Conditions for existence of neutral strange quark matter *

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July 28, 2005

Abstract

Breached pairing solutions to the gap equation are obtained analytically in for two and three quarks and for low and high temperatures. We compare the energy of these states to that of other homogeneous states under the condition of electric neutrality. We found the two-flavor BP and the three flavor mixed BCS-BP phases, which are stable over a wide range of parameters. Both phases contain four BP modes in the quasiparticle spectrum.

1 Introduction

Recently there has been considerable interest in a class of possible new states of matter featuring coexistence of superfluid and normal components. Examples have appeared in several variants, under various names (“Sarma state” [9], “interior gap” [4], “breached pair” [5] - which we adopt here, “gapless superconductor” [7]). Breached pair states are candidates to arise when there are interactions favoring pairing between fermions with fermi surfaces of different size. They are quite different from the famous LOFF phases [1], and need not break translation invariance. The basic ansatz for this class of states goes back to old work of Sarma [9]. There are two main causes for the recent upsurge in interest. First, new candidate applications have emerged. These include notably cold atom systems, where there can be great flexibility in manipulating densities, effective masses, and interactions [2]; and high-density QCD, relevant to neutron star interiors, as will be our concern below. Second, and importantly, parameter regimes have been identified where the new states are likely to be stable. General heuristic arguments, based on extrapolation from clear limiting cases (ultrastrong coupling, flat bands) were presented

*This work was completed in November, 2003
Quantitative comparisons between breached pairing and other homogeneous states, identifying favorable cases, have been presented in [5]. It has also been suggested [3] that inhomogeneous (phase separated superfluid and normal) states can compete favorably with breached pair states. This is an interesting question, that receives further investigation. Our preliminary conclusion is that such phase separation can occur, but only in rather special parameter regimes. We shall not discuss it further here.

In this paper we shall compare the energy of breached pair to conventional BCS and normal state alternatives, in some models suggested by high density QCD. Our calculations may have application to neutron stars, but in view of the difficulty of interpreting astrophysical observations and the idealized nature of the models perhaps their main interest is methodological. We illustrate in this specific context how one establishes the (in)stability of breached pairing, and the influence of different physical conditions.

2 Breached pairing superconductor at zero temperature

Here we summarize properties of the breached pairing superconductor concentrating on the stability condition of the breached pairing phase. We present calculations in a toy model [5] that has a superfluid ground state of up-strange quark pairs. The relation of more realistic cases to this model is worked out in the Appendix.

We consider massive \( s \) and massless \( u \) quarks both being relativistic,

\[
\varepsilon_p^u = p - p_F^u, \quad \varepsilon_p^s = p - p_F^s
\]

where the fermi momenta are related to the chemical potentials as \( p_F^u = \mu_u \) and \( p_F^s = \sqrt{\mu_s^2 - m_s^2} \), and the fermi momentum for the \( s \)-quark is smaller than that for the \( u \)-quark, \( p_F^s < p_F^u \). We suppress color indices and postulate a a weak attractive interaction

\[
g(\psi_u \sigma_2 \psi_s)(\psi_u^\dagger \sigma_2 \psi_s^\dagger)\]

between the light and heavy species. In a basis of light particles and heavy holes the quadratic part of the action is

\[
S = \sum_p \left( \begin{array}{cc} \psi_{u\vec{p}}^\dagger & \psi_{s\vec{p}} \\ \psi_{u\vec{p}} & \psi_{s\vec{p}}^\dagger \end{array} \right) \left( \begin{array}{cc} -\varepsilon_p^u & \Delta \\ \Delta^* & \varepsilon_p^s \end{array} \right) \left( \begin{array}{c} \psi_{u\vec{p}} \\ \psi_{s\vec{p}}^\dagger \end{array} \right)
\]

where the gap parameter is defined as \( \Delta^* = g/V \sum_p \langle \psi_{u\vec{p}} \psi_{s\vec{p}}^\dagger \rangle_{BP} \) in a breached pairing (BP) superconducting ground state, and momentum sum is \( \sum_p = V \int d^3p/(2\pi)^3 \). Since the gap parameter \( \Delta \) is defined as a \( c \)-number, this action can be diagonalized. The quasiparticle energies are \( \delta p_F = \sqrt{\varepsilon_p^2 + \Delta^2} \), where \( \varepsilon_p = (\varepsilon_p^u + \varepsilon_p^s)/2 = p - p_F \) with the average Fermi momentum \( p_F = (p_F^u + p_F^s)/2 \) and the mismatch in Fermi momenta \( \delta p_F = (\varepsilon_p^s - \varepsilon_p^u)/2 = (p_F^u - p_F^s)/2 > 0 \). We obtain the following free energy density [5]

