About the counting of the Goldstone bosons

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Summary

- Introduction
- Proof of the Goldstone theorem
- The Goldstone theorem in the non relativistic case. Examples: scalar fields at finite density
- The case of space-time symmetries
- Conclusions

Introduction

Spontaneous symmetry breaking plays a crucial role in the formulation of the fundamental interactions, both for QCD (**pions**) and electroweak interactions. A consequence of spontaneous breaking of continuous symmetries is the existence of massless particles (**Goldstone bosons**). This is the main content of the Goldstone theorem (J. Goldstone, Nuovo Cimento, **9**, 154 (1961); Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960): J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. **127**, 965 (1962)). Many authors have studied the validity of the theorem with the conclusion that it can be proved rigorously under the following basic hypotheses (more technical assumptions will be considered later)

- Spontaneous breaking of a global continuous symmetry, that is a degenerate vacuum not invariant under a subgroup H of the symmetry group of the hamiltonian, G.

- Manifest Lorentz covariance of the theory.

- Positivity of the Hilbert space.

If the symmetries considered are internal ones (that is not considering space-time symmetries) under the previous hypotheses it follows:

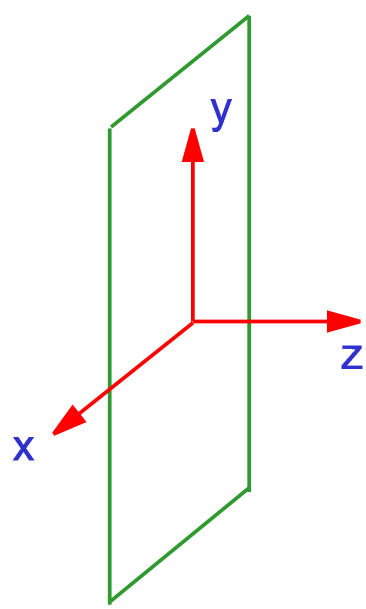
For each broken symmetry there exists one massless scalar particle

However there are exceptions to this counting rule. We recall a few cases:

- In the case of gauge theories, for any spontaneously broken local symmetry, one Goldstone boson disappears from the physical spectrum of the states and the corresponding gauge boson acquires a mass. Here the manifest covariance and the positivity of the Hilbert space cannot be maintained at the same time

- In the non-relativistic case the situation appears to be somewhat confuse. Both for the ferromagnet and the antiferromagnet we have the breaking $O(3) \to O(2)$, but in the first case we have only one Goldstone boson (the magnon) whereas in the second case there are two goldstones

- Also for the breaking of space-time symmetries the simple counting rule does not apply. Consider a plane in a fixed position in 3-dimensional space



There are 3 broken symmetries (P_3, L_1, L_2) (the transformations changing the position of the plane) but only 1 Goldstone boson. Another example is the breaking of conformal symmetry in 4-dimensions. Here there are 5 broken generators (dilatation and the special conformal transformations) but only 1 Goldstone boson, the dilaton.

Whereas the first case (gauge symmetries) is very well known, the other two cases are not so well known. The non-relativistic case has been treated in an almost forgotten paper (H.B. Nielsen and S. Chada, Nucl. Phys. **B105**, 445 (1976)) and more recently in (T. Schäfer, D.T. Son, M.A. Stephanov, D. Toublan and J.J.M. Verbaarschot, hep-ph/0108210; V.A. Miransky and I.A. Shovkovy, hep-ph/0108178 . The case of space-time symmetries has been considered more recently for a particular case (R.C., R. Gatto, M. Mannarelli and G. Nardulli, Phys. Lett. **B511**, 218, 2001) and in a more general setting by I. Low and A. Manohar, hep-th/0110285.

We will start recalling the proof of the Goldstone theorem and than we will review the previous two cases, trying to establish the

RIGHT COUNTING RULES

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Proof of the Goldstone theorem

We recall here a standard proof of the Goldstone theorem reviewing some subtleties. We start showing that if a current is conserved

$$\partial_\mu j^\mu(x) = 0$$

than for a local observable A

$$\lim_{V\to\infty} \frac{d}{dt}[Q_V(t), A] = 0$$

where

$$Q_V(t) = \int_V d^3\vec{x}\, j^0(\vec{x}, t)$$

In fact, we have

$$0 = \int_V d^3\vec{x}\, [\partial_\mu j^\mu(\vec{x}, t), A] =$$

$$= \frac{d}{dt}\int_V d^3\vec{x}\, [j^0(\vec{x}, t), A] + \int_S d\vec{S}\cdot[\vec{j}(\vec{x}, t), A]$$

For $V \to \infty$ the surface integral vanishes since the two local operators are separated by a very large space-like interval.

Consider now a set of local operators $\phi_i(x)$ (order parameters) not invariant under a continuous symmetry generated by the charge $Q = \int d^3\vec{x}\, j^0$, then

$$\lim_{V\to\infty} \langle 0|[Q_V(t), \phi_i(0)]|0\rangle \neq 0$$

Inserting a complete set of intermediate states

$$\lim_{V\to\infty} \sum_n \int_V d^3\vec{x} \Big[\langle 0|j_0(0)|n\rangle\langle n|\phi_i(0)|0\rangle e^{-iP_n\cdot x} -$$

$$-\langle 0|\phi_i(0)|n\rangle\langle n|j_0(0)|0\rangle e^{iP_n\cdot x}\Big] =$$

$$= \sum_n (2\pi)^3 \delta^3(\vec{P}_n) \Big[\langle 0|j_0(0)|n\rangle\langle n|\phi_i(0)|0\rangle e^{-iE_n t} -$$

$$\langle 0|\phi_i(0)|n\rangle\langle n|j_0(0)|0\rangle e^{iE_n t}\Big] \neq 0$$

Since we have shown that this equation does not depend on t, there must be a state $|n\rangle$ such that (just take the time derivative of the previous expression)

$$\langle 0|\phi_i(0)|n\rangle\langle n|j_0(x)|0\rangle \neq 0, \quad \text{for} \quad E_n \delta^3(\vec{P}_n) = 0$$

The state $|n\rangle$ must have the same quantum number as $\phi_i(y)$ and $j_0(x)$.

In particular this state must have the same Lorentz properties of the charge Q. For internal symmetries it is a boson. For supersymmetries, since the charges are spinors, the Goldstone particle is a spin 1/2 fermion. A caveat of the theorem is the use of translational invariance. Notice also that up to now we have only shown that there exists a state with a dispersion relation

$$\lim_{\vec{P}_n \to 0} E_n = 0$$

One can get informations about the dispersion relations assuming Lorentz invariance. In this case we introduce the FT of the commutator

$$J^i_\mu(k) = \int d^4x \langle 0|[j_\mu(x), \phi_i(0)]|0\rangle e^{-ik\cdot x}$$

From current conservation

$$k^\mu J^i_\mu(k) = 0 \;\Rightarrow\; J_\mu(k)^i = k_\mu \delta(k^2)(a + b\,\theta(k_0))f^i$$

Also

$$\int dk_0 J^i_0(k_0, \vec{0}) = (2\pi) \int d^3\vec{x} \langle 0|[j_0(\vec{x}, 0), \phi_i(0)]|0\rangle \equiv$$

$$\equiv 2\pi v^i$$

$$v^i = \langle 0|[Q, \phi_i(0)]|0\rangle \neq 0$$

It follows easily that

$$b f^i = 4\pi v^i$$

Therefore

$$J^i_\mu(k) = 4\pi v^i k_\mu \delta(k^2) \left(\frac{a}{b} + \theta(k_0) \right) \neq 0$$

and to match the Goldstone particle contribution we need a linear dispersion relation

$$k_0 = |\vec{k}|$$

The Goldstone theorem in the non-relativistic case

I will review here briefly the paper of Chada and Nielsen. Defining
$G =$ symmetry group of the theory
$Q_a =$ conserved charges $(a = 1, \cdots, n)$
and assuming

- The generators Q_α, $\alpha = 1, \cdots, m$ are spontaneously broken. That is, there exist m fields ϕ_i, $i = 1, \cdots, m$, and a vacuum state such that

$$\det\langle 0|[Q_\alpha, \phi_i]|0\rangle \neq 0, \quad \alpha, i = 1, \cdots, m$$

- If $A(x)$ and $B(x)$ are any two local operators

$$|\langle 0|[A(\vec{x}, t), B(0)]|0\rangle|_{|\vec{x}|\to\infty} \to e^{-\tau|\vec{x}|}, \quad \tau > 0$$

