\vec{\xi}\cdot\vec{\tau}/v} + \frac{i}{g}\left(\partial_\mu e^{-i\vec{\xi}\cdot\vec{\tau}/v}\right) e^{i\vec{\xi}\cdot\vec{\tau}/v} \tag{2.226}$$

$$Y_\mu \to Y_\mu \tag{2.227}$$

the Higgs lagrangian remains invariant in form, except for the substitution of Φ with $(0, (v+h)/\sqrt{2})$. We will also denote old and new gauge fields with the same symbol. Defining

$$\Phi_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \Phi_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.228}$$

the mass terms for the scalar fields can be read by substituting, in the kinetic term, Φ with its expectation value

$$\Phi \to \frac{v}{\sqrt{2}}\Phi_d \tag{2.229}$$

We get

$$\begin{aligned} -i\left(\frac{g}{2}\vec{\tau}\cdot\vec{W} + \frac{g'}{2}Y\right)\frac{v}{\sqrt{2}}\Phi_d &= -i\left(\frac{g}{\sqrt{2}}(\tau_- W^- + \tau_+ W^+) + \frac{g}{2}\tau_3 W^3 + \frac{g'}{2}Y\right)\frac{v}{\sqrt{2}}\Phi_d \\ &= -i\frac{v}{\sqrt{2}}\left[\frac{g}{\sqrt{2}}W^+\Phi_u - \frac{1}{2}(gW^3 - g'Y)\Phi_d\right] \end{aligned} \tag{2.230}$$

Since Φ_d and Φ_u are orthogonals, the mass term is

$$\frac{v^2}{2}\left[\frac{g^2}{2}|W^+|^2 + \frac{1}{4}(gW^3 - g'Y)^2\right] \tag{2.231}$$

From eq. (2.114) we have $\tan\theta = g'/g$, and therefore

$$\sin\theta = \frac{g'}{\sqrt{g^2 + g'^2}}, \qquad \cos\theta = \frac{g}{\sqrt{g^2 + g'^2}} \tag{2.232}$$

allowing us to write the mass term in the form

$$\frac{v^2}{2}\left[\frac{g^2}{2}|W^+|^2 + \frac{g^2 + g'^2}{4}(W^3\cos\theta - Y\sin\theta)^2\right] \tag{2.233}$$

From eq. (2.112) we see that the neutral fields combination is just the Z field, orthogonal to the photon. In this way we get

$$\frac{g^2 v^2}{4}|W^+|^2 + \frac{1}{2}\frac{g^2 + g'^2}{4}v^2 Z^2 \tag{2.234}$$

Finally the mass of the vector bosons is given by

$$M_W^2 = \frac{1}{4} g^2 v^2, \qquad M_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2 \tag{2.235}$$

Notice that the masses of the $W^\pm$ and of the Z are not independent, since their ratio is determined by the Weinberg angle, or from the gauge coupling constants

$$\frac{M_W^2}{M_Z^2} = \frac{g^2}{g^2 + g'^2} = \cos^2\theta \tag{2.236}$$

Let us now discuss the parameters that we have so far in the theory. The gauge interaction introduces the two gauge couplings g and g', which can also be expressed in terms of the electric charge (or the fine structure constant), and of the Weinberg angle. The Higgs sector brings in two additional parameters, the mass of the Higgs, m_h, and the expectation value of the field Φ, v. The Higgs particle has not been yet discovered, and at the moment we have only an experimental lower bound on m_h, from LEP, which is given by $m_h > 60\ GeV$. The parameter v can be expressed in terms of the Fermi coupling constant G_F. In fact, from eq. (2.3)

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2} \tag{2.237}$$

using the expression for M_W^2, we get

$$v^2 = \frac{1}{\sqrt{2} G_F} \approx (246\ GeV)^2 \tag{2.238}$$

Therefore, the three parameters g, g' and v can be traded for e, $\sin^2\theta$ and G_F. Another possibility is to use the mass of the Z

$$M_Z^2 = \frac{1}{4} \frac{1}{\sqrt{2} G_F} \frac{e^2}{\sin^2\theta \cos^2\theta} = \frac{\pi\alpha}{\sqrt{2} G_F \sin^2\theta \cos^2\theta} \tag{2.239}$$

to eliminate $\sin^2\theta$

$$\sin^2\theta = \frac{1}{2}\left[1 - \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2} G_F M_Z^2}}\,\right] \tag{2.240}$$

The first alternative has been the one used before LEP. However, after LEP1, the mass of the Z is very well known

$$M_Z = (91.1863 \pm 0.0020)\ GeV \tag{2.241}$$

Therefore, the parameters that are now used as input in the SM are α, G_F and M_Z.

The last problem we have to solve is how to give mass to the electron, since it is impossible to construct bilinear terms in the electron field which are invariant under the gauge group. The solution is that we can build up trilinear invariant terms in the electron field and in Φ. Once the field Φ acquires a non vanishing expectation value, due to the breaking of the symmetry, the trilinear term generates the electron mass. Recalling the behaviour of the fields under $SU(2) \otimes U(1)$

$$L \in (\underline{2}, -1), \qquad R \in (\underline{1}, -2), \qquad \Phi \in (\underline{2}, +1) \tag{2.242}$$

we see that the following coupling (Yukawian coupling) is invariant

$$\mathcal{L}_Y = g_e \bar{L} \Phi R + \text{h.c.} = g_e \left[(\bar{\psi}_\nu)_L, \quad (\bar{\psi}_e)_L \right] \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} (\psi_e)_R + \text{h.c.} \tag{2.243}$$

In the unitary gauge, which is obtained by using the transformation (2.225) both for Φ and for the lepton field L, we get

$$\mathcal{L}_Y = g_e \left[(\bar{\psi}_\nu)_L, \quad (\bar{\psi}_e)_L \right] \begin{bmatrix} 0 \\ (v+h)/\sqrt{2} \end{bmatrix} (\psi_e)_R + \text{h.c.} \tag{2.244}$$

from which

$$\mathcal{L}_Y = \frac{g_e v}{\sqrt{2}} (\bar{\psi}_e)_L (\psi_e)_R + \frac{g_e}{\sqrt{2}} (\bar{\psi}_e)_L (\psi_e)_R h + \text{h.c.} \tag{2.245}$$

Since

$$[(\bar{\psi}_e)_L (\psi_e)_R]^\dagger = (\bar{\psi}_e \frac{1+\gamma_5}{2} \psi_e)^\dagger = \psi_e^\dagger \frac{1+\gamma_5}{2} \gamma_0 \psi_e = \bar{\psi}_e \frac{1-\gamma_5}{2} \psi_e = (\bar{\psi}_e)_R (\psi_e)_L \tag{2.246}$$

we get

$$\mathcal{L}_Y = \frac{g_e v}{\sqrt{2}} \bar{\psi}_e \psi_e + \frac{g_e}{\sqrt{2}} \bar{\psi}_e \psi_e h \tag{2.247}$$

Therefore the symmetry breaking generates an electron mass given by

$$m_e = -\frac{g_e v}{\sqrt{2}} \tag{2.248}$$

In summary, we have been able to reproduce all the phenomenological features of the $V-A$ theory and its extension to neutral currents. This has been done with a theory that, before spontaneous symmetry breaking is renormalizable. In fact, also the Yukawian coupling has only dimension 3. The proof that the renormalizability of the theory holds also in the case of spontaneous symmetry breaking ($\mu^2 < 0$) is absolutely non trivial. In fact the proof was given only at the beginning of the seventies by 't Hooft [22].

2.9 The electroweak interactions of quarks and the Cabibbo-Kobayashi-Maskawa matrix

So far we have formulated the SM for an electron-neutrino pair, but it can be extended in a trivial way to other lepton pairs. The extension to the quarks is a little bit less obvious and we will follow the necessary steps in this Section. First of all we recall that in the quark model the nucleon is made up of three quarks, with composition, $p \approx uud$, $n \approx udd$. Then, the weak transition $n \to p + e^- + \bar{\nu}_e$ is obtained through

$$d \to u + e^- + \bar{\nu}_e \tag{2.249}$$

and the assumption is made that quarks are point-like particles as leptons. Therefore, it is natural to write the $\Delta S = 0$ hadronic current (see Section 1.3) in a form analogous to the leptonic current (see eqs. (2.84) and (2.86))

$$J^{\mu(\pm)}_{\Delta S=0} = 2\bar{\Psi}_L \gamma^\mu \tau_\mp \Psi_L \tag{2.250}$$

with

$$\Psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \frac{1-\gamma_5}{2} \begin{pmatrix} u \\ d \end{pmatrix} \tag{2.251}$$

Remember that for the nucleon we wrote

$$J^\mu_{\Delta S=0} = \bar{p}\gamma^\mu (c_V + c_A \gamma_5) n \tag{2.252}$$

This is interpreted saying that the axial-current coupling is renormalized due to the bound state effects, and therefore $c_A \approx -1.25$. On the other hand the vector current does not get renormalized due to its conservation, implying $c_V = 1$. This is the same as for the electric charge, which is +2/3 for the u quark, -1/3 for the d quark, and +1 for the proton. In a similar way we can write down the $\Delta S = 1$ part of the hadronic current

$$J^{\mu(\pm)}_{\Delta S=1} = 2\bar{\Psi}'_L \gamma^\mu \tau_\mp \Psi'_L \tag{2.253}$$

with

$$\Psi'_L = \begin{pmatrix} u_L \\ s_L \end{pmatrix} = \frac{1-\gamma_5}{2} \begin{pmatrix} u \\ s \end{pmatrix} \tag{2.254}$$

Recalling that the hadronic current (see eq. 1.61) has the form

$$J^\mu_h = J^\mu_{\Delta S=0} \cos\theta_C + J^\mu_{\Delta S=1} \sin\theta_C \tag{2.255}$$

where θ_C is the Cabibbo angle, and collecting together the two currents we get

$$J^{\mu(\pm)}_h = 2\bar{Q}_L \gamma^\mu \tau_\mp Q_L \tag{2.256}$$

with

$$Q_L = \begin{pmatrix} u_L \\ d_L \cos\theta_C + s_L \sin\theta_C \end{pmatrix} \equiv \begin{pmatrix} u_L \\ d^C_L \end{pmatrix} \tag{2.257}$$

and

$$d_L^C = d_L \cos\theta_C + s_L \sin\theta_C \tag{2.258}$$

This allows us to assign Q_L to the representation $(\underline{2}, 1/3)$ of $SU(2) \otimes U(1)$. In fact, from

$$Q_{\rm em} = Q^3 + \frac{1}{2} Q^Y \tag{2.259}$$

we get

$$Q^Y(u_L) = 2(\frac{2}{3} - \frac{1}{2}) = \frac{1}{3} \tag{2.260}$$

and $(Q_{\rm em}(s) = Q_{\rm em}(d) = -1/3)$

$$Q^Y(d_L^C) = 2(-\frac{1}{3} + \frac{1}{2}) = \frac{1}{3} \tag{2.261}$$

As far as the right-handed components are concerned, we assign them to $SU(2)$ singlets, obtaining

$$u_R \in (\underline{1}, \frac{4}{3}) \qquad d_R^C \in (\underline{1}, -\frac{2}{3}) \tag{2.262}$$

Therefore, the hadronic neutral currents contribution from quarks $u,\ d,\ s$ are given by

$$j_\mu^3 = \bar{Q}_L \gamma_\mu \frac{\tau_3}{2} Q_L \tag{2.263}$$

$$j_\mu^Y = \frac{1}{3} \bar{Q}_L \gamma_\mu Q_L + \frac{4}{3} \bar{u}_R \gamma_\mu u_R - \frac{2}{3} \bar{d}_R^C \gamma_\mu d_R^C \tag{2.264}$$

Since the Z is coupled to the combination $j_\mu^3 - \sin^2\theta j_\mu^Y$, it is easily seen that the coupling contains a bilinear term in the field d^C given by

$$-\frac{\cos^2\theta}{2} Z^\mu \bar{d}_L^C \gamma_\mu d_L^C + \frac{1}{3} \sin^2\theta Z^\mu \bar{d}_R^C \gamma_\mu d_R^C \tag{2.265}$$

This gives rise to terms of the type $\bar{d}s$ and $\bar{s}d$ which produce neutral current transitions with $\Delta S = 1$. But we saw in Section 2.1 that these transitions are strongly suppressed. In a more general way we can get this result from the simple observation that W_3 is coupled to the current j_μ^3 which has an associated charge Q^3 which can be obtained by commuting the charges Q^1 and Q^2

$$\begin{aligned} [Q^1, Q^2] &= \int d^3\vec{x} d^3\vec{y} [Q_L^\dagger \tau_1 Q_L, Q_L^\dagger \tau_2 Q_L] = \int d^3\vec{x}\, Q_L^\dagger [\tau_1, \tau_2] Q_L \\ &= i \int d^3\vec{x}\, Q_L^\dagger \tau_3 Q_L = i \int d^3\vec{x} (u_L^\dagger u_L - d_L^{C\dagger} d_L^C) \end{aligned} \tag{2.266}$$