\[
\Omega_{BP} = \frac{\Delta^2}{g} + \int \frac{d^3p}{(2\pi)^3} \left( \varepsilon_p - \sqrt{\varepsilon_p^2 + \Delta^2} \right) - \int \frac{d^3p}{(2\pi)^3} \left( \delta p_F - \sqrt{\varepsilon_p^2 + \Delta^2} \right)
\]
where there is no pairing in the momentum range \( R = \{ |\varepsilon_p| \leq \sqrt{\delta p_F^2 - \Delta^2} \} \), which is singly occupied by \( u \)-quarks. The gap equation is found either by minimizing the free energy density or as a self-consistent condition on \( \Delta \)

\[
\frac{2\Delta}{g} = \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\sqrt{\varepsilon_p^2 + \Delta^2}} - \int_R \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\sqrt{\varepsilon_p^2 + \Delta^2}}
\]

(4)

Writing \( \int \frac{d^3p}{(2\pi)^3} = N(0) \int \omega d\varepsilon_p \), where the density of states is \( N(0) = \int \frac{d^3p}{(2\pi)^3} \delta(\varepsilon_p) \) and the UV cutoff is \( \omega \sim p_F \), we have

\[
\frac{1}{g N(0)} = \ln \left( \frac{2\omega}{\Delta} \right) - \ln \left( \frac{\delta p_F + \sqrt{\delta p_F^2 - \Delta^2}}{\Delta} \right)
\]

(5)

Introducing the BCS gap at zero Fermi momentum mismatch, \( \Delta_0 \), we find the breached pairing and the BCS solutions

\[
\Delta = [\Delta_0(2\delta p_F - \Delta_0)]^{1/2} \quad (\delta p_F > \Delta)
\]

\[
\Delta = \Delta_0 \quad (\delta p_F \leq \Delta)
\]

(6)

Stability of the breached pairing state depends on the physical conditions imposed on the system [5]. Solving these constraints leads to the dependence of the Fermi momentum mismatch after pairing on the gap parameter \( \delta p_F(\Delta) \), which is parametrized [5]

\[
\delta p_F(\Delta) = \frac{\delta p_F}{1 - \alpha^2 \frac{2\Delta^2}{\Delta_0^2 + \Delta^2}}
\]

(7)

with a constant \( \alpha^2 \) depending on a condition and \( \delta p_F \) is the mismatch before pairing. Substituting \( \delta p_F \to \delta p_F(\Delta) \) in the solution of the gap equation, Eq. (6), we obtain

\[
\Delta = \left[ \Delta_0(\Delta_0 - 2\delta p_F)/(2\alpha^2 - 1) \right]^{1/2}
\]

(8)

For \( \alpha^2 > 1/2 \) one has a stable breached pairing solution [5].

One can check stability of the BP state also directly calculating the condensation energy, which is obtained by integrating the gap equation, Eq. (4), over the gap parameter \( \Delta \), we obtain

\[
\Omega_{BP} - \Omega_N = \int_0^\Delta d\Delta' \left( -\frac{2\Delta'}{g} + \int \frac{d^3p}{(2\pi)^3} \frac{\Delta'}{\sqrt{\varepsilon_p^2 + \Delta^2}} - \int_R \frac{d^3p}{(2\pi)^3} \frac{\Delta'}{\sqrt{\varepsilon_p^2 + \Delta^2}} \right)
\]

(9)

where \( \Delta \) is a solution of the gap equation, \( \Omega_N \) is the energy density in the normal state. We use the gap equation to eliminate \( g \),

\[
\Omega_{BP} - \Omega_N = 2N(0) \int_0^\Delta \Delta' d\Delta' \left( \ln \left( \frac{\Delta'}{\Delta} \right) - \ln \left( \frac{\Delta'}{\Delta} \right) + \ln \left( \frac{\delta p_F(\Delta') + \sqrt{\delta p_F(\Delta')^2 - \Delta'^2}}{\delta p_F(\Delta) + \sqrt{\delta p_F(\Delta)^2 - \Delta^2}} \right) \right)
\]