- Translational invariance is not entirely broken

It is possible to show that there are two types of Goldstone excitations ($n =$ integer)

1. type I: $E_I \approx |\vec{k}|^{2n+1}$ in number g_I

2. type II: $E_{II} \approx |\vec{k}|^{2n}$ in number g_{II}

with

- \# goldstones $= g_I + g_{II}$

- $m = g_I + 2g_{II}$

We will illustrate the second and the third assumption. Notice that we are not assuming Lorentz invariance, but the theory can be embedded in a relativistic framework introducing a time-like vector $n^\mu = (1, \vec{0})$

The most general expression for the Fourier transform, $J_\mu^{\alpha i}$ of $\langle 0|[j_\mu^\alpha(x), \phi_i(0)]|0\rangle$ is

$$J_\mu^{\alpha i} = A^{\alpha i} k_\mu + B^{\alpha i} n_\mu$$

Using the current conservation

$$0 = k^\mu J_\mu^{\alpha i} = k^2 A^{\alpha i} + k \cdot n B^{\alpha i}$$

we get

$$A^{\alpha i} = \delta(k^2)\chi^{\alpha i}(n \cdot k) - (n \cdot k)\rho^{\alpha i}(k^2, n \cdot k)$$

$$B^{\alpha i} = \delta(n \cdot k)\Delta^{\alpha i}(k^2) + k^2 \rho^{\alpha i}(k^2, n \cdot k) + C^{\alpha i}\delta^4(k)$$

or

$$J_\mu^{\alpha i} = k_\mu \delta(k^2)\chi^{\alpha i}(n \cdot k) +$$

$$+(n_\mu k^2 - k^\mu n \cdot k)\rho^{\alpha i}(k^2, n \cdot k) +$$

$$+n_\mu \delta(n \cdot k)\Delta^{\alpha i}(k^2) + C^{\alpha i} n_\mu \delta^4(k)$$

Using

$$\langle 0|[j_\mu^\alpha(x), \phi_i(0)]|0\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x} J_\mu^{\alpha i}$$

If α and i are chosen in such a way that the v.e.v. of the commutator is $\neq 0$, then the contribution of the last term in $J_\mu^{\alpha i}$, if present is

$$\langle 0|[j_\mu^\alpha(x), \phi_i(0)]|0\rangle \;\Rightarrow\; n_\mu C^{\alpha i}$$

But by the second assumption the commutator cannot be a constant, therefore $C^{\alpha i} = 0$. The term $n_\mu \delta(n \cdot k)\Delta^{\alpha i}(k^2)$ tells us that $k_0 = 0$ for all $\vec{k}$. Therefore from the vacuum it is possible to generate another vacuum ($E = 0$) with $\vec{k} \neq 0$, **breaking completely translational invariance** against the third assumption. It follows $\Delta^{\alpha i}(k^2) = 0$. We get

$$J_\mu^{\alpha i} = k_\mu \delta(k^2)\chi^{\alpha i}(n \cdot k) +$$

$$+(n_\mu k^2 - k^\mu n \cdot k)\rho^{\alpha i}(k^2, n \cdot k)$$

The first term gives rise to the relativistic dispersion relation $k^2 = 0$. The second term contributes generically to states such that

$$\lim_{\vec{k}\to 0} k_0 = 0$$

In fact, it may contribute if and only if

$$\int dk_0 J_0^{\alpha i} \;\mapsto\; \lim_{\vec{k}\to 0} \int dk_0 (k^2 - k_0^2)\rho^{\alpha i}(k^2, k_0) \neq 0$$

This requires

$$\rho^{\alpha i} = \frac{1}{|\vec{k}|^2}\bar{\rho}^{\alpha i}, \qquad \lim_{\vec{k}\to 0} \bar{\rho}^{\alpha i} \neq 0$$

On the other hand, since Q^α is conserved we must have $\int \vec{\nabla}\cdot\vec{j}^\alpha d^3\vec{x} = 0$, that is

$$0 = \int \langle 0|[\vec{\nabla}\cdot\vec{j}^\alpha(x), \phi_i(0)]|0\rangle d^3\vec{x} =$$

$$\int d^3\vec{x} \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}(i\vec{k}\cdot\vec{J}^{\alpha i})$$

$$= \int \frac{d^4k}{2\pi}\delta^3(\vec{k})e^{ik_0x_0}(i\vec{k}\cdot\vec{J}^{\alpha i})$$

This implies, in particular

$$-\delta^3(\vec{k})\,\vec{k}\cdot(n\cdot k)\,\vec{k}\,\rho^{\alpha i} = -\delta^3(\vec{k})\,k_0\,\bar{\rho}^{\alpha i} = 0$$

or

$$\delta^3(\vec{k})\,k_0 = 0 \quad \rightarrowtail \quad \lim_{\vec{k}\to 0} k_0 = 0$$

Complex field at finite density

The simplest way to introduce the chemical potential is through the substitution

$$\partial_0 \to \partial_0 - i\mu$$

For a complex scalar field we get

$$\mathcal{L} = (\partial_0 + i\mu)\phi^\dagger(\partial_0 - i\mu)\phi - \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi$$

$$-m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

$$= \partial_\mu\phi^\dagger\partial^\mu\phi - (m^2 - \mu^2)\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

$$+ i\mu(\phi^\dagger\partial_0\phi - \partial_0\phi^\dagger\phi)$$

Notice that the last term breaks charge conjugation invariance splitting the ϕ and $\phi^\dagger$ masses. Assuming $m^2 > 0$ we have two phases

Normal phase, $\mu < m$: The mass spectrum is given by (Q is the charge, $Q_\phi = -Q_{\phi^\dagger} = -1$)

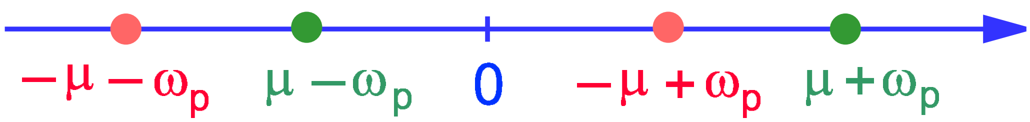
$$p^2 - (m^2 - \mu^2) - 2\mu Q p_0 = 0$$

$$(E - \mu Q)^2 = m^2 + |\vec{p}|^2$$

with solutions

$$E = \mu Q \pm \omega_p\,, \quad \omega_p = \sqrt{m^2 + |\vec{p}|^2}$$

At the critical point, $\mu = m$, the mode with $Q = -1$ becomes gapless ($E \to 0$ for $\vec{p} \to 0$), and by continuity becomes the Goldstone boson in the broken phase.



Broken phase, $\mu > m$: The minimum of the potential is given by

$$\langle\phi\rangle_0 = \frac{v}{\sqrt{2}}, \qquad v^2 = \frac{\mu^2 - m^2}{\lambda}$$

With the usual replacement

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))e^{i\phi(x)/v}$$

We get ($\sigma = v + h$)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma + \frac{\sigma^2}{2v^2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}(m^2 - \mu^2)\sigma^2$$

$$-\frac{\lambda}{4}\sigma^4 - \frac{\mu}{v}\sigma^2\,\partial_0\phi$$

and for the quadratic part

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \lambda v^2 h^2 - 2\mu h \partial_0 \phi$$