The last term is just the charge associated to a Flavour Changing Neutral Current (FCNC). We can compare the strength of a FCNC transition, which from

$$d^{C\dagger} d^C = d^\dagger d \cos^2\theta_C + s^\dagger s \sin^2\theta_C + (d^\dagger s + s^\dagger d) \sin\theta_C \cos\theta_C \tag{2.267}$$

is proportional to $\sin\theta_C \cos\theta_C$, with the strength of a flavour changing charged current transition as $u \to s$, which is proportional to $\sin\theta_C$. From this comparison we expect the two transitions to be of the same order of magnitude. But if we compare $K^0 \to \mu^+\mu^-$ induced by the neutral current , with $K^+ \to \mu^+\nu_\mu$ induced by the charged current (see Fig. 2.3), using $BR(K^0 \to \mu^+\mu^-)$, we find

$$\frac{\Gamma(K^0 \to \mu^+\mu^-)}{\Gamma(K^+ \to \mu^+\nu_\mu)} \leq 6 \times 10^{-5} \tag{2.268}$$

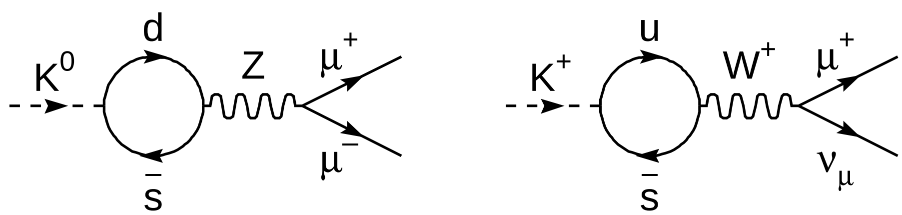

Fig. 2.3 - *The neutral current flavour changing process $K^0 \to \mu^+\mu^-$ and he charged current flavour changing process $K^+ \to \mu^+\nu_\mu$.*

This problem was solved in 1970 by the suggestion (Glashow Illiopoulos and Maiani, GIM) that another quark with charge +2/3 should exist, the charm [23]. Then, one can form two left-handed doublets

$$Q_L^1 = \begin{pmatrix} u_L \\ d_L^C \end{pmatrix}, \qquad Q_L^2 = \begin{pmatrix} c_L \\ s_L^C \end{pmatrix} \tag{2.269}$$

where

$$s_L^C = -d_L \sin\theta_C + s_L \cos\theta_C \tag{2.270}$$

is the combination orthogonal to d_L^C. As a consequence the expression of Q^3 gets modified

$$Q^3 = \int d^3\vec{x}\,(u_L^\dagger u_L + c_L^\dagger c_L - d_L^{C\dagger} d_L^C - s_L^{C\dagger} s_L^C) \tag{2.271}$$

But

$$d_L^{C\dagger} d_L^C + s_L^{C\dagger} s_L^C = d_L^\dagger d_L + s_L^\dagger s_L \tag{2.272}$$

since the transformation matrix

$$\begin{pmatrix} d_L^C \\ s_L^C \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d_L \\ s_L \end{pmatrix} \tag{2.273}$$

is orthogonal. The same considerations apply to Q^Y. This mechanism of cancellation turns out to be effective also at the second order in the interaction, as it should be, since we need a cancellation at least at the order G_F^2 in order to cope with the experimental bounds. In fact, one can see that the two second order graphs contributing to $K^0 \to \mu^+\mu^-$ cancel out except for terms coming from the mass difference between the quarks u and c. One gets

$$A(K^0 \to \mu^+\mu^-) \approx G_F^2(m_c^2 - m_u^2) \tag{2.274}$$

allowing to get a rough estimate of the mass of the charm quark, $m_c \approx 1 \div 3\ GeV$. The charm was discovered in 1974, when it was observed the bound state $\bar{c}c$, with $J^{PC} = 1^{--}$, known as J/ψ. The mass of charm was evaluated to be around $1.5\ GeV$. In 1977 there was the observation of a new vector resonance, the Υ, interpreted as a bound state $\bar{b}b$, where b is a new quark, the bottom or beauty, with $m_b \approx 5\ GeV$. Finally in 1995 the partner of the bottom, the top, t, was discovered at Fermi Lab. The mass of the top is around 175 GeV. We see that in association with three leptonic doublets

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \qquad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \qquad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \tag{2.275}$$

there are three quark doublets

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \qquad \begin{pmatrix} c \\ s \end{pmatrix}_L, \qquad \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{2.276}$$

We will show later that experimental evidences for the quark b to belong to a doublet came out from PEP and PETRA much before the discovery of top. Summarizing we have three generations of quarks and leptons. Each generation includes two left-handed doublets

$$\begin{pmatrix} \nu_A \\ e_A^- \end{pmatrix}_L, \qquad \begin{pmatrix} u_A \\ d_A \end{pmatrix}_L \qquad A = 1, 2, 3 \tag{2.277}$$

and the corresponding right-handed singlets. Inside each generation the total charge is equal to zero. In fact

$$Q_i^{\rm tot} = \sum_{f \in {\rm gen}} Q_f = 0 - 1 + 3 \times \left(\frac{2}{3} - \frac{1}{3}\right) = 0 \tag{2.278}$$

where we have also taken into account that each quark comes in three colors. This is very important because it guarantees the renormalizability of the SM. In fact, in general there are quantum corrections to the divergences of the currents destroying their conservation. However, the conservation of the currents coupled to the Yang-Mills fields is crucial for the renormalization properties. In the case of the SM for the electroweak interactions one can show that the condition for the absence of these corrections (Adler, Bell, Jackiw anomalies [24]) is that the total electric charge of the fields is zero.

We will complete now the formulation of the SM for the quark sector, showing that the mixing of different quarks arises naturally from the fact that the quark mass eigenstates are not necessarily the same fields which couple to the gauge fields. We will denote the last ones by

$$\ell'_{AL} = \begin{pmatrix} \nu'_A \\ e'_A \end{pmatrix}_L, \qquad q'_{AL} = \begin{pmatrix} u'_A \\ d'_A \end{pmatrix}_L, \qquad A = 1,2,3 \tag{2.279}$$

and the right-handed singlets by

$$e'_{AR}, \quad u'_{AR}, \quad d'_{AR} \tag{2.280}$$

We assume that the neutrinos are massless and that they do not have any right-handed partner. The gauge interaction is then

$$\begin{aligned} \mathcal{L} &= \sum_{f_L} \bar{f}_{AL} \left(i\partial_\mu + g\frac{\vec{\tau}}{2}\cdot \vec{W}_\mu + g'\frac{Q^Y(f_L)}{2} Y_\mu \right) \gamma^\mu f_{AL} \\ &+ \sum_{f_R} \bar{f}_{AR} \left(i\partial_\mu + g'\frac{Q^Y(f_R)}{2} \right) \gamma^\mu f_{AR} \end{aligned} \tag{2.281}$$

where $f_{AL} = \ell'_{AL},\ q'_{AL}$, and $f_{AR} = e'_{AR},\ u'_{AR},\ d'_{AR}$. We recall also the weak hypercharge assignments

$$\begin{aligned} &Q^Y(\ell'_{AL}) = -1, \qquad Q^Y(q'_{AL}) = \frac{1}{3} \\ &Q^Y(e'_{AR}) = -2, \qquad Q^Y(u'_{AR}) = \frac{4}{3} \\ &Q^Y(d'_{AR}) = -\frac{2}{3} \end{aligned} \tag{2.282}$$

Let us now see how to give mass to quarks. We start with up and down quarks, and a Yukawian coupling given by

$$g_d \bar{q}_L \Phi d_R + \text{h.c.} \tag{2.283}$$

where Φ is the Higgs doublet. When this takes its expectation value, $\langle \Phi \rangle = (0, v/\sqrt{2})$, a mass term for the down quark is generated

$$\frac{g_d v}{\sqrt{2}} \left(\bar{d}_L d_R + \bar{d}_R d_L \right) = \frac{g_d v}{\sqrt{2}} \bar{d} d \tag{2.284}$$

corresponding to a mass $m_d = -g_d v/\sqrt{2}$. In order to give mass to the up quark we need to introduce the conjugated Higgs doublet defined as

$$\tilde{\Phi} = i\tau_2 \Phi^\star = \begin{pmatrix} (\phi^0)^\star \\ -(\phi^+)^\star \end{pmatrix} \tag{2.285}$$

Observing that $Q^Y(\tilde{\Phi}) = -1$, another invariant Yukawian coupling is given by

$$g_u \bar{q}_L \tilde{\Phi} u_R + \text{h.c.} \tag{2.286}$$

giving mass to the up quark, $m_u = -g_u v/\sqrt{2}$. The most general Yukawian coupling among quarks and Higgs field is then given by

$$\mathcal{L}_Y = \sum_{AB} \left(g^e_{AB} \bar{\ell}'_{AL} \Phi e'_{BR} + g^u_{AB} \bar{q}'_{AL} \tilde{\Phi} u'_{BR} + g^d_{AB} \bar{q}'_{AL} \Phi d'_{BR} \right) \tag{2.287}$$

where g^i_{AB}, with $i = e,\ u,\ d$, are arbitrary 3×3 matrices. When we shift Φ by its vacuum expectation value we get a mass term for the fermions given by

$$\mathcal{L}_{\text{fermion mass}} = \frac{v}{\sqrt{2}} \sum_{AB} \left(g^e_{AB} \bar{\ell}'_{AL} e'_{BR} + g^u_{AB} \bar{u}'_{AL} u'_{BR} + g^d_{AB} \bar{d}'_{AL} d'_{BR} \right) \tag{2.288}$$

Therefore we obtain three 3×3 mass matrices

$$M^i_{AB} = -\frac{v}{\sqrt{2}} g^i_{AB}, \quad i = e,\ u,\ d \tag{2.289}$$

These matrices give mass respectively to the charged leptons and to the up and down of quarks. They can be diagonalized by a biunitary transformation

$$M^i_d = S^\dagger M^i T \tag{2.290}$$

where S and T are unitary matrices (depending on i) and M^i_d are three diagonal matrices. Furthermore one can take the eigenvalues to be positive. This follows from the polar decomposition of an arbitrary matrix

$$M = HU \tag{2.291}$$

where $H^\dagger = H = \sqrt{MM^\dagger}$ is a hermitian definite positive matrix and $U^\dagger = U^{-1}$ is a unitary matrix. Therefore, if the unitary matrix S diagonalizes H we have

$$H^i_d = S^\dagger H S = S^\dagger M U^{-1} S = S^\dagger M T \tag{2.292}$$

with $T = U^{-1} S$ is a unitary matrix. Each of the mass terms has the structure $\bar{\psi}'_L M \psi'_R$. Then we get

$$\bar{\psi}'_L M \psi'_R = \bar{\psi}'_L S (S^\dagger M T) T^\dagger \psi'_R \equiv \bar{\psi}_L M_d \psi_R \tag{2.293}$$

where we have defined the mass eigenstates

$$\psi'_L = S \psi_L, \qquad \psi'_R = T \psi_R \tag{2.294}$$

We can now express the charged current in terms of the mass eigenstates. We have

$$J^{+(h)}_\mu = 2 \bar{q}'_{AL} \gamma_\mu \tau_- q'_{AL} = 2 \bar{u}'_{AL} \gamma_\mu d'_{AL} = 2 \bar{u}_{AL} \gamma_\mu ((S^u)^{-1} S^d)_{AB} d_{BL} \tag{2.295}$$

Defining

$$V = (S^u)^{-1} S^d, \qquad V^\dagger = V^{-1} \tag{2.296}$$

we get

$$J_\mu^{h(+)} = 2\bar{u}_{AL}\gamma_\mu d_{AL}^{(V)} \tag{2.297}$$

with

$$d_{AL}^{(V)} = V_{AB} d_{BL} \tag{2.298}$$

The matrix V is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix [25], and it describes the mixing among the down type of quarks (this is conventional we could have chosen to mix the up quarks as well). We see that its physical origin is that, in general, there are no relations between the mass matrix of the up quarks, M^u, and the mass matrix of the down quarks, M^d. The neutral currents are expressions which are diagonal in the "primed" states, therefore their expression is the same also in the basis of the mass eigenstates, due to the unitarity of the various S matrices. In fact

$$j_\mu^{h(3)} \approx \bar{u}'_L u'_L - \bar{d}'_L d'_L = \bar{u}_L (S^u)^{-1} S^u u_L - \bar{d}_L (S^d)^{-1} S^d d_L = \bar{u}_L u_L - \bar{d}_L d_L \tag{2.299}$$

For the charged leptonic current, since we have assumed massless neutrinos we get

$$J_\mu^{\ell(+)} = 2\bar{\nu}_{AL}\gamma_\mu (S^e)_{AB} e_{BL} \equiv 2\bar{\nu}'_{AL}\gamma_\mu e_{AL} \tag{2.300}$$

where $\nu'_L = (S^e)^\dagger \nu_L$ is again a massless eigenstate. Therefore, there is no mixing in the leptonic sector, among different generations. However this is tied to our assumption of massless neutrinos. By relaxing this assumption we may generate a mixing in the leptonic sector producing a violation of the different leptonic numbers.