(10)
where the first term gives the BCS condensation energy, \(-N(0)\Delta^2/2\), which is cancelled by the second term arising from integrating in the BP region \(R\). Integrating by parts, we find

\[
\Omega_{BP} - \Omega_N = -N(0) \int_0^\Delta \frac{\Delta' d\Delta'}{\sqrt{\delta p_F(\Delta')^2 - \Delta'^2}} \left[ \frac{d\delta p_F(\Delta')}{d\Delta'} - \frac{\Delta'}{\delta p_F(\Delta') + \sqrt{\delta p_F(\Delta')^2 - \Delta'^2}} \right]
\]

(11)

Using in Eq. (11) the parametrization \(\delta p_F(\Delta) = \delta p_F(1 + \alpha^2 \Delta^2 / 2\delta p_F^2)\), which is valid for small \(\Delta\) and is equivalent to Eq. (7), we find the condensation energy of the BP state in the leading order \(O(\Delta^2/\delta p_F^2)\)

\[
\Omega_{BP} - \Omega_N = -N(0) \frac{\Delta^4(2\alpha^2 - 1)}{8\delta p_F^2}
\]

(12)

that shows a stable state when \(\alpha^2 > 1/2\). In the third section, we calculate \(\alpha^2\) for different BP solutions under condition of the electric neutrality, checking stability of these solutions.

Differentiating the free energy density, Eq. (3), with respect to the quark chemical potentials, \(\mu_u\) and \(\mu_s\), we obtain the corresponding quark number densities. Namely, \(n_u + n_s = \partial \Omega_{BP}/\partial p_F\) and \(n_u - n_s = \partial \Omega_{BP}/\partial \delta p_F\), which are equal to

\[
n_u + n_s = \int \frac{d^3p}{(2\pi)^3} \left(1 - \frac{\varepsilon_p}{\varepsilon_p^2 + \Delta^2} + \frac{\varepsilon_p}{\varepsilon_p^2 + \Delta^2}\right), \quad n_u - n_s = \int \frac{d^3p}{R(2\pi)^3} 1
\]

(13)

Integrating, we obtain

\[
n_u + n_s = 2N(0) \frac{p_F^3}{3}, \quad n_u - n_s = 2N(0) \sqrt{\delta p_F(\Delta)^2 - \Delta^2}
\]

(14)

where the density of states is \(N(0) = p_F^2/\pi^2\). When \(\Delta = 0\), \(n_u = p_F^2/(3\pi^2)(p_F + 3\delta p_F) \approx p_F^3/(3\pi^2)\) and \(n_s = p_F^2/(3\pi^2)(p_F - 3\delta p_F) \approx p_F^3/(3\pi^2)\), since \(\delta p_F \ll p_F\), and we recover the free gas limit. This means that the correct way to construct the BP state is to first fill noninteracting quark states up to the corresponding Fermi momenta and to then pair that produces the condensation energy, \(\Omega_{cond}\). In contrast to that, in the BCS state we first fill free quark states up to the common Fermi momentum, \(p_F\), and then pair. Therefore, for the BCS state, we gain in the condensation energy, \(-N(0)\Delta_0^2/2\), but lose by bringing two Fermi surfaces together, \(N(0)\delta p_F^2\). However, the condensation energy of the BP state is parametrically smaller, \(\sim \Delta^4\) Eq. (12), than that of the BCS state.

### 3 Breached pairing at finite temperature

#### 3.1 Gap equation and quark densities

We generalize our model to the case of nonzero temperature. The partition function at finite \(T = 1/\beta\) is

\[
Z = \text{Tr} e^{H - \mu_u n_u - \mu_s n_s} = i \int D\psi_{\alpha,n}^\dagger(p) D\psi_{\alpha,n}(p) e^S
\]

(15)
where \( n_u, n_s \) are the number of \( u \) and \( s \)-quarks, and functional integral is performed in momentum-frequency and in flavor spaces, \( \prod_n \prod_p \prod_\alpha d\psi_{\alpha,n}^\dagger(p) d\psi_{\alpha,n}(p) \). Assuming a superconducting ansatz for the ground state, in a basis of light particles and heavy holes the quadratic part of the action takes the form

\[
S = \sum_n \sum_p i \left( \psi_{u,n}^\dagger(p) \psi_{s,n}(-p) \right) K_{\alpha\beta} \left( \psi_{u,n}(p) \psi_{s,n}^\dagger(-p) \right)
\]

\[
K_{\alpha\beta} = (i\beta) \left( \begin{array}{cc} i\omega_n - \varepsilon_n^u & \Delta \\ \Delta^* & i\omega_n + \varepsilon_n^s \end{array} \right)
\]