giving rise to the mass spectrum condition

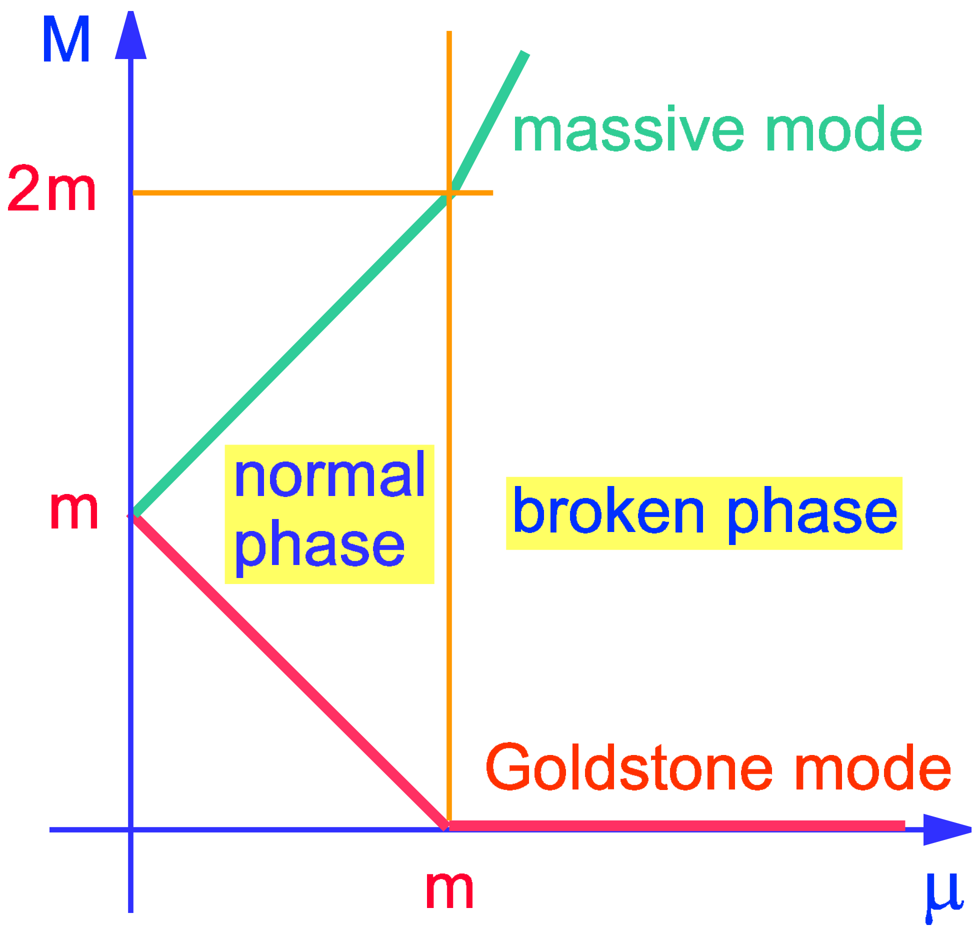
$$\det \begin{pmatrix} p^2 - 2\lambda v^2 & 2i\mu p_0 \\ -2i\mu p_0 & p^2 \end{pmatrix} = 0$$

The solutions for small momenta are

$$E^2 \approx \frac{\mu^2 - m^2}{3\mu^2 - m^2}|\vec{p}|^2 + \frac{2\mu^4}{(3\mu^2 - m^2)^3}|\vec{p}|^4$$

$$E^2 \approx 6\mu^2 - 2m^2 + \frac{9\mu^2 - m^2}{6\mu^2 - 2m^2}|\vec{p}|^2$$

The masses in terms of the chemical potential are given in the following figure:



The first solution corresponds to a Goldstone boson of type I, $E \propto |\vec{p}|$, whereas the second one to a massive mode. In this case one could guess from the beginning that everything should work as in the relativistic case. In fact the global symmetry of the model is $U(1)$, therefore we expect one goldstone of type I, simply because to get a goldstone of type II we need to break at least two symmetries.

O(4)-symmetric scalar field

The first interesting model showing dispersion relations for the goldstones of the type $E \propto |\vec{p}|^2$ is the $O(4)$ symmetric scalar model. In the symmetric phase at $\mu = 0$, ($\vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$)

$$\mathcal{L}_{\vec{\phi}} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{2}m^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(|\vec{\phi}|^2)^2$$

To introduce the chemical potential it is convenient to define a doublet of complex fields

$$\Phi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_2 - i\phi_1\\ \phi_4 - i\phi_3\end{pmatrix}$$

The lagrangian at finite density becomes formally identical to the previous case

$$\mathcal{L}_\Phi = \partial_\mu\Phi^\dagger\partial^\mu\Phi - (m^2 - \mu^2)\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$$

$$+ i\mu(\Phi^\dagger\partial_0\Phi - \partial_0\Phi^\dagger\Phi)$$

In this form the theory shows explicitly an invariance $U(2)$, $\Phi \to U\Phi$, $U \in U(2)$, which is a subgroup of $O(4)$.

A third form can be obtained in terms of the 2×2 matrix

$$M = \frac{1}{\sqrt{2}}\left(\phi_4 - i\vec{\tau}\cdot\vec{\phi}\right) = \begin{pmatrix} \chi_1 & -\chi_2^\dagger \\ \chi_2 & \chi_1^\dagger \end{pmatrix}$$

satisfying the pseudo-reality property

$$M^* = \epsilon M \epsilon^{-1}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the property

$$M^\dagger M = (|\chi_1|^2 + |\chi_2|^2) 1_2 = \frac{1}{2}|\vec{\phi}|^2 1_2$$

where 1_2 is the 2×2 identity matrix. We get

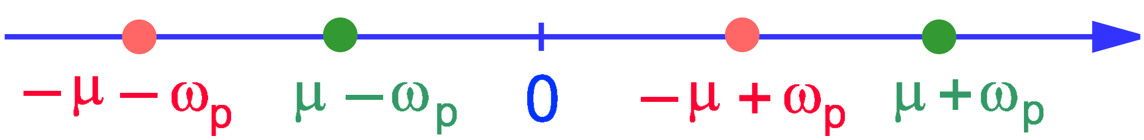
$$\mathcal{L}_M = \frac{1}{2}Tr[\partial_\mu M^\dagger \partial^\mu M] - \frac{m^2 - \mu^2}{2}Tr[M^\dagger M]$$

$$-\frac{\lambda}{4}(Tr[M^\dagger M])^2$$

$$+i\frac{\mu}{2}Tr[M^\dagger \partial_0 M \frac{1+\tau_3}{2} - M\frac{1+\tau_3}{2}\partial_0 M^\dagger]$$

The invariance of the first three pieces is $SU(2) \otimes SU(2)$, $M \to UMV^\dagger$, $U, V \in SU(2)$, which is essentially the same as $O(4)$. This symmetry is broken by the last term to $SU(2) \otimes U(1)$.

For $m^2 > 0$ the theory is in the symmetric phase for $\mu < m$ and in the broken phase for $\mu > m$. The spontaneous breaking is $U(2) \to U(1)$, giving rise to 3 broken symmetries. In the symmetric phase it is convenient to use $\mathcal{L}_\Phi$. The result for the masses is exactly as for the complex field (again the C-invariance is broken), except that each pole is doubly degenerate (two complex fields)



$$\omega_p = \sqrt{m^2 + |\vec{p}|^2}$$

The mass spectrum is given by (Q is the charge, $Q_\phi = Q_{\phi^\dagger} = -1$)

$$p^2 - (m^2 - \mu^2) - 2\mu Q p_0 = 0$$

$$(E - \mu Q)^2 = m^2 + |\vec{p}|^2$$

with solutions

$$E = \mu Q \pm \omega_p\,, \quad \omega_p = \sqrt{m^2 + |\vec{p}|^2}$$

At the critical point, $\mu = m$, the two modes with $Q = -1$ become gapless and by continuity become the Goldstone bosons in the broken phase Since there are 3 broken symmetry we expect one of the goldstones to have a quadratic dispersion relation. The minimum of the potential is given by

$$\langle \Phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v^2 = \frac{\mu^2 - m^2}{\lambda}$$

To study the broken phase we use the standard parameterization

$$\Phi = \frac{1}{\sqrt{2}} e^{-i\vec{\tau}\cdot\vec{\pi}/v} \begin{pmatrix} 0 \\ v + h \end{pmatrix} \approx \frac{1}{\sqrt{2}} \begin{pmatrix} -\pi_2 - i\pi_1 \\ v + h + i\pi_3 \end{pmatrix}$$

leading to the identification

$$\vec{\phi} = (\pi_1, -\pi_2, -\pi_3, v + h)$$

The quadratic part of the lagrangian turns out to be

$$\mathcal{L}_{\text{quad.}} = \frac{1}{2} \partial_\mu h \, \partial^\mu h + \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \lambda v^2 h^2$$

$$+ \mu(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1 + \pi_3 \partial_0 h - h \partial_0 \pi_3)$$