It is interesting to discuss in a more detailed way the structure of the CKM matrix. In the case we have n generations of quarks and leptons, the matrix V, being unitary, depends on n^2 parameters. However one is free to choose in an arbitrary way the phase for the $2n$ up and down quark fields. But an overall phase does not change V. Therefore the CKM matrix depends only on

$$n^2 - (2n-1) = (n-1)^2 \tag{2.301}$$

parameters. We can see that, in general, it is impossible to choose the phases in such a way to make V real. In fact a real unitary matrix is nothing but an orthogonal matrix which depends on

$$\frac{n(n-1)}{2} \tag{2.302}$$

real parameters. This means that V will depend on a number of phases given by

$$\text{number of phases} = (n-1)^2 - \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2} \tag{2.303}$$

Then, in the case of two generations V depends only on one real parameter (the Cabibbo angle). For three generations V depends on three real parameters and one phase. Since the invariance under the discrete symmetry CP implies that all the couplings in the lagrangian must be real, it follows that the SM, in the case of three generations, implies a CP violation. A violation of CP has been observed experimentally by Christensen et al. in 1964 [26]. These authors discovered that the eigenstate of CP, with $CP = -1$

$$|K_2^0\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \tag{2.304}$$

decays into a two pion state with $CP = +1$ with a BR

$$BR(K_2^0 \to \pi^+\pi^-) \approx 2 \times 10^{-3} \tag{2.305}$$

It is not clear yet, if the SM is able to explain the observed CP violation in quantitative terms.

To complete this Section we give the experimental values for some of the matrix elements of V. The elements V_{ud} and V_{us} are determined through nuclear β-, $K-$ and hyperon-decays, using the muon decay as normalization. One gets

$$|V_{ud}| = 0.9736 \pm 0.0010 \tag{2.306}$$

and

$$|V_{us}| = 0.2205 \pm 0.0018 \tag{2.307}$$

The elements V_{cd} and V_{cs} are determined from charm production in deep inelastic scattering $\nu_\mu + N \to \mu + c + X$. The result is

$$|V_{cd}| = 0.224 \pm 0.016 \tag{2.308}$$

and

$$|V_{cs}| = 1.01 \pm 0.18 \tag{2.309}$$

The elements V_{ub} and V_{cb} are determined from the $b \to u$ and $b \to c$ semi-leptonic decays as evaluated in the spectator model, and from the B semi-leptonic exclusive decay $B \to \bar{D}^\star \ell \nu_\ell$. One gets

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.08 \pm 0.02 \tag{2.310}$$

and

$$|V_{cb}| = 0.041 \pm 0.003 \tag{2.311}$$

More recently from the branching ratio $t \to Wb$, CDF (1996) has measured $|V_{tb}|$

$$|V_{tb}| = 0.97 \pm 0.15 \pm 0.07 \tag{2.312}$$

A more recent comprehensive analysis made by the Particle Data Group [27] gives the following 90 % C.L. range for the various elements of the CKM matrix

$$V = \begin{pmatrix} 0.9745 - 0.9757 & 0.219 - 0.224 & 0.002 - 0.005 \\ 0.218 - 0.224 & 0.9736 - 0.9750 & 0.036 - 0.046 \\ 0.004 - 0.014 & 0.034 - 0.046 & 0.9989 - 0.9993 \end{pmatrix} \tag{2.313}$$

2.10 The parameters of the SM

It is now the moment to count the parameters of the SM. We start with the gauge sector, where we have the two gauge coupling constants g and g'. Then, the Higgs sector is specified by the parameters μ and the self-coupling λ. We will rather use $v = \sqrt{-\mu^2/\lambda}$ and $m_h^2 = -2\mu^2$. Then we have the Yukawian sector, that is that part of the interaction between fermions and Higgs field giving rise to the quark and lepton masses. If we assume three generations and that the neutrinos are all massless we have three mass parameters for the charged leptons, six mass parameters for the quarks (assuming the color symmetry) and four mixing angles (see Section 2.9). In order to test the structure of the SM the important parameters are those relative to the gauge and to the Higgs sectors. As a consequence one assumes that the mass matrix for the fermions is known. This was not the case till two years ago, before the top discovery, and its mass was unknown. The masses of the vector bosons can be expressed in terms of the previous parameters. However, the question arises about the better choice of the input parameters. Before LEP1, the better choice was to use quantities known with great precision related to the parameters (g, g', v). The choice was to use the fine structure constant

$$\alpha = \frac{1}{137.0359895(61)} \tag{2.314}$$

and the Fermi constant

$$G_F = 1.166389(22) \times 10^{-5}\ GeV^{-2} \tag{2.315}$$

as measured from the β-decay of the muon. Then, most of the experimental research of the seventies and eighties was centered about the determination of $\sin^2\theta$. The other parameters as (m_t, m_h) affect only radiative corrections, which, in this type of experiments can be safely neglected due to the relatively large experimental errors. Therefore, in order to relate the set of parameters $(\alpha, G_F, \sin^2\theta)$ to (g, g', v) we can use the tree level relations. The situation has changed a lot after the beginning of running of LEP1. In fact, the mass of the Z has been measured with great precision, $\delta M_Z/M_Z \approx 2 \times 10^{-5}$. In this case the most convenient set is (α, G_F, M_Z), with

$$M_Z = (91.1863 \pm 0.0020)\ GeV \tag{2.316}$$

Also the great accuracy of the experimental measurements requires to take into account the radiative corrections. We will discuss this point in more detail at the end of these lectures. However we notice that one gets informations about parameters as (m_t, m_h) just because they affect the radiative corrections. In particular, the radiative corrections are particularly sensitive to the top mass. In fact, LEP1 was able to determine the top mass, in this indirect way, before the top discovery.

Chapter 3

The phenomenology of the SM in the low-energy limit

3.1 The low energy limit of the SM

Let us recall from Section 2.4 that the interaction of the fermions with the gauge fields is given by

$$\mathcal{L}_{\rm int} = g j^i_\mu W^\mu_i + \frac{g'}{2} j^Y_\mu Y^\mu \tag{3.1}$$

or, in terms of the vector boson mass eigenstates

$$\mathcal{L}_{\rm int} = g(j^1_\mu W^\mu_1 + j^2_\mu W^\mu_2) + \frac{g}{\cos\theta} j^Z_\mu Z^\mu + e j^{\rm em}_\mu A^\mu \tag{3.2}$$

where

$$j^Z_\mu = j^3_\mu - \sin^2\theta j^{\rm em}_\mu \tag{3.3}$$

and

$$j^{\rm em}_\mu = j^3_\mu + \frac{1}{2} j^Y_\mu \tag{3.4}$$

In the formal limit $M_W,\ M_Z \to \infty$, corresponding to processes with typical energies much below the mass of the massive vector bosons, one gets an effective current$\times$current interaction at the second order in the gauge couplings g and g'

$$\mathcal{L}^{\rm eff} \approx \frac{1}{2}\int \mathcal{L}_{\rm int} \otimes \mathcal{L}_{\rm int} \tag{3.5}$$

In the limit one can substitute to the vector boson propagator a δ-function (see eq. (2.28)) with the result

$$\mathcal{L}^{\rm eff} = \frac{g^2}{2M_W^2} \sum_{i=1}^{2} j^i_\mu j^{i\mu} + \frac{g^2}{2\cos^2\theta M_Z^2} j^Z_\mu j^{Z\mu} + e j^{\rm em}_\mu A^\mu \tag{3.6}$$

(notice that we have not eliminated, or integrated out, the photon field). The identification of the Fermi constant can be made by noticing that the charged currents defined in eqs. (2.84) and (2.86), are related to the currents j_μ^i, $i = 1, 2$, by

$$J_\mu^{(\pm)} = 2(j_\mu^1 \mp i j_\mu^2) \tag{3.7}$$

We get, for the charged effective interaction

$$\mathcal{L}_{\text{eff}}^{\text{charged}} = \frac{g^2}{8M_W^2} J_\mu^{(+)} J^{(-)\mu} \tag{3.8}$$

from which

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{3.9}$$

The expression for $\mathcal{L}_{\text{eff}}$ is generally written in the form

$$\mathcal{L}_{\text{eff}} = 4\frac{G_F}{\sqrt{2}} \left(j_\mu^{(+)} j^{(-)\mu} + \rho j_\mu^Z j^{Z\mu} \right) + e j_\mu^{\text{em}} A^\mu \tag{3.10}$$

where, in the case of the SM one has

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta} = 1 \tag{3.11}$$

This relation follows from the choice of the Higgs field as an $SU(2)$ doublet. if the Higgs field is assigned to a representation of spin t of $SU(2)$ ($\vec{T}^2 = t(t+1)$), then one has

$$\begin{aligned} M_W^2 &= \frac{g^2}{2}\left[t(t+1) - t_3^2\right] v^2 \\ M_Z^2 &= (g^2 + g'^2) t_3^2 v^2 \end{aligned} \tag{3.12}$$

This result can be easily extended to the case of several Higgs multiplets. In any case, if all the Higgs fields are chosen to be doublets, then the relation $\rho = 1$ follows. Therefore the low-energy study of the neutral couplings allows us to test the representation to which the Higgs field belongs to, as well as to measure the value of $\sin^2\theta$, as it follows from the expression of j_μ^Z. On the other hand, the charged interaction gives us only the measure of the Fermi constant G_F.

In the case of the SM, from eq. 3.3 we get

$$\mathcal{L}_{\text{eff}} = 4\frac{G_F}{\sqrt{2}} \left((j_\mu^1)^2 + (j_\mu^2)^2 + (j_\mu^3 - \sin^2\theta j_\mu^{\text{em}})^2 \right) + e j_\mu^{\text{em}} A^\mu \tag{3.13}$$

showing, for $\theta \to 0$, a symmetry $SU(2)$. This symmetry is called the custodial $SU(2)$.

From the expression of j_μ^Z we can write

$$j_\mu^Z = \sum_f \left(\bar{f}_L g_L^f \gamma_\mu f_L + \bar{f}_R g_R^f \gamma_\mu f_R \right) \tag{3.14}$$

where

$$\begin{aligned} g_L^f &= T_3^f - \sin^2\theta\, Q_{\rm em}^f \\ g_R^f &= -\sin^2\theta\, Q_{\rm em}^f \end{aligned} \tag{3.15}$$

Another parameterization is in terms of Dirac spinors. Recalling that

$$f_{L,R} = \frac{1 \mp \gamma_5}{2} f \tag{3.16}$$

we get

$$j_\mu^Z = \frac{1}{2} \sum_f \bar{f} \left(\gamma_\mu g_V^f - \gamma_\mu \gamma_5 g_A^f \right) f \tag{3.17}$$

where

$$\begin{aligned} g_V^f &= g_L^f + g_R^f = T_3^f - 2\sin^2\theta\, Q_{\rm em}^f \\ g_A^f &= g_L^f - g_R^f = T_3^f \end{aligned} \tag{3.18}$$

3.2 Electron-neutrino scattering

The electron-neutrino scattering is an ideal process to test the structure of the neutral currents due to the fact that it involves only point-like objects (as far as we know, that is up to LEP energies). However it is a very difficult experiment because the cross-sections are very tiny. Since we will consider this process at low energy, it can be described as a current$\times$current interaction. Therefore the cross-section should be proportional to G_F^2, and taking into account the dimensionality and the Lorentz invariance of the cross-section we expect

$$\sigma \approx s\, G_F^2 \tag{3.19}$$

where $s = (p_1 + p_2)^2$ is the usual Mandelstam invariant. In the laboratory frame, if $p_1 = (E_\nu, \vec{p}_\nu)$ is the neutrino four-momentum, and $p_2 = (m_T, \vec{0})$ is the target four-momentum, we get

$$\sigma \approx 2 m_T E_\nu G_F^2 \tag{3.20}$$

Therefore, the typical νe cross-section, for neutrinos with energy of 1 GeV, is of the order of

$$\frac{\sigma}{E_\nu} \approx m_e G_F^2 = 2.7 \times 10^{-41} {\rm cm}^2 GeV^{-1} = 2.7 \times 10^{-8} {\rm nbarn} GeV^{-1} \tag{3.21}$$

For comparison, remember that the typical electromagnetic cross-section, as for $e^+e^- \to \mu^+\mu^-$, is of the order of

$$\sigma \approx \frac{87}{s(GeV^2)} \text{nbarn} \tag{3.22}$$

We see that the difficulty of the experiment is in its very low statistics. In the case of neutrino-nucleon scattering, one gains a factor of about 2000 in the cross-section, but the theoretical interpretation is much more complex (see later). The cleanest processes are $\nu_\mu + e^- \to \nu_\mu + e^-$, or $\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$, because they are pure neutral current processes. In fact they are generated by the Z exchange. For the experiment one needs beams of neutrinos. These are produced by letting decay the pions and the kaons produced from high-energy proton scattering on a fixed target. The typical decays are $\pi \to \mu + \nu_\mu$, $K \to \mu + \nu_\mu$, $K \to \pi^0 \mu\nu_\mu$, etc. After their production, pions and kaons are focused and let decay inside a vacuum tunnel. After that the muons are absorbed. For the experiment one does not need to know the energy of the incoming neutrino, if one measures energy and momentum of the outgoing electron. In fact the unknown quantities of the problem are the energy of the incoming neutrino (the direction is known), and the momentum of the outgoing neutrino. Since the scattering happens on a plane we need three relations to determine these variables. The relations are the energy conservation and the conservation of the spatial momentum on the scattering plane. In principle there is a problem with the background due to the neutrino nucleon interactions. However this can be eliminated by noticing that the outgoing electrons are strongly peaked in the forward direction. In fact, one can simply show that the electron scattering angle is given by

$$\sin^2 \frac{\phi}{2} = \frac{m_e}{E}\left(1 - \frac{E}{E_\nu}\right) \tag{3.23}$$

where E is the energy of the outgoing electron. Since $m_e \ll E$, one has $\phi \ll 1$. By the same argument the neutrino nucleon scattering has a flat distribution and the background can be easily subtracted. In 1973 in the bubble chamber Gargamelle, at CERN, the first event $\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$ was observed. This was the discovery of the neutral currents. In a period of two-years a total of three such events were observed in 1.4 million pictures (with $\approx 10^9$ antineutrinos per pulse). After 7 years of observations in six experiments, only about 100 events were found.