(16)

where the gap parameter is defined as \( \Delta^* = g/V \sum_n \sum_p \langle \psi_{u,n}^\dagger(p) \psi_{s,n}^\dagger(-p) \rangle_{BP} \). The momentum sum is \( \sum_p = V \int d^3p/(2\pi)^3 \) and the frequency sum involves \( \omega_n = (2n + 1)\pi T \). Integrating over Grassmann variables, we have \( Z = \det K \), where the determinant is carried out over the flavor indices and in momentum-frequency space. Employing the identity \( \ln \det K = \text{Tr} \ln K \), we find

\[
\ln Z = \sum_n \sum_p \ln \left[ \beta^2 \left( (\omega_n - i\delta\varepsilon_p)^2 + \varepsilon_p^2 + \Delta^2 \right) \right]
\]

(17)

where \( \delta\varepsilon_p = (\varepsilon_p^u - \varepsilon_p^s)/2 \) and \( \varepsilon_p = (\varepsilon_p^u + \varepsilon_p^s)/2 \). Since both positive and negative frequencies are summed over, Eq. (17) can be rewritten

\[
\ln Z = \frac{1}{2} \sum_n \sum_p \left( \ln \left[ \beta^2 \left( \omega_n^2 + (\sqrt{\varepsilon_p^2 + \Delta^2} + \delta\varepsilon_p)^2 \right) \right] + \ln \left[ \beta^2 \left( \omega_n^2 + (\sqrt{\varepsilon_p^2 + \Delta^2} - \delta\varepsilon_p)^2 \right) \right] \right)
\]

(18)

Following (14) we write

\[
\ln \left[ (2n + 1)^2\pi^2 + \beta^2(\omega_p \pm \delta\varepsilon_p)^2 \right] = \int_1^{\beta^2(\omega_p \pm \delta\varepsilon_p)^2} \frac{d\theta^2}{\theta^2 + (2n + 1)^2\pi^2} + \ln \left[ 1 + (2n + 1)^2\pi^2 \right]
\]

(19)

where \( \omega_p = \sqrt{\varepsilon_p^2 + \Delta^2} \). The sum over \( n \) may be carried out by using the summation formula (14), (15)

\[
\sum_{n=-\infty}^{\infty} \frac{1}{(2n + 1)^2\pi^2 + \theta^2} = \frac{1}{\theta} \left( \frac{1}{2} - \frac{1}{e^\theta + 1} \right)
\]

(20)

Integrating over \( \theta \), and dropping terms independent of \( \beta \) and \( \delta\varepsilon_p \), we finally obtain

\[
\ln Z = \frac{1}{2} V \int \frac{d^3p}{(2\pi)^3} \left\{ \beta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta\varepsilon_p) + \beta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta\varepsilon_p) \right\} + 2 \ln \left[ 1 + e^{-\beta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta\varepsilon_p)} \right] + 2 \ln \left[ 1 + e^{-\beta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta\varepsilon_p)} \right]
\]

(21)

The thermodynamic potential is defined as \( \Omega = -P = -T \ln Z/V \). Adding the contribution from the mean field potential, \( \Delta^2/g \), and the vacuum energy arising from the normal ordering, \( \varepsilon_p \), we obtain

\[
\Omega_{BP} = \frac{\Delta^2}{g} + \int \frac{d^3p}{(2\pi)^3} \left\{ \varepsilon_p - \sqrt{\varepsilon_p^2 + \Delta^2} \right\} - T \ln \left[ 1 + e^{-\beta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta_{PF})} \right] - T \ln \left[ 1 + e^{-\beta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta_{PF})} \right]
\]

(22)
where \( \varepsilon_p = p - p_F \), \( \delta \varepsilon_p = \delta p_F \), with \( p_F = (p^+_F + p^-_F)/2 \) and \( \delta p_F = (p^+_F - p^-_F)/2 > 0 \). The thermodynamic potential in the four-dimensional notations, Eq. (17), is

\[
\Omega_{BP} = \frac{\Delta^2}{g} + T \sum_n \int \frac{d^3p}{(2\pi)^3} \left\{ \varepsilon_p - \ln \left[ \beta^2 \left( (\omega_n - i\delta p_F)^2 + \varepsilon_p^2 + \Delta^2 \right) \right] \right\}
\]