We get now two different conditions for the mass spectrum in the sectors (π_3, h) and (π_1, π_2). The first sector is as for the complex case, with a would be massive and a would be massless mode, whereas in the second case we have two would be massless modes. However these two were splitted at the critical point, and they remain as such after the transition. More specifically we get

$$\text{sector } (\pi_3, h): \quad \det \begin{pmatrix} p^2 - 2\lambda v^2 & 2i\mu p_0 \\ -2i\mu p_0 & p^2 \end{pmatrix} = 0$$

$$\text{sector } (\pi_1, \pi_2): \quad \det \begin{pmatrix} p^2 & 2i\mu p_0 \\ -2i\mu p_0 & p^2 \end{pmatrix} = 0$$

The solutions for small momenta in the first sector are as before

$$E^2 \approx \frac{\mu^2 - m^2}{3\mu^2 - m^2} |\vec{p}|^2$$

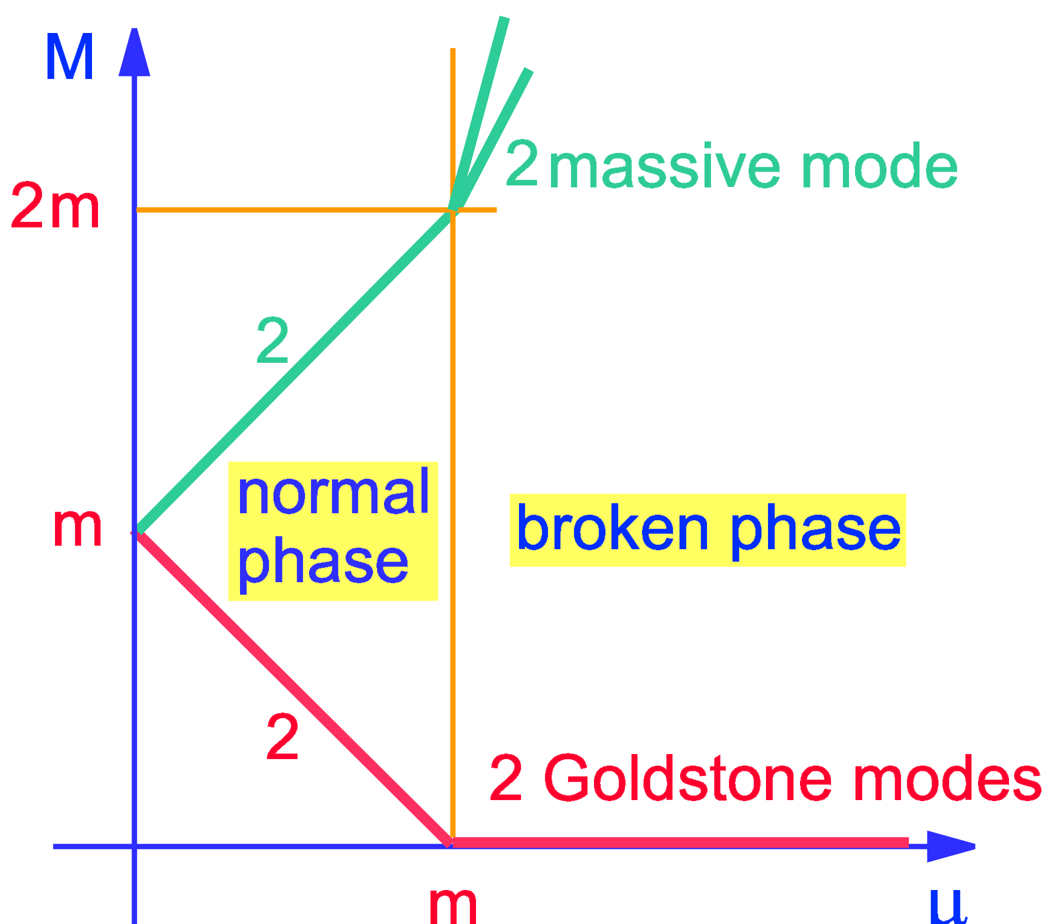
$$E^2 \approx 6\mu^2 - 2m^2 + \frac{9\mu^2 - m^2}{6\mu^2 - 2m^2} |\vec{p}|^2$$

In the second sector we get

$$E^2 \approx \frac{|\vec{p}|^4}{4\mu^2}$$

$$E^2 \approx 4\mu^2 + 2|\vec{p}|^2$$

To summarize we have two gapless modes, one, in the sector (π_3, h) with linear dispersion relation. The other in the sector (π_1, π_2) with quadratic dispersion relation in agreement with the Chada-Nielsen theorem.



However, notice that both π_1 and π_2 correspond to two flat directions in the potential, but they describe only one Goldstone mode. This can be understood in the following way

- For $E \ll \mu$, neglect $(\partial_0\pi_1)^2 + (\partial_0\pi_2)^2$ in the lagrangian. One remains with $\mu(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)$, then

$$\Pi_{\pi_1} = -\mu\pi_2, \quad \Pi_{\pi_2} = \mu\pi_1$$

  π_1 and π_2 are conjugate variables $\Rightarrow$ only one degree of freedom

- if a field satisfy a first order wave equation, it must be complex

$$\phi\partial_0\phi = \frac{1}{2}\partial_0(\phi^2), \quad \text{if } \phi \text{ real}$$

  ϕ contains only annihilation operators, and $\phi^\dagger$ only creation operators. We need two fields for describing a physical particle.

A recent theorem helps to clarify the situation (T. Schäfer, D.T. Son, M.A. Stephanov, D. Toublan and J.J.M. Verbaarschot, hep-ph/0108210):

If Q_α, $\alpha = 1, \cdots, \mathrm{m}$ are the broken generators, and $\langle 0|[\mathrm{Q}_\alpha, \mathrm{Q}_\beta]|0\rangle = 0$ for any pair $\alpha, \beta = 1, \cdots, \mathrm{m}$, then the number of Goldstone bosons is equal to m, the number of broken generators.

As a consequence we have: Since $[Q_\alpha, Q_\beta]$ is a linear combination of the generators of the symmetry group, if the vev of all the conserved charges is zero, the number of goldstones is equal to the number of broken generators. For a mismatch to arise it is necessary (but not sufficient) that some of the broken generators have vev $\neq 0$

If the goldstones are less than m, since they are generated acting with the broken generators upon the vacuum, there must exist some linear combination of Q_α such that

$$\sum_\alpha c_\alpha Q_\alpha |0\rangle = 0$$

Notice that since c_α is generally complex, the previous combination does not need to be a broken generator, but

$$Q_a = \sum_\alpha (\mathrm{Re}\, c_\alpha) Q_\alpha \text{ and } Q_b = \sum_\alpha (\mathrm{Im}\, c_\alpha) Q_\alpha$$

are in the algebra, therefore $Q_a|0\rangle \neq 0$ and $Q_b|0\rangle \neq 0$. However

$$(Q_a + iQ_b)|0\rangle = 0$$

Define the state $|\psi\rangle$

$$Q_a|0\rangle = |\psi\rangle, \quad Q_b|0\rangle = -i|\psi\rangle$$

We get

$$\langle 0|[Q_a, Q_b]|0\rangle = -2i\langle\psi|\psi\rangle \neq 0$$

which contradicts the assumption.

Several applications are

- In the actual case we have the generator τ_3 which have vev not zero in the ground state $(0, v)$.

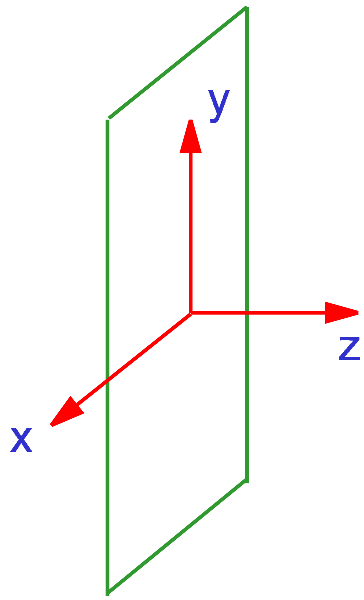
- The ferromagnet has a ground state with all the spins aligned. The corresponding spin density is not zero, and we have one magnon for two broken generators ($O(3) \to O(2)$).

- The antiferromagnet, a system of two coupled sublattices, has a ground state with all the spin of one sublattice along one direction and all the spins of the other sublattice along the opposite direction. All the spin densities vanish and there are two magnons.

- If the vacuum is Lorentz invariant, all the densities vanish (fourth component of a four-vector).