By a standard calculation one gets the relevant cross-sections

$$\sigma(\nu_\mu e) = \frac{2m_e G_F^2}{\pi} E_\nu \left[(g_L^e)^2 + \frac{1}{3}(g_R^e)^2\right] \tag{3.24}$$

and

$$\sigma(\bar{\nu}_\mu e) = \frac{2m_e G_F^2}{\pi} E_{\bar{\nu}} \left[\frac{1}{3}(g_L^e)^2 + (g_R^e)^2\right] \tag{3.25}$$

or, in terms of g_V^e and g_A^e

$$\sigma(\nu_\mu e) = \frac{2m_e G_F^2}{3\pi} E_\nu \left[{g_V^e}^2 + g_V^e g_A^e + {g_A^e}^2\right] \tag{3.26}$$

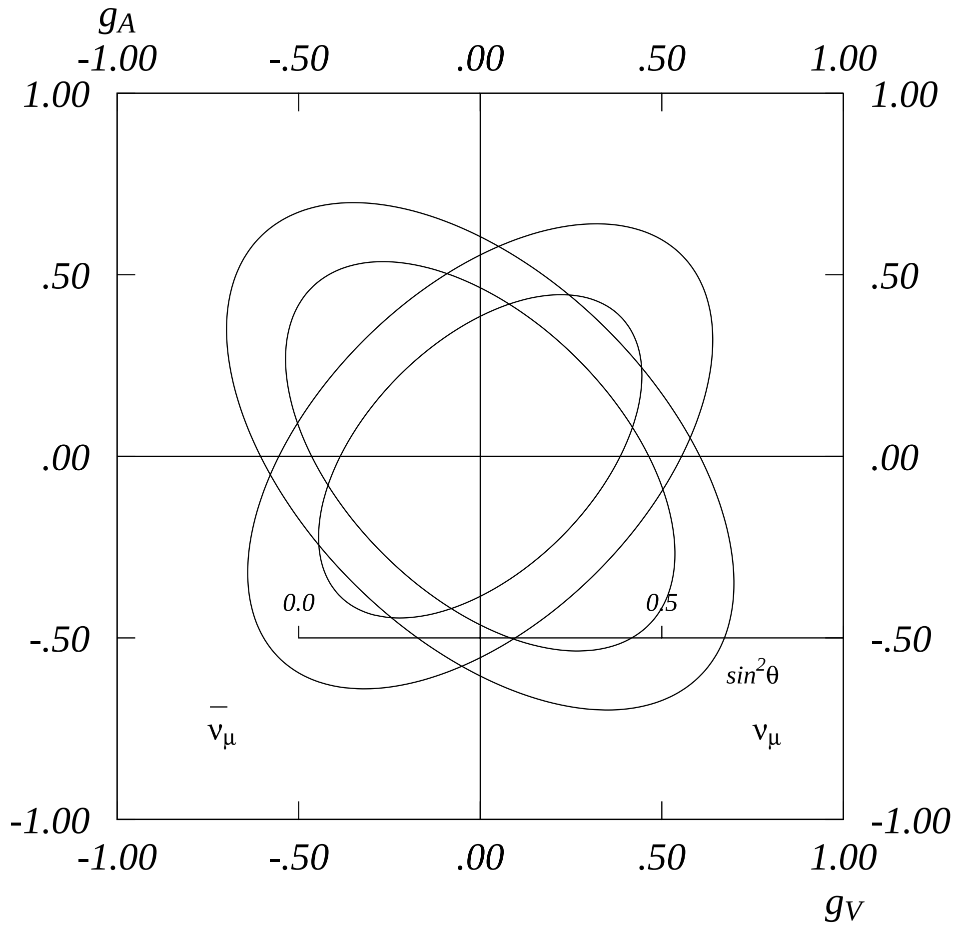

Fig. 3.1 - *68% C.L. contours in the plane (g_V, g_A) from the elestic scattering processes $\nu_\mu e$ and $\bar{\nu}_\mu e$.*

and

$$\sigma(\bar{\nu}_\mu e) = \frac{2m_e G_F^2}{3\pi} E_{\bar{\nu}} \left[g_V^{e\,2} - g_V^e g_A^e + g_A^{e\,2} \right] \tag{3.27}$$

In this calculation we have assumed that the coupling of the neutrino to the Z is the one of the SM, that is

$$g_V^\nu = g_A^\nu = \frac{1}{2} \tag{3.28}$$

If we do the further assumption that the couplings of the electron are the ones of the SM, that is

$$g_V^e = -\frac{1}{2} + 2\sin^2\theta, \qquad g_A^e = -\frac{1}{2} \tag{3.29}$$

we can fit the value of $\sin^2\theta$. For instance, in [28] the result was

$$\sin^2\theta = 0.231 \pm 0.023 \tag{3.30}$$

If one leaves the parameter ρ arbitrary (see the Section 3.1), one can fit both ρ and $\sin^2\theta$

$$\sin^2\theta = 0.231 \pm 0.024, \qquad \rho = 0.989 \pm 0.052 \tag{3.31}$$

If one leaves the couplings of the electrons arbitrary, the measure of the two cross-sections gives a four-fold ambiguity, see Fig. 3.1. This can be partially removed by using the reaction $\bar{\nu}_e + e \to \bar{\nu}_e + e$. This was studied by Reines in 1976 by antineutrinos produced in a reactor. The process involves also the charged current. However one still remains with a two-fold ambiguity. The situation was completely settled down at SLAC in 1978 using the scattering $eD \to eD$ with polarized electrons. Notice that by using the SM one has the relation

$$\frac{\sigma(\nu_\mu e \to \nu_\mu e)}{\sigma(\bar{\nu}_\mu e \to \bar{\nu}_\mu e)} = 3\frac{1 - 4\sin^2\theta + \frac{16}{3}\sin^4\theta}{1 - 4\sin^2\theta + 16\sin^4\theta} \tag{3.32}$$

The advantage is that the systematic errors (as those coming from the neutrino flux) cancel in the ratio. Furthermore, the error in $\sin^2\theta$ is strongly reduced

$$\delta\sin^2\theta \approx \frac{1}{8}\frac{\delta R}{R} \tag{3.33}$$

One of the last evaluations of $\sin^2\theta$ by these methods was done by CHARM-II at CERN in 1991. Using about 1300 events from $\nu_\mu e$ and about 1500 from $\bar{\nu}_\mu e$ they got

$$\sin^2\theta = 0.237 \pm 0.009 \text{ (stat.)} \pm 0.007 \text{ (syst.)} \tag{3.34}$$

3.3 Neutrino-nucleon scattering

We have already said that the cross-section in the neutrino reactions depend on the mass of the target. In particular, the cross-section of νN processes is about 2000 times bigger than the cross-section of the νe reactions. However, the elementary process takes place at the level of quarks, so one has to describe the nucleon as a bound state of point-like objects. Due to the non-perturbative nature of the problem its theoretical description is much less clear than the one we have in pure leptonic processes. We recall here very briefly how one deals with this problem in the context of the parton model. The nucleon is thought of as a composite object made of quarks and gluons (the partons). The typical reaction one considers is the so-called deep-inelastic scattering, where an ingoing lepton (electron or neutrino) exchange a vector boson (γ, W or Z) of high momentum with the nucleon. The main idea is that the interaction time of the vector boson with a single parton is much bigger than the typical interaction time of the hadronic interaction, or that the momentum of the vector boson is much higher than the typical scale of the strong interactions ($\Lambda_{QCD} \approx 200\ MeV$). In this approximation we can treat the partons as free objects. In the phenomenological description of these processes there are various problems. For instance, the momentum and the energy of the target (the parton inside the nucleon) are not known. Then one introduces the momentum distribution of the parton i inside the nucleon, N

$$q_i^N(x)dx \tag{3.35}$$

the probability that the parton i has a fraction of the momentum of the nucleon, N, between x and $x + dx$. Also the flavour of the target is not known. In fact, when including QCD corrections, one has to take into account the possibility that a quark inside the nucleon emits a gluon producing a quark-antiquark pair. In this case the vector boson may interact with the antiquark. This can be taken into account by introducing also a density of antiquarks inside the nucleon. For simplicity we will neglect the presence of antiquarks in our discussion. In this hypothesis the charged current, in a process like $\nu N \to \mu X$, will interact necessarily with quarks of type d. However, the neutral current, in processes like $\nu N \to \nu X$, will interact with quarks of type d and u. It is then convenient to consider isoscalar targets, that is targets having the same number of protons and neutrons. From isospin invariance it is easily found that

$$q_u^p(x) = q_d^n(x) \equiv u(x), \qquad q_d^p(x) = q_u^n(x) \equiv d(x) \tag{3.36}$$

Therefore for isoscalar targets the average structure function observed both from charged and neutral currents is

$$F(x) = \frac{1}{2}\left(u(x) + d(x)\right) \tag{3.37}$$

Using this description one can evaluate the double differential cross-sections for charged and neutral currents

$$\frac{d\sigma^{CC}(\nu_\mu N \to \mu^- X)}{dxdy} = \frac{2G_F^2 M E_\nu}{\pi} x F(x) \tag{3.38}$$

and

$$\frac{d\sigma^{NC}(\nu_\mu N \to \nu_\mu X)}{dxdy} = \frac{2G_F^2 M E_\nu}{\pi} x F(x) \left[({g_L^u}^2 + {g_L^d}^2) + ({g_R^u}^2 + {g_L^u}^2)(1-y)^2\right] \tag{3.39}$$

where M is the nucleon mass and y is the ratio of the energy of the final lepton over the energy of the initial one. Integrating over x and y and taking the ratio the dependence on $F(x)$ cancels out and we get

$$R_\nu = \frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} = ({g_L^u}^2 + {g_L^d}^2) + \frac{1}{3}({g_R^u}^2 + {g_L^u}^2) \tag{3.40}$$

In analogous way, by considering the reactions by antineutrinos one gets

$$R_{\bar\nu} = \frac{\sigma^{NC}(\bar\nu)}{\sigma^{CC}(\bar\nu)} = ({g_L^u}^2 + {g_L^d}^2) + 3({g_R^u}^2 + {g_L^u}^2) \tag{3.41}$$

Of course these equations do not allow us to determine the various couplings, but within the SM we get

$$R_\nu = \frac{1}{2} - \sin^2\theta + \frac{20}{27}\sin^4\theta \tag{3.42}$$

and

$$R_{\bar{\nu}} = \frac{1}{2} - \sin^2\theta + \frac{20}{9}\sin^4\theta \tag{3.43}$$

These two measures are not completely equivalent, in fact R_ν is more sensitive to $\sin^2\theta$ than $R_{\bar{\nu}}$. In fact, for $\sin^2\theta \approx 1/4$, one gets

$$\delta R_\nu \approx -\frac{17}{27}\delta\sin^2\theta, \qquad \delta R_{\bar{\nu}} \approx \frac{1}{9}\delta\sin^2\theta \tag{3.44}$$

Also experiments had a better statistics for the neutrino reactions, since the available flux of neutrinos was about 5 times higher then the one of antineutrinos. A very clean theoretical observable is the Paschos-Wolfenstein ratio [29]

$$\frac{\sigma^{NC}(\nu) - \sigma^{NC}(\bar{\nu})}{\sigma^{CC}(\nu) - \sigma^{CC}(\bar{\nu})} = \frac{1}{2} - \sin^2\theta \tag{3.45}$$

The reason is that in the differences the antiquarks structure functions contribution cancels out. Unfortunately a good experimental determination would require a flux of antineutrinos comparable to the one of neutrinos. In any case the combined results of these measures gave (see, for instance, [30])

$$\sin^2\theta = 0.233 \pm 0.006 \tag{3.46}$$

with an error which is about half of the one obtained in the νe scattering (see eq. (3.34)). These measures were mostly performed at CERN and FNAL by various collaborations.