(23)

where \( T \sum_n 1 = T \int_0^\beta d\tau = 1 \). Variation of the thermodynamic potential with respect to the gap parameter, \( \partial \Omega_{BP}/\partial \Delta = 0 \), gives the gap equation in the 3-d notations

\[
\frac{2\Delta}{g} = \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\sqrt{\varepsilon_p^2 + \Delta^2}} \left[ 1 - \frac{1}{e^{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta p_F)}} \right] - \frac{1}{e^{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta p_F)}} \]
\]

rewritten as

\[
\frac{2\Delta}{g} = \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\sqrt{\varepsilon_p^2 + \Delta^2}} \left[ \tanh \left( \frac{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta p_F)}{2} \right) \right] + \tanh \left( \frac{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta p_F)}{2} \right)
\]

(25)

and the gap equation in the 4-d notations

\[
\frac{2\Delta}{g} = T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{(\omega_n - i\delta p_F)^2 + \varepsilon_p^2 + \Delta^2}
\]

(26)

where \( \int \frac{d^3p}{(2\pi)^3} = N(0) \int_0^\infty d\varepsilon_p \), with the density of states \( N(0) = \int \frac{d^3p}{(2\pi)^3} \delta(\varepsilon_p) \) and the UV cutoff \( \omega \sim p_F \).

The poles of the anomalous propagator are located at \( p_0 = i\omega_n = \pm \sqrt{\varepsilon_p^2 + \Delta^2 - \delta p_F} \), which define the quasiparticle energies. They match energies at \( T = 0 \).

Differentiating the thermodynamic potential with respect to the quark chemical potentials, \( n_u + n_s = \partial \Omega_{BP}/\partial p_F \) and \( n_u - n_s = \partial \Omega_{BP}/\partial \delta p_F \), we obtain the quark number densities in the 3-d

\[
n_u + n_s = \int \frac{d^3p}{(2\pi)^3} \left[ 1 - \frac{\varepsilon_p}{\sqrt{\varepsilon_p^2 + \Delta^2}} \left( 1 - \frac{1}{e^{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta p_F)}} \right) - \frac{1}{e^{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta p_F)}} \right]
\]

\[
n_u - n_s = \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} - \delta p_F)}} - \frac{1}{e^{\beta(\sqrt{\varepsilon_p^2 + \Delta^2} + \delta p_F)}} \]
\]

(27)

and in the 4-d,

\[
n_u + n_s = 2T \sum_n \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2} \varepsilon_p \left( \frac{1}{(\omega_n - i\delta p_F)^2 + \varepsilon_p^2 + \Delta^2} \right) \right]
\]

\[
n_u - n_s = 2T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{i(\omega_n - i\delta p_F)}{(\omega_n - i\delta p_F)^2 + \varepsilon_p^2 + \Delta^2}
\]

(28)

Both the 3-d representations Eq. (24, 27) and the 4-d Eq. (26, 28) representations will be useful later.
3.2 Solving the gap equation

Small temperatures, $T ≪ T_c$.

We now examine how the size of the gap in the energy spectrum depends on the mismatch of the fermi momenta and on the temperature.

First consider the case of low temperatures, $T ≪ T_c$ where $T_c$ is the critical temperature for the BCS superconductor, and make a suitable expansion of the gap equation, Eq. (24). We have

$$1 = gN(0) \int_0^\omega \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \left[ 1 - \frac{1}{e^{(\sqrt{\varepsilon^2 + \Delta^2 + \delta p_F})/T} + 1} - \frac{1}{e^{(\sqrt{\varepsilon^2 + \Delta^2 - \delta p_F})/T} + 1} \right]$$

(29)

where $N(0) = p_{F}^2/(2\pi^2)$ is the density of states, $\varepsilon = p - p_F$, and $\omega$ is the ultraviolet cutoff. Introducing a dimensionless variable $x = \sqrt{\varepsilon^2 + \Delta^2}/\Delta$, we rewrite the gap equation as

$$\frac{1}{gN(0)} = \int_1^{\omega/\Delta} \frac{dx}{\sqrt{x^2 - 1}} \left[ 1 - \frac{1}{e^{(\Delta/T)(x+\delta p_F/\Delta)} + 1} - \frac{1}{e^{(\Delta/T)(x-\delta p_F/\Delta)} + 1} \right]$$

(30)