The case of space-time symmetries

As we shall see, in the case of breaking of space-time symmetry, the counting of the associated NGB is not trivial and it depends on the nature of the space-time group. We will begin considering a 3-dimensional theory with a ground state corresponding to a plane in the space, or in a modern language a 2-brane. This may arise in string theory or, in a more complicated version (a lattice of parallel planes), in finite density QCD for the so-called LOFF phase (M. Alford, J.A. Bowers and K. Rajagopal, hep-ph/0008208)). We have a breaking of the 3-space symmetry group, G (rotations + translations) down to the group H of transformations leaving the plane invariant



The plane is invariant under the transformations generated by

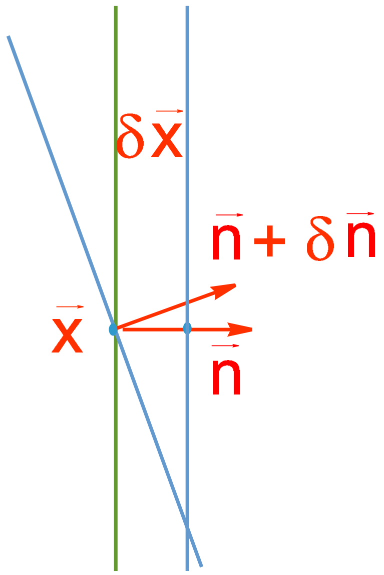
$$\text{Lie H} = (P_1, P_2, L_3)$$

whereas it changes under

$$\text{Lie G - Lie H} = (P_3, L_1, L_2)$$

We will be interested in the fluctuations of the plane, which will be described by NGB fields, phonons (R.C., R. Gatto, M. Mannarelli and G. Nardulli, Phys. Lett. **B511**, 218, 2001) . Denoting by $\vec{n} = (0, 0, 1)$ the normal to the plane, they are given by $(\vec{x}_\perp = \vec{x} - \vec{n}(\vec{n}\cdot\vec{x}))$

$$\vec{n}\cdot\vec{x} \to \vec{n}\cdot\vec{x} + \delta(\vec{n}\cdot\vec{x}) = \vec{n}\cdot\vec{x} + \frac{1}{f}\phi(\vec{x}_\perp) \equiv \frac{1}{f}\Phi(\vec{x})$$



Fluctuations of the lattice planes

The total fluctuation, $\delta(\vec{n}\cdot\vec{x})$, can be decomposed in two pieces described by the following fields

$$T(x) = \vec{n}\cdot\delta\vec{x}, \quad \vec{R}(x) = \vec{n} + \delta\vec{n}$$

such that

$$\frac{1}{f}\Phi = \vec{R}\cdot\vec{x} + T(\vec{x})$$

and satisfying ($\langle\Phi\rangle_0 = \vec{n}\cdot\vec{x}$)

$$\langle T(x)\rangle_0 = 0, \quad |\vec{R}(x)|^2 = 1, \quad \langle\vec{R}(x)\rangle_0 = \vec{n}$$

The field $\vec{R}(x)$ can be defined as

$$\vec{R}(x) = \left[e^{i(\xi_1 L_1 + \xi_2 L_2)}\right]_{i3}, \quad (L_i)_{jk} = -i\epsilon_{ijk}$$

We now get

$$\frac{\phi}{f} = \delta(\vec{n}\cdot\vec{x}) = (\vec{R}-\vec{n})\cdot\vec{x} + T$$

In order the field ϕ describes small fluctuations we must have the fields T and $\vec{R}$ functionally related, meaning that $\vec{R}(x)$ can be expressed in terms of $\Phi(x)$, $(|\vec{R}|^2 = 1$, $\langle \vec{R} \rangle_0 = \vec{n})$

$$\vec{R}(x) = \frac{\vec{\nabla}\Phi(x)}{|\vec{\nabla}\Phi(x)|}$$

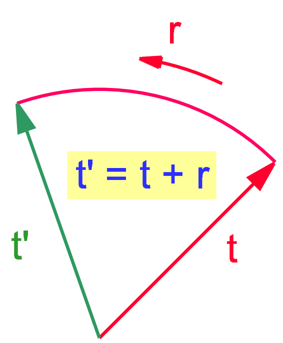
In terms of the ξ_i fields

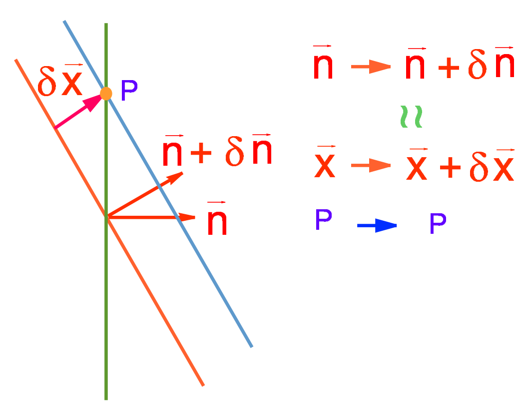
$$R_i \approx n_i + \epsilon_{ijk} n_j \xi_k \approx n_i + \frac{1}{f}(\partial_i - n_i(\vec{n}\cdot\vec{\nabla}))\phi$$

that is

$$\vec{\nabla}_\perp \phi(\vec{x}_\perp) = f\,\vec{n} \wedge \vec{\xi}$$

The physical origin for this is that the NG fields are nothing but the local parameters of the broken symmetries and locally there is no difference between translations and rotations





Therefore only one NGB is present: the phonon

As a further example, consider the conformal group. To get an action, S, invariant under the conformal group we follow B. Zumino, Brandeis lectures, (1970):

- Introduce $g_{\mu\nu}$ and make S E-invariant

$$S(\phi) \rightarrow S(g_{\mu\nu}, \phi)$$

- Introduce σ and make S W-invariant:

$$\delta g = 2\Lambda g,\ \delta\phi = -\Lambda\phi,\ \delta\sigma = -\Lambda/f$$

$$S(g_{\mu\nu}, \phi) \rightarrow S(g_{\mu\nu} e^{2f\sigma}, \phi e^{-f\sigma})$$

- Restrict S to the flat space $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$

- Add the kinetic term for the σ field

$$\frac{1}{2}\int d^4x e^{2f\sigma} \partial_\mu\sigma\partial^\mu\sigma$$

The resulting theory breaks spontaneously the conformal symmetry in the vacuum $\langle\sigma\rangle_0 = 0$.

Example:

$$S=\int d^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4}\phi^4\right]$$

1) - E-invariance

$$S=\int d^4x\sqrt{|g|}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4}\phi^4\right]$$

2) - W-invariance

$$S=\int d^4x\sqrt{|g|}e^{4f\,\sigma}\Big[\frac{1}{2}g^{\mu\nu}e^{-2f\,\sigma}\partial_\mu\left(e^{-f\,\sigma}\phi\right)\partial_\nu\left(e^{-f\,\sigma}\phi\right)$$

$$-\tfrac{1}{2}m^2e^{-2f\,\sigma}\phi^2-\tfrac{\lambda}{4}e^{-4f\,\sigma}\phi^4\Big]$$

$$=\int d^4x\sqrt{|g|}\Big[\frac{1}{2}g^{\mu\nu}(\partial_\mu-f\partial_\mu\sigma)\phi(\partial_\nu-f\partial_\nu\sigma)\phi$$

$$-\tfrac{1}{2}m^2e^{2f\,\sigma}\phi^2-\tfrac{\lambda}{4}\phi^4\Big]$$

3) - $g_{\mu\nu}=\eta_{\mu\nu}$ + kinetic term for σ

$$S=\int d^4x\Big[\frac{1}{2}(\partial_\mu-f\partial_\mu\sigma)\phi(\partial^\mu-f\partial^\mu\sigma)\phi$$

$$-\tfrac{1}{2}m^2e^{2f\,\sigma}\phi^2-\tfrac{\lambda}{4}\phi^4+\tfrac{1}{2}e^{2f\,\sigma}\partial_\mu\sigma\partial^\mu\sigma\Big]$$

The result is a theory with SB of conformal symmetry. Only a single NGB σ appears against $D+1$ charges

We review now the paper by I. Low and A. Manohar, hep-th/0110285. Let us recall our notations:

$$Q_A \in \text{Lie G},\ A = 1, \cdots, n+m,\ \text{charges of G}$$

$$Q_a \in \text{Lie H},\ a = 1, \cdots n,\ \text{unbroken charges}$$

$$Q_\alpha \in \text{Lie H},\ \alpha = 1, \cdots m,\ \text{broken charges}$$

We will denote generically the set of order parameters by $\phi(x)$

$$\langle\phi(x)\rangle_0 \neq 0$$

and

$$Q_a\langle\phi(x)\rangle_0 = 0, \quad Q_\alpha\langle\phi(x)\rangle_0 \neq 0$$

By construction of the vacuum symmetry group H, it follows that for constant c_α's

$$c_\alpha Q_\alpha\langle\phi(x)\rangle_0 = 0 \quad \Rightarrow \quad c_\alpha = 0$$