3.4 e^+e^- scattering

Before the experimentation in e^+e^- at the Z-peak, as done at LEP and SLC, the neutral current has been studied in this process by using the interference between the electromagnetic and the weak amplitudes (the one due to the Z exchange) which contribute to $e^+e^- \to \bar{f}f$, as illustrated in Fig. 3.2 [32]. This interference was studied at various laboratories as SLAC (Stanford), with a total energy of $E \approx 30\ GeV$, DESY (Hamburg) with $E \approx 47\ GeV$, and KEK (Japan) with $E \approx 55\ GeV$. Here we will consider only processes with final fermions different from electrons, in order to avoid the complications due to two more diagrams corresponding to the γ and the Z exchange in the crossed channel (t-channel). At energies $E \ll M_Z$, and recalling that $e = g\sin\theta$, we have roughly that the electromagnetic and the weak amplitudes behave as

$$\mathcal{M}_{QED} \approx \frac{e^2}{s}, \qquad \mathcal{M}_{WEAK} \approx \frac{e^2}{M_Z^2} \tag{3.47}$$

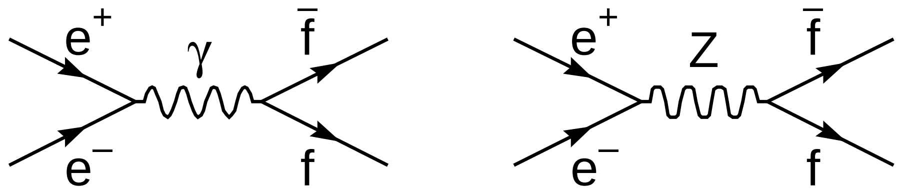

Fig. 3.2 -*The electromagnetic and the weak contributions to the scattering* $e^+e^- \to \bar{f}f$ *(*$f \neq e$*).*

Therefore the comparison of the interference contribution to the cross-section to the pure electromagnetic one gives

$$\frac{\mathcal{M}_{QED}\mathcal{M}_{WEAK}}{\mathcal{M}^2_{QED}} \approx \frac{s}{M_Z^2} \tag{3.48}$$

whereas for the pure weak contribution we get

$$\frac{\mathcal{M}^2_{WEAK}}{\mathcal{M}^2_{QED}} \approx \left(\frac{s}{M_Z^2}\right)^2 \tag{3.49}$$

Therefore, at the energies we have listed above, the interference can vary between $10 \div 35\%$. On the other hand the pure weak contribution is practically observable only at the Z resonance.

It is convenient to consider the cross-section for polarized fermions. Since we have conservation of helicity at high energies (where we can neglect the fermion masses) it turns out that there are only 4 independent cross-sections. For instance, if we specify the initial electron to be left-handed, the cross-section is different from zero only if the initial positron is right-handed, and so on. One has, in the center of mass reference frame

$$\left.\frac{d\sigma_{ij}}{d\Omega}\right|_{\rm cm} = \frac{1}{64\pi^2 s}|\mathcal{M}_{ij}|^2, \qquad i,j = L,R \tag{3.50}$$

where the indices i,j refer to the helicity of the initial electron and of the final fermion respectively, and

$$|\mathcal{M}_{LL}|^2 \;\; = \;\; 4e^4|\epsilon_{LL}|^2\left(\frac{u}{s}\right)^2$$

$$
\begin{aligned}
|\mathcal{M}_{RR}|^2 &= 4e^4|\epsilon_{RR}|^2\left(\frac{u}{s}\right)^2 \\
|\mathcal{M}_{LR}|^2 &= 4e^4|\epsilon_{LR}|^2\left(\frac{t}{s}\right)^2 \\
|\mathcal{M}_{RL}|^2 &= 4e^4|\epsilon_{RL}|^2\left(\frac{t}{s}\right)^2
\end{aligned} \tag{3.51}
$$

where s, t and u are the usual Mandelstam invariant variables. The ϵ_{ij} are defined as follows

$$
\epsilon_{ij}(s) = Q_f - 8g_i^f g_j^e f(s) \tag{3.52}
$$

with

$$
f(s) = \frac{1}{8\sin^2\theta\cos^2\theta}\frac{s}{s - M_Z^2 + iM_Z\Gamma_Z} \tag{3.53}
$$

We have assumed here a Breit-Wigner form for the Z-propagator (see later on). Notice that the factors u and t in the invariant amplitudes M_{ij} reflect the conservation of helicity as can be seen using

$$
\frac{t}{s} = -\frac{1-\cos\psi}{2}, \qquad \frac{u}{s} = -\frac{1+\cos\psi}{2} \tag{3.54}
$$

The unpolarized cross-section for the production of a single fermion f is given by

$$
\frac{d\sigma}{d\Omega}\Big|_{\mathrm{cm}} = \frac{\alpha^2}{4s}[C_S(1+\cos^2\psi) + 2C_A\cos\psi] \tag{3.55}
$$

where

$$
\begin{aligned}
C_S &= \frac{1}{4}(|\epsilon_{LL}(s)|^2 + |\epsilon_{RR}(s)|^2 + |\epsilon_{LR}(s)|^2 + |\epsilon_{RL}(s)|^2) \\
C_A &= \frac{1}{4}(|\epsilon_{LL}(s)|^2 + |\epsilon_{RR}(s)|^2 - |\epsilon_{LR}(s)|^2 - |\epsilon_{RL}(s)|^2)
\end{aligned} \tag{3.56}
$$

The unpolarized cross-section is parity invariant, but it brings track of the axial-couplings present in the neutral current. These produce a forward-backward asymmetry due to the term proportional to $\cos\psi$ in the cross-section. This can be seen explicitly by writing the coefficients C_S and C_A in terms of the vector and axial couplings g_V and g_A

$$
C_S = Q_f^2 - 4Q_F g_V^e g_V^f Re(f(s)) + 4({g_V^e}^2 + {g_A^e}^2)({g_V^f}^2 + {g_A^f}^2)|f(s)|^2 \tag{3.57}
$$

and

$$
C_A = -4Q_f g_A^e g_A^f Re(f(s)) + 16 g_V^e g_A^e g_V^f g_A^f |f(s)|^2 \tag{3.58}
$$

where $Re(f(s)$ is the real part of $f(s)$. We see that C_A goes to zero when we turn off the axial couplings of the fermions to the Z. This effect is better quantified by defining a forward-backward asymmetry, which is obtained by comparing the forward and the backward cross-sections. That is, we define

$$\sigma_F = \int_0^{2\pi} d\varphi \int_0^1 d\cos\psi \frac{d\sigma}{d\Omega}\Big|_{\rm cm} = \frac{\pi\alpha^2}{2s}\left(\frac{4}{3}C_S + C_A\right) \tag{3.59}$$

and

$$\sigma_B = \int_0^{2\pi} d\varphi \int_{-1}^0 d\cos\psi \frac{d\sigma}{d\Omega}\Big|_{\rm cm} = \frac{\pi\alpha^2}{2s}\left(\frac{4}{3}C_S - C_A\right) \tag{3.60}$$

and the forward-backward asymmetry

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4}\frac{C_A}{C_S} \tag{3.61}$$

By using these results, combining the data from $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \tau^+\tau^-$ one can determine g_V^ℓ and g_A^ℓ. In particular, in this way, one is able to eliminate the ambiguity on the couplings that was left from the νe scattering. Also, using the full expression for $Ref(s)$ one can try a simultaneous fit to $\sin^2\theta$ and M_Z, with the result (see, for instance, [30])

$$\sin^2\theta = 0.195 \pm 0.017, \qquad M_Z = 89.2 \pm 2.7\ GeV \tag{3.62}$$

One can get also informations about the ρ parameter defined in eq. (3.11). In fact, from there one can express M_Z in terms of ρ and the other known parameters, and perform a simultaneous fit to $\sin^2\theta$ and ρ obtaining

$$\sin^2\theta = 0.209 \pm 0.031, \qquad \rho = 1.003 \pm 0.053 \tag{3.63}$$

An important piece of physics was also obtained by measuring A_{FB} for the process $e^+e^- \to \bar{b}b$. Up to energies of about 50 GeV we can roughly estimate the factor $f(s)$

$$f(s) \approx -\frac{1}{8\sin^2\theta\cos^2\theta}\frac{s}{M_Z^2} = -\frac{1}{8M_W^2\sin^2\theta}s = -\frac{1}{2g^2\sin^2\theta v^2}s = -\frac{G_F}{\sqrt{2}}\frac{s}{4\pi\alpha} \tag{3.64}$$

This turns out to be less than ≈ -0.22. Furthermore, from $\sin^2\theta \approx 1/4$ we can approximate C_A and C_S to

$$C_S \approx Q_f^2, \qquad C_A \approx -4Q_f g_A^e g_A^f Ref(s) \tag{3.65}$$

Then we get from eqs. (3.61) and (3.65)

$$A_{FB} = -3\frac{g_A^e g_A^b}{Q_b}Ref(s) \tag{3.66}$$

This allows to measure g_A^b [30] (assuming $g_A^e = -1/2$)

$$g_A^b = -.50 \pm 0.14 \tag{3.67}$$

Recalling that $g_A^f = T_3^f$, we see that $T_3^b = -1/2$. This has been the first indirect evidence that the bottom quark has an isospin partner, the top.

Chapter 4

The physics of the massive vector bosons

4.1 Properties of the massive vector bosons

The massive vector bosons, $W^\pm$ and Z were discovered at CERN in 1983 at the collider $\bar{p}p$. The charged bosons were discovered by looking at the decay $W \to \ell\bar{\nu}_\ell$, and in particular to the case $\ell = \tau$, using the jet produced in the decay $\tau \to \nu_\tau + hadrons$. The two experiments (UA1 and UA2) gave the following average result

$$M_W = 81.3 \pm 1.4\ GeV \qquad \Gamma_W < 6.5\ GeV\ (90\% C.L.) \tag{4.1}$$

The neutral vector boson was discovered through the decay $Z \to e^+e^-$. The result was

$$M_Z = 92.1 \pm 1.7\ GeV \qquad \Gamma_Z < 4.6\ GeV\ (90\% C.L.) \tag{4.2}$$

A detailed study of the processes used for the identification

$$\bar{p}p \to (W \to \ell\nu_\ell) + X, \qquad \bar{p}p \to (Z \to e^+e^-) + X$$

would require a detailed parton model analysis and it will not pursued here. In this Section we will study only the decay properties of $W^\pm$ and Z at tree level. As we have already pointed out in Section 2.10, our input parameters will be (α, G_F, M_Z). Then using

$$M_W^2 = \frac{g^2v^2}{4} = \frac{e^2}{\sin^2\theta}\frac{1}{4}\frac{1}{\sqrt{2}G_F} = \frac{1}{\sin^2\theta}\frac{\pi\alpha}{\sqrt{2}G_F} \tag{4.3}$$

and

$$M_W^2 = M_Z^2\cos^2\theta \tag{4.4}$$

we can eliminate M_W^2 obtaining $\sin^2\theta$ in terms of the input parameters

$$\sin^2\theta = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}G_F M_Z^2}}\right) \tag{4.5}$$

and therefore

$$M_W = \frac{M_Z}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2} G_F M_Z^2}}} \tag{4.6}$$