In the limit $\Delta/T \gg 1$ and when $\delta p_F > \Delta$, we have

$$\frac{1}{gN(0)} = \int_1^{\omega/\Delta} \frac{dx}{\sqrt{x^2 - 1}} - \int_1^{\infty} \frac{dx}{\sqrt{x^2 - 1}} \sum_{n=1}^\infty (-1)^{n+1} e^{-n(\Delta/T)(x+\delta p_F/\Delta)}$$

$$- \int_{\delta p_F/\Delta}^{\infty} \frac{dx}{\sqrt{x^2 - 1}} \sum_{n=1}^\infty (-1)^{n+1} e^{-n(\Delta/T)(x-\delta p_F/\Delta)}$$

$$- \int_1^{\delta p_F/\Delta} \frac{dx}{\sqrt{x^2 - 1}} \left( 1 - e^{(\Delta/T)(x-\delta p_F/\Delta)} + ... \right)$$

(31)

When $\delta p_F \leq \Delta$ the third integral is over the range $[1, \infty)$, and there is no fourth integral. Since the second and third integrals converge, we can set their upper limit equal to $\infty$.

For a not very large mismatch, $\delta p_F \sim \Delta$, we get

$$\ln \left( \frac{\Delta_0}{\Delta} \right) = 2 \sum_{n=1}^\infty (-1)^{n+1} \cosh \left( \frac{n\delta p_F}{T} \right) K_0 \left( \frac{n\Delta}{T} \right) + \ln \left( \frac{\delta p_F + \sqrt{\delta p_F^2 - \Delta^2}}{\Delta} \right)$$

(32)

when $\delta p_F > \Delta$, while the last term on the right-hand side is absent when $\delta p_F \leq \Delta$. Here, we introduced the BCS gap at zero temperature, $\Delta_0 = \Delta_0(T = 0)$, and we invoke the Bessel function $K_0(z) = \int_1^\infty e^{-zt}/\sqrt{t^2 - 1} dt$ [15].

Using the asymptotic expansion of Bessel functions at large arguments and solving Eq. (32) we find

$$\Delta(T) = [\Delta_0(T) \left( 2\delta p_F - \Delta_0(T) \right)]^{1/2} \quad (\delta p_F > \Delta)$$

$$\Delta_0(T) = \Delta_0 - \cosh \left( \frac{\delta p_F}{T} \right) \sqrt{2\pi T\Delta_0} \left( 1 - \frac{T}{8\Delta_0} \right) e^{-(\Delta_0/T)} \quad (\delta p_F \leq \Delta)$$

(33)

where $\Delta(T)$ is the BP and $\Delta_0(T)$ is the BCS solutions of the gap equation at low temperatures and finite Fermi momenta mismatch. At $\delta p_F = 0$ we recover the known dependence of the BCS gap on the temperature [13].
Next we calculate the condensation energy by integrating the gap equation, Eq. \([23]\), over the gap parameter. We use the gap equation to eliminate \(g\),

\[
\Omega_{BP} - \Omega_N = 2N(0) \int_0^\Delta \Delta' d\Delta' \left( \ln \left( \frac{\Delta'}{\Delta} \right) - \ln \left( \frac{\Delta'}{\Delta} \right) + \ln \left( \frac{\delta p_F(\Delta') + \sqrt{\delta p_F(\Delta')^2 - \Delta'^2}}{\delta p_F(\Delta)} \right) + \sqrt{\delta p_F(\Delta)^2 - \Delta^2} \right) + 2 \sum_{n=1}^\infty (-1)^{n+1} \left[ \cosh \left( \frac{n\delta p_F(\Delta')}{T} \right) K_0 \left( \frac{n\Delta'}{T} \right) - \cosh \left( \frac{n\delta p_F(\Delta)}{T} \right) K_0 \left( \frac{n\Delta}{T} \right) \right] \quad (34)
\]

Integrating by parts, we obtain

\[
\Omega_{BP} - \Omega_N = -N(0) \int_0^\Delta \frac{\Delta'^2 d\Delta'}{\sqrt{\delta p_F(\Delta')^2 - \Delta'^2}} \left[ \frac{d\delta p_F(\Delta')}{d\Delta'} - \frac{\Delta'}{\delta p_F(\Delta') + \sqrt{\delta p_F(\Delta')^2 - \Delta'^2}} \right] + N(0) \int_0^\Delta \Delta' d\Delta' \frac{2}{T} \sum_{n=1}^\infty (-1)^{n+1} \left[ \cosh \left( \frac{n\delta p_F(\Delta')}{T} \right) K_1 \left( \frac{n\Delta'}{T} \right) \right. \\
\left. - \sinh \left( \frac{n\delta p_F(\Delta')}{T} \right) K_0 \left( \frac{n\Delta'}{T} \right) \right] \left( \frac{d\delta p_F(\Delta')}{d\Delta'} \right) \quad (35)
\]