The NGB are nothing but small amplitude long-wavelenght fluctuations of the order parameters

$$\delta\langle\phi(x)\rangle_0 = c_A(x)Q_A\langle\phi(x)\rangle_0 = c_\alpha(x)Q_\alpha\langle\phi(x)\rangle_0$$

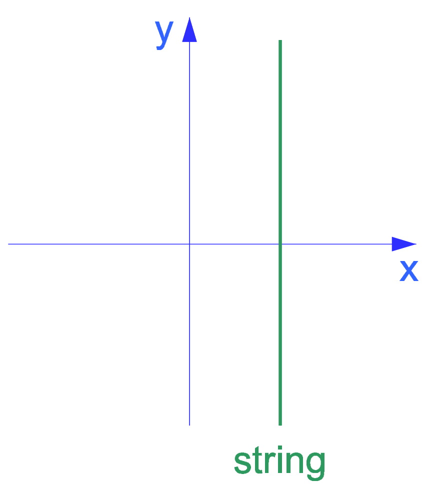
The number of Goldstone boson fields is the number of independent $c_\alpha(x)$ appearing in this equation

For internal symmetries the number of these fields is the same as the number of broken generators. However **in the case of space-time symmetry there might be non-constant solutions to the equation**

$$c_\alpha(x) Q_\alpha \langle \phi(x) \rangle_0 = 0$$

The number of NGB is equal to the number of broken generators minus the number of the solutions of this equation

As an example consider a 3-dimensional string



We have (G = space-symmetry group)

$$\text{Lie H} = (P_y, L_{xz})$$

$$\text{Lie G - Lie H} = (P_x, P_z, L_{xy}, L_{yz})$$

Consider the following fluctuation associated to the broken generators:

$$\delta\langle\phi(x)\rangle_0 = (\epsilon(y)P_x + \theta(y)L_{xy})\langle\phi(x)\rangle_0$$

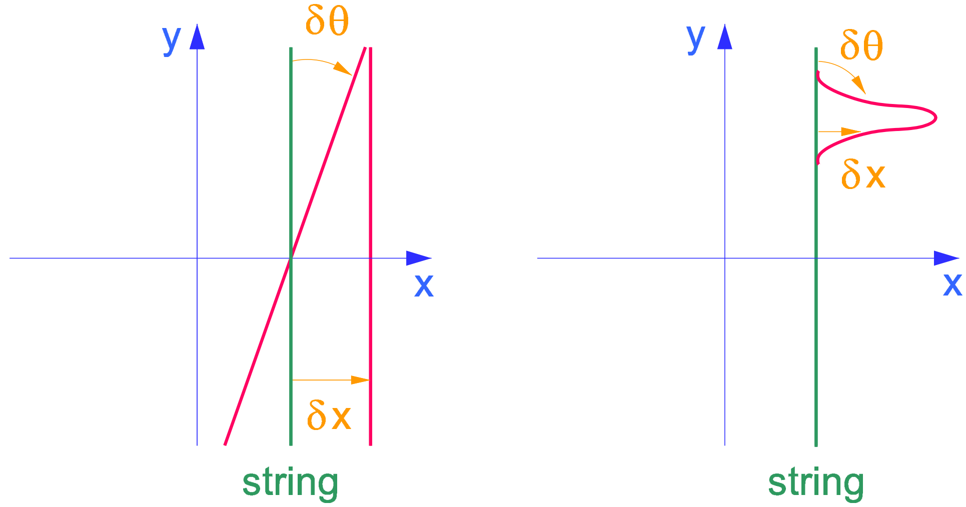
and using

$$L_{xy}\langle\phi(x)\rangle_0 = (xP_y - yP_x)\langle\phi(x)\rangle_0 = -yP_x\langle\phi(x)\rangle_0$$

we get: $\delta\langle\phi(x)\rangle_0 = (\epsilon(y) - y\theta(y))\, P_x\langle\phi(x)\rangle_0$. We may require this particular combination to vanish

$$\epsilon(y) = y\theta(y)$$

eliminating one Goldstone boson. No solution would be possible for ϵ and θ constant. Since the NGB are the parameters of broken transformations, for space-time symmetries it is not possible to distinguish among certain transformations. No local difference between a translation P_x and a rotation L_{xy}



To look for solutions to $c_\alpha Q_\alpha \langle \phi(x) \rangle_0 = 0$, we act with an unbroken momentum P_μ (the Goldstone fields depend only on the coordinates corrisponding to unbroken momenta), obtaining

$$P_\mu \left(c_\alpha(x) Q_\alpha \right) \langle \phi(x) \rangle_0 = [P_\mu, c_\alpha(x) Q_\alpha] \langle \phi(x) \rangle_0 = 0$$

that is

$$\left(-i\partial_\mu c_\alpha(x) Q_\alpha + c_\alpha(x)[P_\mu, Q_\alpha] \right) \langle \phi(x) \rangle_0 = 0$$

Using

$$[P_\mu, Q_\alpha] = i f_{\mu\alpha\beta} Q_\beta + i f_{\mu\alpha b} Q_b$$

we obtain

$$\boxed{\partial_\mu c_\beta(x) - f_{\mu\alpha\beta} c_\alpha(x) = 0}$$

The Goldstone boson $\mathbf{c_\alpha(x)}$ can be eliminated in favor of $\mathbf{c_\beta(x)}$, if we may invert $\mathbf{f_{\mu\alpha\beta}}$. In the case of the string, consider the Goldstone fields $c_\beta = (c_x, c_y)$ associated to the broken momenta. The only non vanishing commutators between the conserved momentum P_y and the broken generators are

$$[P_y, L_{xy}] = iP_x, \quad [P_y, L_{yz}] = -iP_z$$

Therefore

$$\partial_y c_x = f_{y,\alpha,x} c_\alpha = c_{xy}, \quad \partial_y c_z = f_{y,\alpha,z} = -c_{yz}$$

The NGB's are 4-2=2

The case of the plane considered before can be treated along the same lines. Remember

$$\text{Lie H} = (P_x, P_y, L_{xy}),$$

$$\text{Lie G - Lie H} = (P_z, L_{yz}, L_{zx})$$

We take now the NG field $c_\beta = c_z$. The non vanishing commutators of the two conserved momenta (P_x, P_y) with the broken generators are

$$[P_x, L_{zx}] = iP_z, \quad [P_y, L_{yz}] = -iP_z$$

Therefore

$$\partial_x c_z = f_{x,\alpha,z} c_\alpha = c_{zx}, \quad \partial_y c_z = f_{y,\alpha,z} = -c_{yz}$$

These relations coincide with the ones we got before

$$\vec{\nabla}_\perp \phi = f\vec{n} \cdot \vec{\xi} \Rightarrow \partial_x \phi = -f\xi_y, \quad \partial_y \phi = f\xi_x$$

via: $c_z = -\phi/f, c_{zx} = \xi_y, c_{yz} = \xi_x$.