Numerically we find

$$\sin^2\theta = 0.2126, \qquad M_W = 80.94\ GeV \tag{4.7}$$

The value of M_W is in a very good agreement with the value measured at the TEVATRON [31]

$$M_W = 80.356 \pm 0.125\ GeV \tag{4.8}$$

The widths of $W^{\pm}$ and Z are easily evaluated. One gets

$$\Gamma(W^- \to \ell^- \bar{\nu}_\ell) = \frac{G_F M_W^3}{6\pi\sqrt{2}} \tag{4.9}$$

$$\Gamma(W^- \to \bar{u}_A d_B) = \frac{G_F M_W^3}{6\pi\sqrt{2}} |V_{AB}|^2 N_C \tag{4.10}$$

$$\Gamma(Z \to f\bar{f}) = \frac{G_F M_Z^3}{6\pi\sqrt{2}} \left({g_V^f}^2 + {g_A^f}^2 \right) N_f \tag{4.11}$$

where $N_\ell = 1$ and $N_q = N_C = 3$. Gluonic corrections to the decay $Z \to q\bar{q}$ and $W^- \to \bar{u}_A d_B$ can be readily included in these equations by defining an **effective** number of colors

$$N_C = 3\left(1 + \frac{\alpha_s(M_Z^2)}{\pi}\right) \approx 3.115 \tag{4.12}$$

where the running coupling constant of the strong interactions evaluated at the Z-mass has been taken $\alpha_s(M_Z^2) = 0.12$. The total width of the $W^{\pm}$ is then given by

$$\Gamma_W = (3 + 2N_C)\Gamma(W^- \to \ell\bar{\nu}_\ell) \tag{4.13}$$

where we have used $m_t > M_W$ and the unitarity of the CKM matrix. From the value found for M_W, eq. (4.7), we get

$$\Gamma(W^- \to \ell\bar{\nu}_\ell) = 232\ MeV \tag{4.14}$$

and

$$\Gamma_W = 2.09\ GeV \quad (\Gamma_W^{QCD} = 2.14\ GeV) \tag{4.15}$$

where we have also given the QCD corrected value for Γ_W. These values can be compared with the experimental result [27]

$$\Gamma_W^{\exp} = 2.07 \pm 0.06\ GeV \tag{4.16}$$

At the same time we get

$$BR(W^- \to \ell\bar{\nu}_\ell) = 11.1\%, \quad (BR^{QCD}(W^- \to \ell\bar{\nu}_\ell) = 10.8\%) \tag{4.17}$$

to be compared with [27]

$$BR^{\exp}(W^- \to \ell\bar{\nu}_\ell) = (10.8 \pm 0.4)\% \tag{4.18}$$

In the case of the Z we have

$$\Gamma(Z \to \nu\bar{\nu}) = \frac{G_F M_Z^3}{6\pi\sqrt{2}}\frac{1}{2} = 165.9\ MeV \tag{4.19}$$

and

$$\Gamma(Z \to \ell\bar{\ell}) = \frac{G_F M_Z^3}{6\pi\sqrt{2}}\left[\left(-\frac{1}{2} + 2\sin^2\theta\right)^2 + \frac{1}{4}\right] = 84.84\ MeV \tag{4.20}$$

to be compared with [27]

$$\Gamma^{\exp}(Z \to \ell\bar{\ell}) = 83.93 \pm 0.14\ MeV \tag{4.21}$$

We get also

$$\begin{aligned}\Gamma(Z \to u\bar{u}) = \frac{G_F M_Z^3}{6\pi\sqrt{2}}\left[\left(\frac{1}{2} - \frac{4}{3}\sin^2\theta\right)^2 + \frac{1}{4}\right] N_C = 295.8\ MeV \\ (\Gamma^{QCD}(Z \to u\bar{u}) = 307.1\ MeV)\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\Gamma(Z \to d\bar{d}) = \frac{G_F M_Z^3}{6\pi\sqrt{2}}\left[\left(-\frac{1}{2} + \frac{2}{3}\sin^2\theta\right)^2 + \frac{1}{4}\right] N_C = 376.8\ MeV \\ (\Gamma^{QCD}(Z \to d\bar{d}) = 391.2\ MeV)\end{aligned} \tag{4.23}$$

From which

$$\begin{aligned}\Gamma_Z = 3(\Gamma(Z \to \nu\bar{\nu}) + \Gamma(Z \to \ell\bar{\ell}) + \Gamma(Z \to d\bar{d})) + 2\Gamma(Z \to u\bar{u}) = 2474.2\ MeV \\ (\Gamma_Z^{QCD} = 2540.0)\end{aligned} \tag{4.24}$$

to be compared with [31]

$$\Gamma_Z^{\exp} = 2494.6 \pm 2.7\ MeV \tag{4.25}$$

Summing over the quark contribution we get

$$\Gamma_h = \Gamma(Z \to hadrons) = 1722.0\ MeV \quad (\Gamma_h^{QCD} = 1787.8\ MeV) \tag{4.26}$$

and defining $\Gamma_{\rm inv}$ as the total width in neutrinos we have

$$\Gamma_{\rm inv} = 497.7\ MeV \tag{4.27}$$

The experimental data give [27]

$$\Gamma_h^{\exp} = 1744.8 \pm 3.0\ MeV \tag{4.28}$$

$$\Gamma_{\rm inv}^{\rm exp} = 499.9 \pm 2.5 \ MeV \tag{4.29}$$

where this last quantity is defined experimentally as the total width minus the total width in charged leptons and hadrons.

We see that the biggest deviations from our SM predictions are in the hadronic quantities. As we shall see this is due mainly to the electromagnetic corrections which can be embodied in the running of α. As a result we get a different result for M_W and $\sin^2\theta$ see eq. (4.7). This has more influence on the hadronic quantities since the dependence on $\sin^2\theta$ is in g_V^2, and

$$\delta g_V^2 = -4Q g_V \delta \sin^2\theta \tag{4.30}$$

In the lepton case we have $g_V \approx 0$ for the charged leptons, and $Q = 0$ for the neutrinos.

4.2 Production of Z at LEP

In this Section we will study the production of the neutral vector boson, the Z, at LEP. This machine started its running in 1990 allowing a high precision test of the SM. In the same period also SLC at SLAC became operational. This machine has not a copious production of Z as LEP, but it has the advantage of producing polarized electrons allowing also precision measurements at the same level of the ones realized at LEP. For the moment we will give the general features of the production of the Z particle without worrying about radiative corrections. We will deal with this subject in the next Section.

From eqs. (3.59) and (3.60) we get the total cross-section for $e^+e^- \to f\bar{f}$

$$\sigma_{f\bar{f}} = \sigma_F + \sigma_B = \frac{4\pi\alpha^2}{3s} N_f C_S \tag{4.31}$$

where N_f is the same factor introduced in eq. (4.11), taking into account the color multiplicity for quarks and the QCD corrections. We want now evaluate this cross-section around the mass of the Z. First of all we have the relation

$$\frac{1}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta}\frac{M_Z^2}{M_W^2} = \frac{g^2}{e^2}\frac{4}{g^2 v^2} M_Z^2 = \frac{\sqrt{2} G_F}{\pi\alpha} M_Z^2 \tag{4.32}$$

Also, around the mass of the Z the ϵ_{ij} are dominated by the Z propagator (see eq. (3.52))

$$\epsilon_{ij} \approx -8 g_i^f g_j^e f(s) \tag{4.33}$$

Therefore

$$C_S = \frac{1}{4}\sum_{ij} |\epsilon_{ij}(s)|^2 = \frac{1}{4}\left(\frac{\sqrt{2} G_F M_Z^2}{\pi\alpha}\right)^2 \frac{s^2}{(s-M_Z)^2 + M_Z^2\Gamma_Z^2} \sum_{ij} |g_i^f|^2 |g_j^e|^2 \tag{4.34}$$

But the term

$$\sum_{ij} |g_i^e|^2 |g_j^f|^2 = \frac{1}{4}[(g_V^e)^2 + (g_A^e)^2][(g_V^f)^2 + (g_A^f)^2] \tag{4.35}$$

can be expressed in terms of the partial widths

$$\sum_{ij} |g_i^e|^2 |g_j^f|^2 = \left(\frac{6\pi}{\sqrt{2} G_F M_Z^3} \right)^2 \frac{1}{N_f} \Gamma_{e^+e^-} \Gamma_{f\bar{f}} \tag{4.36}$$

and substituting in $\sigma_{f\bar{f}}$ we find

$$\sigma_{f\bar{f}} = 12\pi \frac{s}{M_Z^2} \frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to f\bar{f})}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \tag{4.37}$$

This expression is rather general and it is also very intuitive. In fact, around the Z-pole the cross-section will be proportional to the probability for an electron-positron pair to create a Z (that is $\Gamma(Z \to e^+e^-)$) times the probability for the Z to decay in the pair $f\bar{f}$ (that is $\Gamma(Z \to f\bar{f})$). The only model-dependent pieces in the previous expression are just the widths. This formula can be used to study the properties of the Z in a model independent way. Let us define the cross-section at the peak energy

$$\sigma_{f\bar{f}}^{\text{peak}} = \frac{12\pi}{M_Z^2} \frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to f\bar{f})}{\Gamma_Z^2} \tag{4.38}$$

The total cross-section at the peak is then given by

$$\sigma_{\text{TOT}}^{\text{peak}} = \frac{12\pi}{M_Z^2} \frac{\Gamma(Z \to e^+e^-)}{\Gamma_Z} \tag{4.39}$$

Using the values of the widths we have calculated in the previous Section we get

$$\sigma_{\text{TOT}}^{\text{peak}} \approx 6 \times 10^{-32} \text{ cm}^2 \tag{4.40}$$

Since the nominal luminosity of LEP is

$$\mathcal{L} = 10^{31} \text{ cm}^{-2} \text{ sec}^{-1} \tag{4.41}$$

we get that the total number of events (that is the number of Z) that we have for second is given by

$$\mathcal{L}\sigma_{\text{TOT}}^{\text{peak}} \approx 0.6 \tag{4.42}$$

with a daily production of Z given by

$$0.60 \times 60 \times 60 \times 24 \approx 5.2 \times 10^4 \tag{4.43}$$

We shall see that electromagnetic corrections reduce the peak cross-section of about 70%. Therefore the daily production of Z at LEP running at its nominal luminosity is rather 3.6×10^4. In terms of the peak cross-section we can write in a different way the cross-section around the peak

$$\sigma_{f\bar{f}} = \frac{s\Gamma_Z^2}{(s-M_Z^2)^2 + M_Z^2\Gamma_Z^2}\sigma_{f\bar{f}}^{\text{peak}} \tag{4.44}$$

From the measure of $\sigma_{f\bar{f}}$ around the Z peak it is possible to extract the widths and the Z mass. What it is done is the following. First one considers a given channel, say $\mu^+\mu^-$. Measuring $\sigma_{\mu^+\mu^-}$ at various energies around the Z mass one can perform a 3 parameter fit determining M_Z, Γ_Z and $\sigma_{\mu^+\mu^-}^{\text{peak}}$. This is done for the various channels. Then one can extract $\Gamma(Z\to e^+e^-)$ from the corresponding peak cross-section

$$\sigma_{e^+e^-}^{\text{peak}} = \frac{12\pi}{M_Z^2}\frac{\Gamma(Z\to e^+e^-)^2}{\Gamma_Z^2} \longrightarrow \Gamma(Z\to e^+e^-) = M_Z\Gamma_Z\sqrt{\frac{\sigma_{e^+e^-}^{\text{peak}}}{12\pi}} \tag{4.45}$$

Finally one can get $\Gamma(Z\to f\bar{f})$ from the peak cross-section

$$\Gamma(Z\to f\bar{f}) = \sigma_{\mu^+\mu^-}^{\text{peak}}\frac{M_Z^2\Gamma_Z^2}{12\pi}\frac{1}{\Gamma(Z\to e^+e^-)} = \frac{M_Z\Gamma_Z}{\sqrt{12\pi}}\frac{\sigma_{f\bar{f}}^{\text{peak}}}{\sqrt{\sigma_{e^+e^-}^{\text{peak}}}} \tag{4.46}$$

With this procedure one gets all the decay widths in the charged channels. Defining, as in the previous Section, the invisible width as

$$\Gamma_{\text{inv}} = \Gamma_Z - \sum_{\text{charged fermions}} \Gamma(Z\to f\bar{f}) \tag{4.47}$$

within the SM, one can use the experimental results for getting a value for the number of neutrinos with mass less than $M_Z/2$

$$N_\nu = \frac{\Gamma_{\text{inv}}}{\Gamma_\nu^{\text{theor}}} \tag{4.48}$$

From the values given in the previous Section we get

$$N_\nu = 3.013 \pm 0.015 \tag{4.49}$$

this number changes a little when radiative corrections to the SM are taken into account (from 1995 data one gets $N_\nu = 2.991\pm 0.016$).