where we used the formula \(K_0(z) = -K_1(z)\). Using the parametrization \(\delta p_F(\Delta) = \delta p_F(1 + \alpha^2 \frac{\Delta^2}{2\delta p_F})\), we obtain in the leading order \(O(\Delta^2/\delta p_F^2)\)

\[
\Omega_{BP} - \Omega_N = -N(0) \frac{\Delta^4(2\alpha^2 - 1)}{8\delta p_F^2} + N(0) 2T^2 \sum_{n=1}^\infty (-1)^{n+1} \left[ \cosh \left( \frac{n\delta p_F}{T} \right) \right. \\
\left. \frac{1}{n^2} \int_0^{\Delta/T} K_1(x)x^2dx \right] - \frac{\alpha^2 T}{\delta p_F} \sinh \left( \frac{n\delta p_F}{T} \right) \frac{1}{n^3} \int_0^{\Delta/T} K_0(x)x^3dx \quad (36)
\]

To evaluate the integrals, we write \(\int_0^{\Delta/T} K_1(x)x^2dx = 2 - \int_0^{\Delta/T} K_1(x)x^2dx\), where in the remaining integral we use the asymptotic expansion of the function \(K_1\) (or equivalently we use the recursion relation for the Bessel functions \(\int x^{p+1}Z_p(x)dx = x^{p+1}Z_{p+1} \quad [1]\); and similar for the second integral. We obtain

\[
\int_0^{\Delta/T} K_1(x)x^2dx = 2 - \left( \frac{n\Delta}{T} \right)^2 K_2 \left( \frac{n\Delta}{T} \right)
\]
\[
\int_0^{\Delta/T} K_0(x)x^3dx = 4 - \left[ \left( \frac{n\Delta}{T} \right)^3 K_3 \left( \frac{n\Delta}{T} \right) - 2 \left( \frac{n\Delta}{T} \right)^2 K_2 \left( \frac{n\Delta}{T} \right) \right] \quad (37)
\]

where we used \(K_2(z) = \frac{2}{\pi} K_1(z) + K_0(z)\), \(\int_0^{\infty} x^\mu K_\nu(ax)dx = 2^{\mu-1}a^{-\mu-1}\Gamma \left( \frac{1+\mu+\nu}{2} \right) \Gamma \left( \frac{1+\mu-\nu}{2} \right) \quad [15]\). Since \(\Delta/T \gg 1\) at low temperatures, the Bessel functions for \(n = 1\) give the dominant contribution (we use the asymptotic expansion of the function \(K_\nu(z)\)). For the constant term in Eq. \([37]\), we need to calculate the sum in Eq. \([36]\). In the leading order \(T/\Delta \ll 1\), we obtain

\[
\Omega_{BP} - \Omega_N = -N(0) \frac{\Delta^4(2\alpha^2 - 1)}{8\delta p_F^2} + N(0) \left[ 2T^2 \left( -\text{Li}_2(-e^{-(\delta p_F/T)}) - \text{Li}_2(-e^{(\delta p_F/T)}) \right) \right]
\]

8
- cosh \left( \frac{\delta p_F}{T} \right) \sqrt{2\pi \Delta^2 T} \left( 1 + \frac{15 T}{8 \Delta} \right) e^{-(\Delta/T)} \\
\left. \right. - N(0) \frac{\alpha^2 T}{\delta p_F} \left[ 4T^2 \left( \text{Li}_3(-e^{-(\delta p_F/T)}) - \text{Li}_3(-e^{(\delta p_F/T)}) \right) \right] \\
- \sinh \left( \frac{\delta p_F}{T} \right) \left( \frac{\Delta}{T} \right) \sqrt{2\pi \Delta^2 T} \left( 1 + \frac{19 T}{8 \Delta} \right) e^{-(\Delta/T)} \right) 

(38)

where the polylogarithmic functions are given by \( \text{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n \), \( \frac{\partial}{\partial z} \text{Li}_n(z) = \frac{1}{n} \text{Li}_{n-1}(z) \), and \( \Delta \) is the BP solution of the gap equation given by Eq. (8) with \( \Delta_0 \) defined in Eq. (33). When \( \delta p_F = 0 \), terms in the first square bracket of Eq. (38) describe the temperature dependent part of the condensation energy in the BCS state [13].