For the symmetry breaking of the conformal group we have

$$\text{Lie H} = (P_\mu, J_{\mu\nu})$$

$$\text{Lie G - Lie H} = (D, K_\mu)$$

Consider the NG field associated to the dilatation, c_D. The commutators of the broken generators with the momentum P_μ are

$$[P_\mu, D] = iP_\mu, \quad [P_\mu, K_\nu] = 2iJ_{\mu\nu} - 2ig_{\mu\nu}D$$

Therefore

$$\partial_\mu c_D = f_{\mu,\alpha,D}\, c_\alpha = -2c_\mu$$

The Goldstone field c_μ associated to the broken conformal generators can be expressed in terms of the dilaton field. Again, this comes from the fact that locally we cannot distinguish between dilatations and conformal transformations

Effective theory

The most efficient way to build up an effective theory is the coset construction by S.R. Coleman, J. Wess and B. Zumino, Phys. Rev, **177**, 2239 (1969) and modified for space-time symmetries by D.V. Volkov, Sov. J. Particles Nucl. **4**, 3 (1973); V.I. Ogievetsky, Proc. of X-th Winter School of Theor. Phys. in Karpacz, Vol 1, Wroclaw (1974) 227. We recall that a given element, g, of a group G has a unique decomposition with respect to a subgroup H, $g = \eta h$ with $h \in$ H and $\eta \in$ G/H. Consider now a group G spontaneously broken to H and define the element

$$\Omega(\xi) = e^{i\xi_\alpha Q_\alpha}$$

Multiplying to the left by an arbitrary element g of the group we have

$$\Omega(\xi) \underset{g}{\rightarrow} g\Omega(\xi) = \Omega(\xi')h(\xi, g)$$

The next step is the construction of the Maurer-Cartan differential form at values in Lie G

$$\omega = \Omega^{-1} d\Omega$$

which under the group transform as

$$\omega \underset{g}{\rightarrow} h^{-1}\Omega^{-1} d(\Omega h) = h^{-1}\omega h + h^{-1} dh$$

Defining $\omega_{\parallel}$ and $\omega_{\perp}$ as the restrictions of ω to Lie G and Lie G - Lie H respectively, we get

$$\omega_{\parallel} \underset{g}{\rightarrow} h^{-1}\omega_{\parallel} h + h^{-1} dh$$

$$\omega_{\perp} \underset{g}{\rightarrow} h^{-1}\omega_{\perp} h$$

since $h^{-1} dh \in$ Lie H.

One makes use of $\omega_{\perp}$ to build up invariant Lagrangians under G from a trace over the group

$$\text{Tr}\left(\omega_{\perp}^{2}\right)$$

In the actual case we enlarge Ω to contain the conserved momenta

$$\Omega(x,\xi) = e^{ix^{\mu}P_{\mu}} e^{i\xi_{\alpha}(x)Q_{\alpha}}$$

The reason being that under translations the coordinates transform as Goldstone bosons $x^{\mu} \rightarrow x^{\mu} + a^{\mu}$. Also we can think to the coordinates as parameters of the coset Poincaré/Lorentz

Examples: Poincaré/Lorentz, $\Omega = e^{ix^\mu P_\mu}$

$$\omega = e^{-ix\cdot P} d\left(e^{ix\cdot P}\right) = i dx^\mu P_\mu$$

from $\omega_\perp = i\omega_\perp^\mu$

$$\omega_\perp^\mu = dx^\mu$$

invariant under translations and in homogeneous way under Lorentz

SUSY/Lorentz, $\Omega = e^{ix\cdot P} e^{i\theta^\alpha Q_\alpha + i\theta^{\dot\alpha} Q_{\dot\alpha}}$, Q_α and $Q_{\dot\alpha}$ are the SUSY generators satisfying

$$[Q_{\dot\alpha}, Q_\beta] = 2P_\mu \sigma^\mu_{\dot\alpha\beta}$$

Using

$$e^{-X} de^X = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}[X,[X,\cdots[X,dX]\cdots]]$$

we get

$$\omega = i\left(dx^\mu - i\theta^{\dot\alpha}\sigma^\mu_{\dot\alpha\beta} d\theta^\beta + i d\theta^{\dot\alpha}\sigma^\mu_{\dot\alpha\beta}\theta^\beta\right) P_\mu$$

$$+ i d\theta^\alpha Q_\alpha + i d\theta^{\dot\alpha} Q_{\dot\alpha}$$

One can recognize the typical SUSY invariant differential forms

Let us go back to our case where

$$\Omega(x, \xi) = e^{ix^\mu P_\mu} e^{i\xi_\alpha(x) Q_\alpha}$$

we get

$$\omega = e^{-i\xi_\alpha Q_\alpha} i dx^\mu P_\mu e^{i\xi_\alpha Q_\alpha} + e^{-i\xi_\alpha Q_\alpha} d e^{i\xi_\alpha Q_\alpha}$$

At the lowest order in the fields, we get

$$\omega_\perp \approx i dx^\mu P_\mu + \xi_\alpha [Q_\alpha, P_\mu] dx^\mu + i d\xi_\alpha Q_\alpha$$

From

$$[P_\mu, Q_\alpha] = i f_{\mu\alpha\beta} Q_\beta + i f_{\mu\alpha b} Q_b$$

we obtain

$$\omega_\perp \approx i dx^\mu P_\mu + i(\partial_\mu \xi_\alpha - f_{\mu\beta\alpha}\xi_\beta) Q_\alpha$$

This shows that the covariant derivative of the field ξ_α coincides with our previous relation and that one can eliminate the Goldstone mode ξ_β from the effective lagrangian.

Conclusions

- We have discussed the counting of NGB's for

  1. Non relativistic theories

  2. Breaking of space-time symmetries

- In case 1, the number of NG fields coincide with the number of broken charges, but the number of NG fields do not coincide with the number of physical NGB's. This relation depends on the dispersion relation

- In case 2, the number of independent NGB's depend on the specific space-time symmetry and on its breaking. A way to find out the independent gapless modes is shown

$\mathbf{O(3)}$-symmetric scalar field

Since the model at $\mu = 0$ is $O(3)$ symmetric is not possible to introduce a chemical potential without breaking explicitly the symmetry. Picking up the conserved current

$$j_\mu^3 = \phi_i \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1$$

one gets

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} m^2 \vec{\phi} \cdot \vec{\phi} - \frac{\lambda}{4}(|\vec{\phi}|^2)^2$$

$$+\frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) + \mu j_0^3$$

The symmetry is now $O(2)$ with the following possibilities:

- $m^2 > 0$

  1. $\mu < m$. The symmetry is unbroken and $\langle \vec{\phi} \rangle_0 = 0$
  2. $\mu > m$. The symmetry is broken and $\langle \phi_1^2 + \phi_2^2 \rangle_0 = (\mu^2 - m^2)/\lambda$, $\langle \phi_3 \rangle_0 = 0$

- $m^2 < 0$. As in 2. the symmetry is broken and $\langle \phi_1^2 + \phi_2^2 \rangle_0 = (\mu^2 - m^2)/\lambda$, $\langle \phi_3 \rangle_0 = 0$

One can show that the case $\langle\phi_3\rangle \neq 0$ is never a minimum of the potential. In fact, in this situation the fields ϕ_1 and ϕ_2, which are the goldstones at zero μ, would acquire a quadratic term in the potential given by $-\mu^2(\phi_1^2 + \phi_2^2)/2$ making the theory unstable. The conclusion is that this model shows the same features of the case of the complex field.

Conformal theories

Conformal group ($D > 2$) (B. Zumino, Brandeis lectures, (1970)): Poincaré + dilatation + proper conformal transformations

$$\text{Dilatation}: \quad x^\mu \rightarrow x^{\mu\prime} = \lambda x^\mu$$

$$\text{Conformal}: \quad x^\mu \rightarrow x^{\mu\prime} = \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2 x^2}$$

$$\#\ \text{generators} = D + \frac{D(D-1)}{2} + 1 + D =$$

$$\frac{(D+2)(D+1)}{2} = \#\ \text{generators of } O(D+1,1)$$

Under infinitesimal transformations ($\lambda = 1 + \epsilon$):

$$\text{Dilatation}: \quad \delta x^\mu = \epsilon x^\mu,$$

$$\delta x^2 = 2\epsilon x^2$$

$$\text{Conformal}: \quad \delta x^\mu = a^\mu x^2 - 2x^\mu a\cdot x,$$

$$\delta x^2 = -2a\cdot x\, x^2$$

For $x^2 = 0 \Rightarrow \delta x^2 = 0 \Rightarrow$ **light-cone invariant**

Conformal field transformations:

Dilatation : $\delta\phi = (\delta x^\mu \partial_\mu + d\epsilon)\phi$

Conformal : $\delta\phi = (\delta x^\mu \partial_\mu - 2d a\cdot x - 2a^\mu x^\nu \Sigma_{\mu\nu})\phi$

For canonical fields:

	scalar	spinor	vector
d	1	3/2	1
$(\Sigma_{\mu\nu})_i^j$	0	$-\frac{i}{2}(\sigma_{\mu\nu})_i^j$	$\eta_{\mu i}\delta_\nu^j - \eta_{\nu i}\delta_\mu^j$

Einstein transformations

Introduce an external gravitational field $g_{\mu\nu}$ and write the action in a generally covariant form. Under coordinate transformations

$$\delta x^\mu = x^{\mu\prime} - x^\mu = \xi^\mu(x)$$

$$\delta g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\lambda\mu}$$

$$\delta\phi = \xi^\lambda \partial_\lambda \phi \qquad \text{(scalar field)}$$

Weyl transformations

$$\delta g_{\mu\nu} = 2\Lambda(x) g_{\mu\nu}, \quad \delta\phi = -\Lambda(x)\phi \quad \text{(scalar field)}$$

Theorem: If the action, S, is invariant under Einstein + Weyl then S, with $g_{\mu\nu} = \eta_{\mu\nu}$, is conformal invariant.