Let us now consider other observable quantities that have been measured at LEP and SLC. First of all we recall the forward-backward asymmetry that has been defined in Section 3.4. We recall the result

$$A_{FB} = \frac{3}{4}\frac{C_A}{C_S} \tag{4.50}$$

where the coefficients C_A and C_S were defined in eqs. (3.57) and (3.58). Therefore using the same approximation as in eq. (4.33), we get

$$A_{FB}^f = 3\frac{g_V^e g_A^e g_V^f g_A^f}{({g_V^e}^2 + {g_A^e}^2)({g_V^f}^2 + {g_A^f}^2)} \tag{4.51}$$

Assuming to know g_V^e/g_A^e this quantity allows to determine g_V^f/g_A^f. The first ratio could be determined (assuming lepton universality) from the process $e^+e^- \to \mu^+\mu^-$ (or $e^+e^- \to \tau^+\tau^-$), but since A_{FB} for leptons depend quadratically on g_V^ℓ, the errors in determining $\sin^2\theta$ are rather big. One can do much better playing with polarized fermions. At LEP it is possible to measure the polarization of the final τ's in $e^+e^- \to \tau^+\tau^-$. Defining the polarization asymmetry as

$$P_f = \frac{\sigma(f_R) - \sigma(f_L)}{\sigma(f_R) + \sigma(f_L)} \tag{4.52}$$

with $\sigma(f_{L,R})$ the total cross-section for the production of left, right fermions, one gets easily, using the results of Section 3.4

$$P_f = -2\frac{g_A^f g_V^f}{{g_V^f}^2 + {g_A^f}^2} \tag{4.53}$$

The measure of P_f gives directly g_V^f/g_A^f allowing a measure of $\sin^2\theta$. At SLD it is possible to produce beams of polarized electrons. Then one can define a left-right asymmetry

$$A_{LR} = \frac{\sigma(e_L) - \sigma(e_R)}{\sigma(e_L) + \sigma(e_R)} \tag{4.54}$$

In the ideal case of 100% polarized beams one gets

$$A_{LR} = 2\frac{g_A^e g_V^e}{{g_V^e}^2 + {g_A^e}^2} \tag{4.55}$$

Otherwise the result is the average polarization times the previous expression. In both last cases the errors in determining $\sin^2\theta$ are much smaller than those coming from A_{FB}. We will present the experimental results only after having discussed the radiative corrections.

4.3 QED corrections at the Z-peak

The experimental precision reached at LEP is of the order of a few per mill for observables as the various Z widths, or a few per cent in the case of the asymmetries. This requires that the theoretical calculations should be at the same order of precision in order to have a meaningful test of the SM. In particular this implies that the tree level formulas are not adequate. In fact, only the electromagnetic corrections are of the order of about 6%, but also the pure weak corrections can be of the order of several per mill. We will not give a complete review of the radiative corrections to the SM, but we will limit ourselves to present the so called **improved Born approximation**, which maintains the same structure of the formulas for the various observables as in the tree approximation, but with couplings embodying radiative corrections. This approximation works with a precision better than 1% (in comparison to the exact loop results). In practice one has to work with the total loop corrections, but the previous approximation is quite simple and very useful in order to illustrate the main effects. In this Section we will start with an analysis of the electromagnetic corrections.

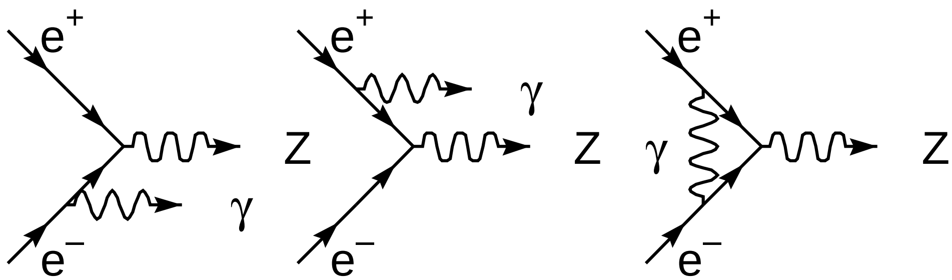

Fig. 4.1 -*QED corrections to the initial electron-positron vertex.*

First of all we have to take into account the initial-photon radiation and the initial vertex photon correction, see Fig. 4.1. In particular the first two diagrams correspond to soft photon emission from the initial lines. The detectors cannot detect photons with energy below a certain cut depending on the apparatus itself. Therefore all these corrections are, at large extent, depending on the experimental details. The physical effect is that if the initial electron, or positron, emits a bremsstrahlung photon its energy will be degraded. In general one can express the real cross-section as a convolution of a radiator function $G(z,s)$ times the theoretical cross-section. Here z represents the fraction of energy available. In equations

$$\sigma^{\rm corr}(s) = \int_0^1 dz\, G(z,s)\sigma(zs) \tag{4.56}$$

The first two diagrams contribute to $G(z,s)$ as the product of the probability for the initial fermions to emit a photon. The total effect, near the resonance, can be parameterized as [33]

$$\sigma^{\rm corr} \approx \sigma(s)\exp(\beta\ln r) = \sigma(s)r^\beta \tag{4.57}$$

Both β and r depend on s, but at the resonance one has

$$r^\beta|_{\rm res} \approx 0.67 \tag{4.58}$$

Also the peak of the cross-section is shifted to

$$(\sqrt{s})_{\rm max} = M_Z + \frac{\pi\beta}{8}\Gamma_Z \tag{4.59}$$

Since $\beta(M_Z^2) \approx 0.11$, the shift is about 100 MeV. Given this situation, what happens is that the experimentalists use dedicated programs as Zfitter [34] to take into account these effects and to deconvolute the theoretical cross-section, from the measured quantity. A further correction taken into account is an improvement of the Breit-Wigner expression for the Z-propagator that we used in eq. (3.53). We will ignore here, for simplicity, the four vector indices. So the Z propagator is given by

$$\Delta_Z(q^2) = \frac{1}{q^2 - M_0^2} + \frac{1}{q^2 - M_0^2}\Pi(q^2)\frac{1}{q^2 - M_0^2} + \cdots = \frac{1}{q^2 - M_0^2 - \Pi(q^2)} \tag{4.60}$$

where M_0 is the mass parameter appearing in the lagrangian and $\Pi(q^2)$ represents the self-energy diagrams, that is the one-particle irreducilble graphs with respect to the Z-lines. If we expand the self-energy around M_0 we obtain

$$\Pi(q^2) = Re(\Pi(M_0^2)) + Re(\Pi'(M_0^2))(q^2 - M_0^2) + iIm(\Pi(M_0^2)) \tag{4.61}$$

from which

$$q^2 - M_0^2 - \Pi(q^2) \approx (1 + Re(\Pi'(M_0^2)))(q^2 - M^2 - iM\Gamma) \tag{4.62}$$

where we have introduced the renormalized mass (the position of the pole)

$$M^2 = M_0^2 + \Pi(M^2) + \cdots \tag{4.63}$$

and the width of the Z

$$M\Gamma = Im(\Pi(M^2)) \tag{4.64}$$

In this way one gets the classical form for the Breit-Wigner form of the propagator

$$\Delta(q^2) = \frac{1}{1 + Re(\Pi')}\frac{1}{q^2 - M^2 - iM\Gamma} \tag{4.65}$$

A more precise description is obtained by expanding also $Im(\Pi(q^2))$ around the pole [35]. Is then possible to show that, neglecting the fermion masses, one must have

$$Im(\Pi(q^2)) \approx q^2 \tag{4.66}$$

Therefore the cross-section around the peak is given by

$$\sigma(e^+e^- \to f\bar{f}) = 12\pi\frac{s}{M_Z^2}\frac{\Gamma(Z \to e^+e^-)\Gamma(Z \to f\bar{f})}{(s - M_Z^2)^2 + \frac{s^2}{M_Z^2}\Gamma_Z^2} \tag{4.67}$$

As a consequence, the maximum of the cross-section is given by

$$\sqrt{s} = M_Z - \frac{1}{4}\frac{\Gamma_Z^2}{M_Z^2} \approx M_Z - 17\ MeV \tag{4.68}$$

One should also notice that the factor $1/(1+Re(\Pi'))$ contributes in this equation to reconstruct the 1-loop corrected widths in the numerator of the cross-section around the peak. The other big electromagnetic correction is in the running of the fine structure constant. If we repeat the same argument that we have used for the Z- propagator, one has that the photon propagator is modified by

$$\frac{1}{s} \Longrightarrow \frac{1}{s}\frac{1}{1+Re(\Pi'_\gamma(s))} \tag{4.69}$$

Consider for instance the process $e^+e^- \to \gamma \to f\bar{f}$. The amplitude is then modified

$$\frac{\alpha}{s} \Longrightarrow \frac{1}{s}\frac{\alpha}{1+Re(\Pi'_\gamma(s))} \equiv \frac{\alpha(s)}{s} \tag{4.70}$$

Therefore one can take into account this kind of QED corrections introducing the concept of running coupling. In the same way, in the previous argument about the Z-propagator, one could introduce running couplings

$${g_i^f(s)}^2 = \frac{{g_i^f}^2}{1+Re(\Pi'(s))} \tag{4.71}$$

This effect is not big for the Z-couplings [36], but is very important in the case of α. In fact,

$$\alpha(M_Z^2) \equiv \frac{\alpha}{1-\Delta\alpha} \approx 1.064\,\alpha \tag{4.72}$$

since

$$\Delta\alpha = 0.0601 \pm 0.0009 \tag{4.73}$$

As a consequence, the relation (4.3) is modified in

$$M_W^2 \sin^2\theta = \frac{\pi\alpha(M_Z^2)}{\sqrt{2}G_F} \tag{4.74}$$

From this and $M_W = M_Z \cos\theta$ we get

$$M_W = 79.94\ GeV, \qquad \sin^2\theta = 0.2314 \tag{4.75}$$

In order to see the relevance of this electromagnetic correction, we compare in Table 4.1 several observables evaluated at tree-level with the same quantities with QED and QCD corrections included and with their experimental values. The table shows quite clearly the relevance of the QED and QCD corrections. The QED corrections are particularly relevant for the asymmetries, mainly due to the modification in the value of $\sin^2\theta$.

Observable	Tree-level	QED+QCD corrections	Experimental value
$M_W(GeV)$	80.94	79.95	80.356 ± 0.125
$\Gamma_W(GeV)$	2.09	2.06	2.07 ± 0.06
$\Gamma_Z(GeV)$	2.4742	2.4860	2.4946 ± 0.0027
$\Gamma(Z \to \bar{\ell}\ell)(MeV)$	83.93	83.40	83.93 ± 0.14
$\Gamma(Z \to had)(MeV)$	1722.0	1738.2	1744.8 ± 3.0
$\Gamma_{\rm inv}(MeV)$	497.63	497.63	499.9 ± 2.5
A^ℓ_{FB}	0.0657	0.0165	0.0174 ± 0.0010
P_τ	-0.296	-0.148	-0.1401 ± 0.0067
A_{LR}	0.296	0.148	0.1551 ± 0.0040

Table 4.1 -*Comparison among the tree-level SM prediction for several observables, their values corrected by QED and QCD effects, and their experimental values (see [27] and [31]).*

4.4 Radiative Corrections and the improved Born approximation

We have already discussed the pure QED and QCD corrections. In this Section we will list the remaining radiative corrections. These come from various sources

- **Oblique**: These corrections come from the boson self-energies. We have already seen in the previous Section their relevance. Their main features are the sensitivity to loop effects from heavy particles, like the top quark. Also these corrections are independent on the particular process, that is they are **universal**.

- **Vertex**: These are corrections to the various couplings. Obviously they are not universal and are usually smaller than the oblique corrections. An exception is the vertex $Z\bar{b}b$ which has a noticeable correction from the top quark contribution.

- **Box**: These are diagrams with two intermediate gauge bosons. Being not resonant, these contributions are generally negligible at the Z-peak. The box contribution at the decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ is not small and it must be considered in the definition of G_F.

- **Higgs**: These corrections come from the exchange of a Higgs particle between two fermionic lines. Usually they are irrelevant since we have a suppression factor given by the product of two fermionic masses.

In the context of radiative corrections is useful to define the Weinberg angle in terms of the input parameters of the theory. At tree level there are various equivalent ways in which the angle enters in the theory. At this point we have also to distinguish among the bare parameters (to be denoted by a tilde) of the theory and the renormalized ones. At tree level all the relations are among bare parameters, so we have the following different definitions

1. The electroweak unification condition
$$\tilde{e} = \tilde{g} \sin\tilde{\theta} = \tilde{g}' \cos\tilde{\theta} \tag{4.76}$$

2. The relation between $\tilde{M}_W$ and $\tilde{M}_Z$ (that is the symmetry breaking via doublets)
$$\sin^2\tilde{\theta} = 1 - \left(\frac{\tilde{M}_W}{\tilde{M}_Z}\right)^2 \tag{4.77}$$

3. The relation with the Fermi constant
$$\sqrt{2} G_F = \frac{\tilde{e}^2}{4 \sin^2\tilde{\theta} \tilde{M}_W^2} \tag{4.78}$$

4. The definition of the neutral current
$$j_\mu^Z = j_\mu^3 - \sin^2\tilde{\theta} j_\mu^{\rm em} \tag{4.79}$$

One can define a renormalized $\sin^2\theta$ by generalizing any of the previous relations. But each of these different definitions will correspond to a different function of the input parameters. One possibility is the Sirlin scheme [37] in which the $\sin^2\theta$ is related to the renormalized values of the W and Z masses
$$s_W^2 \equiv 1 - \frac{M_W^2}{M_Z^2} \tag{4.80}$$

Now the relation in eq. (4.78), which allows to determine $\sin^2\theta$ in terms of the input parameters, becomes
$$\sqrt{2} G_F = \frac{e^2}{4 s_W^2 c_W^2 M_Z^2} \frac{1}{1 - \Delta r} \tag{4.81}$$

Where Δr takes into account the radiative corrections. Here $c_W^2 = 1 - s_W^2$. To see how this equation comes about, let us remember that the vector boson masses are defined through the radiatively corrected propagators

$$\Delta_V^{-1}(q^2) = q^2 - \tilde{M}_V^2 - \Pi_V(q^2) \tag{4.82}$$

by

$$M_V^2 = \tilde{M}_V^2 + \Pi_V(M_V^2) \tag{4.83}$$

Also remember that the Fermi constant is defined through the μ-decay. Then we have (recall the discussion in Section 3.1)

$$\begin{aligned} \frac{G_F}{\sqrt{2}} &= \frac{\tilde{e}^2}{8\sin^2\tilde{\theta}}\left(\frac{-1}{-\tilde{M}_W^2 - \Pi_W(0)} + (\text{vertex, box})\right) \\ &= \frac{\tilde{e}^2}{8\sin^2\tilde{\theta}\tilde{M}_W^2}\left(1 - \frac{\Pi_W(0)}{M_W^2} + (\text{vertex, box})\right) \end{aligned} \tag{4.84}$$

To evaluate the radiative corrections we expand the bare parameters in terms of the renormalized ones

$$\begin{aligned} \tilde{e}^2 &= e^2\left(1 + 2\frac{\delta e}{e}\right) \\ \tilde{M}_W^2 &= M_W^2\left(1 + \frac{\delta M_W^2}{M_W^2}\right) \\ \sin^2\tilde{\theta} &= 1 - \frac{M_W^2 + \delta M_W^2}{M_Z^2 + \delta M_Z^2} = s_W^2 + c_W^2\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) \end{aligned} \tag{4.85}$$

Notice that we have defined $\tilde{M}_V^2 = M_V^2 + \delta M_V^2$, therefore

$$\delta M_V^2 = -\Pi_V(M_V^2) \tag{4.86}$$

By keeping only the first order terms in the corrections (we are evaluating the 1-loop corrections) we get

$$\begin{aligned} \frac{G_F}{\sqrt{2}} &= \frac{e^2}{8s_W^2 M_W^2}\Big[1 + 2\frac{\delta e}{e} - \frac{c_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) - \frac{\Pi_W(0) + \delta M_W^2}{M_W^2} \\ &+ (\text{vertex, box})\Big] \end{aligned} \tag{4.87}$$

All the corrections are generally divergent, however the divergences cancel out in the final result, as it should be in the evaluation of an observable quantity. The correction Δr can be written as

$$\Delta r = \Delta\alpha - \frac{c_W^2}{s_W^2}\Delta\rho + \text{small terms} \tag{4.88}$$

with

$$\Delta\rho = \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \tag{4.89}$$

The small terms mean that they do not contain large logs and/or leading powers of $G_F m_t^2$. In particular they include the vertex and the box contribution, and the quantity $\Pi_W(0) + \delta M_W^2 = \Pi_W(0) - \Pi(M_W^2)$. Although Π_W has a quadratic contribution from the top mass, this cancels out in the difference. At the leading order in m_t one gets

$$\Delta\rho = \frac{3G_F}{8\sqrt{2}\pi^2} m_t^2 \tag{4.90}$$

Therefore, neglecting small terms we can write

$$s_W^2 c_W^2 = \frac{\pi\alpha(M_Z^2)}{\sqrt{2}G_F M_Z^2} \frac{1}{1 + \frac{c_W^2}{s_W^2}\Delta\rho} \tag{4.91}$$

Of course, this relation really gives the expression of M_W in terms of the input parameters

$$M_W^2\left(1 - \frac{M_W^2}{M_Z^2}\right) = \frac{\pi\alpha(M_Z^2)}{\sqrt{2}G_F} \frac{1}{1 + \frac{c_W^2}{s_W^2}\Delta\rho} \tag{4.92}$$

In analogous way we can get an expression for the Weinberg angle entering in the couplings of fermions to the Z. We have

$$\sin^2\tilde{\theta} = 1 - \frac{\tilde{M}_W^2}{\tilde{M}_Z^2} + \text{small terms} = s_W^2 + c_W^2\Delta\rho + \text{small terms} \tag{4.93}$$

This allows a definition of an effective Weinberg angle

$$\sin^2\bar{\theta} = s_W^2 + c_W^2\Delta\rho \tag{4.94}$$

The small terms include also the vertex corrections which are small except for the case of the coupling $Z\bar{b}b$, which we will discuss later. Notice that in terms of this effective angle we have [38]

$$\cos^2\bar{\theta} = c_W^2(1 - \Delta\rho) = \frac{M_W^2}{\rho M_Z^2} \tag{4.95}$$

where we have defined

$$\rho = \frac{1}{1 - \Delta\rho} \tag{4.96}$$

This parameter is analogous to the one defined in eq. (3.11). The origin of $\Delta\rho$ is in the big splitting between the top and the bottom quarks. In fact, the contribution of an $SU(2)$ doublet of fermions to $\Delta\rho$ is given by [38, 33]

$$\frac{3G_F}{8\sqrt{2}\pi^2}\left[m_u^2 + m_d^2 - 2\frac{m_u^2 m_d^2}{m_u^2 - m_d^2}\log\frac{m_u^2}{m_d^2}\right] \tag{4.97}$$

This expression goes to zero for $m_d \to m_u$, and gives eq. (4.90) for $m_u \gg m_d$. As discussed in Section 3.1, for $\rho = 1$ the effective weak lagrangian shows a symmetry $SU(2)$. This is related to the fact that when the Higgs fields belong to a doublet, the vacuum of the theory has a global symmetry group $SU(2)$ (broken by the gauge interactions). This is the symmetry that gives rise to the relation between M_W and M_Z. An explicit breaking of this symmetry, as a non degenerate doublet of fermions, brings to a parameter $\rho \neq 1$. As a consequence also the size of the neutral couplings are affected by this correction. The radiative corrections depend also on the Higgs mass. However due to the previous symmetry the virtual production of Higgs particles does not generate quadratic corrections in the Higgs mass at one-loop. This is known as the Veltman screening [39]. The dependence on the Higgs mass is only logarithmic.

With these considerations we are now in the position to understand practically all the electroweak effects at the Z peak by means of the so called improved Born approximation. This gives the amplitude for $e^+e^- \to f\bar{f}$ in a form very similar to the tree one, but taking into account the main radiative corrections we have discussed above. One has

$$\mathcal{M}(e^+e^- \to \bar{f}f) \approx \frac{4\pi\alpha(M_Z^2)}{s} Q_e Q_f j_\mu^{\rm em}(e) j^\mu_{\rm em}(f) + 4\sqrt{2} G_F M_Z^2 \rho \frac{j_\mu^Z(e) j_Z^\mu(f)}{s - M_Z^2 + is\Gamma_Z/M_Z} \tag{4.98}$$

where

$$j_\mu^Z = \frac{1}{2}\gamma_\mu \left(T_3^f (1-\gamma_5) - 2Q_f \sin^2\bar{\theta} \right) \tag{4.99}$$

with ρ and $\sin^2\bar{\theta}$ defined before. This approximation works with a precision better than 1%, except for the case of the $Z\bar{b}b$ vertex. In fact, as we have already noticed there are vertex corrections to $Z \to \bar{b}b$ depending on the top, as illustrated in Fig. 4.2.

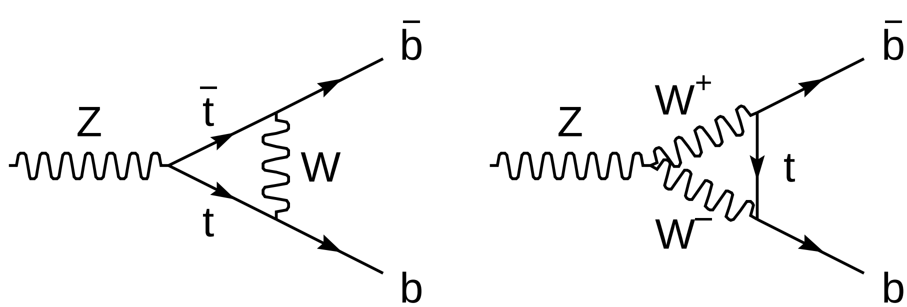

Fig. 4.2 -*Corrections to the vertex $Z \to \bar{b}b$ depending on m_t.*

These corrections can be neatly taken into account by the following substitutions

$$\rho \to \sqrt{\rho \rho_b} = \rho \left(1 - \frac{2}{3}\Delta\rho\right), \qquad \sin^2\bar{\theta} \to \sin^2\bar{\theta}\left(1 + \frac{2}{3}\Delta\rho\right) \tag{4.100}$$

It is worth to stress the main modifications in the improved Born approximation. These are

- The replacement of α with its value at the mass of the Z.
- The inclusion of the s dependence in the width appearing in the propagator.
- The presence of the factor ρ in front of the Z-exchange term.
- The use of the effective Weinberg angle $\sin^2\bar{\theta}$ in the Z-couplings.

Finally we mention that the improved Born approximation for the inclusive widths $\Gamma(Z \to \bar{f}f(\gamma, g))$, that is including gluon and photons in the final state (we have mentioned so far only the first correction) is given by

$$\Gamma(Z \to \bar{f}f(\gamma, g)) = N_f \frac{G_F \rho M_Z^3}{24\sqrt{2}\pi}\left[1 + (1 - 4|Q_f|\sin^2\bar{\theta})^2\right] \tag{4.101}$$

where

$$\begin{aligned} N_\ell &= 1 \times \left(1 + \frac{3\alpha}{4\pi}Q_f^2\right) \\ N_q &= 3 \times \left(1 + \frac{3\alpha}{4\pi}Q_f^2\right)\left(1 + \frac{\alpha_s(M_Z^2)}{\pi}\right) \end{aligned} \tag{4.102}$$

4.5 Comparison with the experimental results

LEP and SLC give a lot of informations, but already from the leptonic width and the forward-backward asymmetry is possible to extract the quantities ρ and $\sin^2\bar{\theta}$. However these quantities depend on various parameters of the SM as the Higgs mass, the value of the QCD coupling constant at M_Z, etc. which are unknown or not known with sufficient accuracy.

Observable	Data (Warsaw '96)	Standard Model	Pull
$M_Z(GeV)$	91.1863(20)	91.1861	0.1
$\Gamma_Z(GeV)$	2.4946(27)	2.4960	-0.5
$\sigma^{\text{peak}}_{(Z\to hadr)}(\text{nb})$	41.508(56)	41.465	0.8
$R_h = \frac{\Gamma(Z\to hadr)}{\Gamma(Z\to\bar{\ell}\ell)}$	20.788(29)	20.757	0.7
$R_b = \frac{\Gamma(Z\to\bar{b}b)}{\Gamma(Z\to hadr)}$	0.2178(11)	0.2158	1.8
$R_c = \frac{\Gamma(Z\to\bar{c}c)}{\Gamma(Z\to hadr)}$	0.1715(56)	0.1723	-0.1
A^{ℓ}_{FB}	0.0174(10)	0.0159	1.4
P_τ	-0.1401(67)	-0.1458	-0.9
A_{LR}	0.1542(37)	0.1458	-2.2
$\sin^2\bar{\theta}$ (LEP-combined)	0.23200(27)	0.23167	1.2
$M_W(GeV)$	80.356(125)	80.353	0.3
$m_t(GeV)$	175(6)	172	0.5

Table 4.2 -*Comparison among the experimental values of several observables [41], and their theoretical values as obtained from a best fit to the SM [31]). The pull is defined as the difference between the experimental and the theoretical values divided by the standard deviation.*

The most convenient thing to do is to try to combine all the available measures and make a best fit of the poorly known parameters. Among the data to be included one should consider also the value of m_t as obtained combining the latest data from [40]

$$m_t = 175 \pm 6\ GeV \tag{4.103}$$

In Table 4.2 we give the experimental values of some of the quantities measured at LEP and SLC, compared with the values obtained from a best fit to the SM. From all the available data, by fitting m_t, m_h and $\alpha_s(M_Z^2)$ one finds (with χ^2/d.o.f.=19/14) [31]

$$\begin{aligned} m_t &= 172 \pm 6 \; GeV \\ m_h &= 149^{+148}_{-82} \; GeV \\ \alpha_s(M_Z^2) &= 0.1202 \pm 0.0033 \end{aligned} \tag{4.104}$$

In particular one gets $m_h < 392 \; GeV$ at 1.64σ. The values of $\sin^2\bar{\theta}$ and M_W corresponding to the best fit are

$$\begin{aligned} \sin^2\bar{\theta} &= 0.23167 \pm 0.0002 \\ M_W &= 80.352 \pm 0.034 \; GeV \end{aligned} \tag{4.105}$$

It is interesting to notice that the error obtained on M_W from the measure of the radiative corrections at LEP and SLC is quite a challenge for the future experiments.

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