Dilatation: Choose $\xi^\lambda = \epsilon x^\lambda$

$$\delta g_{\mu\nu} = \epsilon(x^\lambda \partial_\lambda g_{\mu\nu} + 2g_{\mu\nu}), \quad \delta\phi = \epsilon x^\lambda \partial_\lambda \phi$$

Then $\Lambda = -\epsilon$

$$\delta g_{\mu\nu} = -2\epsilon g_{\mu\nu}, \quad \delta\phi = \epsilon\phi$$

Combining E+W

$$\delta g_{\mu\nu} = \epsilon x^\lambda \partial_\lambda \, g_{\mu\nu}, \quad \delta\phi = \epsilon(x^\lambda \partial_\lambda + 1)\phi$$

For $g_{\mu\nu} = \eta_{\mu\nu}$

$$\delta g_{\mu\nu} = 0, \quad \delta\phi = \epsilon(x^\lambda \partial_\lambda + 1)\phi$$

we recover the dilatation transformation. The same is true for proper conformal transformations with the choice

$$\xi^\lambda = a^\lambda x^2 - 2x^\lambda a \cdot x, \quad \Lambda = 2a \cdot x$$

Construction of the action through E+W simpler than using conformal invariance

Example: Consider a mass term for a scalar field

$$-\frac{1}{2}\int d^4x\, m^2\phi^2$$

Make it E-invariant

$$-\frac{1}{2}\int d^4x\,\sqrt{|g|}\,m^2\phi^2$$

To make it W-invariant consider a scalar field

$$\chi = \frac{1}{f}e^{f\,\sigma}, \quad \mathrm{dim}\,[f] = -1$$

Under W-transformations we get

$$\delta\sigma = -\Lambda/f$$

If we assume $\langle\sigma\rangle_0 = 0$ the σ-field behaves as a Goldstone boson breaking W-invariance, since $\langle e^{f\sigma}\rangle_0 \neq 0$. Then

$$-\frac{1}{2}\int d^4x\,\sqrt{|g|}\,m^2\phi^2\,e^{2f\,\sigma}$$

is E+W-invariant

$\sqrt{|g|} \to e^{4\Lambda}\sqrt{|g|},\ \phi^2 \to e^{-2\Lambda}\phi^2,\ e^{2f\,\sigma} \to e^{-2\Lambda}e^{2f\,\sigma}$

Therefore

$$-\frac{1}{2}\int d^4x\, m^2\phi^2\,e^{2f\,\sigma}$$

is conformal invariant.

Theorem:

If the action $S(g_{\mu\nu}, \phi)$ is E-invariant, then

$$S\left(g_{\mu\nu}\, e^{2f\,\sigma}, \phi\, e^{-f\,\sigma}\right)$$

is W-invariant. On the other side, any action $S(g_{\mu\nu}, \phi, \sigma)$, E+W-invariant, is necessarily of the previous form, since

$$S\left(g_{\mu\nu}, \phi, \sigma\right) = S\left(g_{\mu\nu}\, e^{2\Lambda}, \phi\, e^{-\Lambda}, \sigma - \frac{\Lambda}{f}\right)$$

Then, choosing $\Lambda = f\,\sigma$ we get the previous expression.

Notice that in this way any term of S containing only σ is gauged away. **The kinetic term for the σ-field cannot be obtained in this way**. However, for any scalar field, χ, the kinetic term is conformal invariant. Therefore the kinetic term for the σ field will be assumed as

$$\frac{1}{2f^2}\int d^4x\, \partial_\mu\left(e^{f\,\sigma}\right)\partial^\mu\left(e^{f\,\sigma}\right) = \frac{1}{2}\int d^4x\, e^{2f\,\sigma}\partial_\mu\sigma\partial^\mu\sigma$$

Summarizing, to get an action, S, invariant under the conformal group:

- Introduce $g_{\mu\nu}$ and make S E-invariant
- Introduce σ and make S W-invariant
- Restrict S to the flat space $g_{\mu\nu} \to \eta_{\mu\nu}$
- Add the kinetic term for the σ field

The resulting theory breaks spontaneously the conformal symmetry in the vacuum $\langle\sigma\rangle_0 = 0$. For instance,

$$-\frac{1}{2}\int d^4x\phi^2 e^{2f\,\sigma} \to -\frac{1}{2}m^2\int d^4x\phi^2$$

Notice that a single NGB, σ, is necessary to break the conformal symmetry, whereas there are $D+1$ broken generators.

Notice about the W-invariance that one could have thought to introduce a vector field χ_μ such to compensate the variation of the derivative terms. This could have been thought both as a gauge field or as the Goldstone field associated to the proper conformal transformations

$$\delta\chi_\mu \propto \partial_\mu \Lambda$$

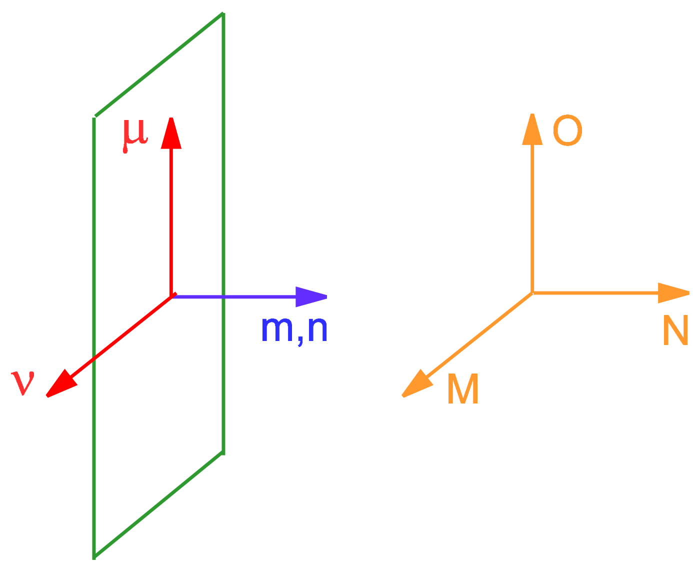
We have shown that such a role is played by $\partial_\mu\sigma$, suggesting a relation of the type

$$\partial_\mu\sigma \propto \chi_\mu$$

Both in this case and in the previous one, a relation is obtained between the derivatives of one Goldstone field along the direction of conserved momenta and other Goldstones. These relations have been formalized in a recent paper by I. Low and A. Manohar, hep-th/0110285.

Branes: We use the following conventions

$$M, N = 0, 1, \cdots, D, \text{ bulk space-time indices}$$
$$\mu, \nu = 1, \cdots, p, \quad p\text{-brane indices}$$
$$m, n = 0, 1, \cdots, D-p \quad \text{remaining indices}$$



The symmetry group is Poincaré in $D+1$ dimensions, and

Lie H $= (P_\mu, J_{\mu\nu}, J_{mn})$, Lie G-Lie H $= (P_m, J_{\mu m})$

The relevant commutator is

$$[P_\mu, J_{\nu m}] = i(\eta_{m\mu} P_\nu - \eta_{\nu\mu} P_m)$$

Therefore

$$\partial_\mu c_m = f_{\mu\alpha m} c_\alpha = -\eta_{\nu\mu} c_{\nu m} = -c_{\mu m}$$

Notice that the number of generators is

$$\#\,\text{Lie H} = p + \frac{p\,(p+1)}{2} + \frac{(D-p)(D-p-1)}{2}$$

$$\#\,\text{Lie G - Lie H} = D - p + p\,(D-p)$$

The only NGB are those associated to the broken translations.

The NGB's associated to a brane can be defined by the following choice of bulk coordinates for the brane. Let x^μ a point on the brane, then the bulk coordinates are defined as follows

$$Y^\mu(x^\mu) = x^\mu, \quad \text{brane coordinates}$$

$$Y^m(x^\mu), \quad \text{NGB fields}$$