\[ \Omega_{\text{BCS}} - \Omega_N = -N(0) \frac{\Delta^2}{2} + N(0) \left[ \frac{\pi^2 T^2}{\Delta^2} - \sqrt{2\pi \Delta^2 T} \left( 1 + \frac{15 T}{8 \Delta} \right) e^{-(\Delta_0/T)} \right] \]

(39)

where the term \( \sim T^2 \) is the negative of the principal term in the expansion of the free energy of the normal state in powers of \( T \); we used \( \sum_{k=1}^{\infty} (-1)^{k+1}/k^2 = \pi^2/12 \). In Eq. (38), terms dependent on the gap parameter contribute to the \( \Omega_{\text{BP}} \), while both cosh and sinh terms give comparable contributions in the BP state when \( T \ll \Delta \sim \delta p_F \). This means that the BP heat capacity differs from that of the BCS state and is modified by imposing different physical conditions.

**Large temperatures, \( T \sim T_c \).** To determine the behavior of the gap for the temperatures near the critical temperature \( T_c \), it is most convenient to start from the 4-d relation Eq. (26). Near \( T_c \) the size of the gap is small, and hence in Eq. (26), we can carry out an expansion in powers of \( \Delta^2/T^2 \ll 1 \)

\[ \frac{2}{gN(0)} = T \sum_{n=-\infty}^{\infty} \int_{-\omega}^{\omega} d\varepsilon \left[ \frac{1}{(\omega_n - i\delta p_F)^2 + \varepsilon^2} \right. \]

\[ - \left. \frac{\Delta^2}{((\omega_n - i\delta p_F)^2 + \varepsilon^2)^2} + \frac{\Delta^4}{((\omega_n - i\delta p_F)^2 + \varepsilon^2)^2} \right] \]

(40)

Interchanging the order of summation over the frequencies and integration over \( \varepsilon \) in the (convergent) terms of the right-hand side, we obtain

\[ \frac{1}{gN(0)} = \int_{-\omega}^{\omega} \frac{d\varepsilon}{\varepsilon} \left( 1 - \frac{1}{e^{(\varepsilon - \delta p_F)/T} + 1} - \frac{1}{e^{(\varepsilon + \delta p_F)/T} + 1} \right) \]

\[ - \frac{\Delta^2}{(\pi T)^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} + 3 \frac{\Delta^4}{4(\pi T)^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \]

To evaluate the integral, we split it into two pieces, \( \int_{-\omega}^{\omega} d\varepsilon = \int_{0}^{1} dx + \int_{1}^{\omega} dx \), where \( x = \varepsilon/T \), and we calculate each piece by doing necessary approximations in the corresponding region. Expressing the series appearing in Eq. (41) in terms of the Riemann zeta function, i.e. writing \( \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 2\zeta(2) - \frac{\pi^2}{2} \zeta(2) \), we find in the leading order

\[ \ln \left( \frac{T}{T_c} \right) = -2Ei(-1) \left[ 1 - \cosh \left( \frac{\delta p_F}{T} \right) \right] - \frac{7\zeta(3)}{8} \frac{\Delta^2}{(\pi T)^2} \]

(42)
where the exponential-integral function is \( \int_1^{\infty} \frac{e^{-\mu x}}{x} dx = -Ei(-\mu) \) and \( \zeta(3) \approx 1.202 \), and \( T_c \) is the critical temperature for the BCS gap at zero mismatch.

Thus, in the leading order, we find that the size of the gap near \( T_c \) is

\[
\Delta(T) = \Delta_0(T) = \pi T_c \sqrt{\frac{8}{7\zeta(3)}} \sqrt{1 - \frac{T}{T_c} - 2Ei(-1)} \left[ 1 - \cosh \left( \frac{\delta p_F}{T_c} \right) \right]
\]

\[
\approx 3.06 T_c \sqrt{1 - \frac{T}{T_c} + 0.44 \left[ 1 - \cosh \left( \frac{\delta p_F}{T_c} \right) \right]} \quad (\forall \delta p_F)
\]

for both the BP (\( \delta p_F > \Delta \)) and the BCS (\( \delta p_F \leq \Delta \)) pairing. At \( \delta p_F = 0 \) we reproduce the known temperature dependence of the BCS gap near \( T_c \). The behavior of the gap as a function of the Fermi momenta mismatch for low Eq. (33) and high Eq. (43) temperatures is shown in Fig.(1) and Fig.(2), correspondingly. To parametrize \( \Delta(T) \) we used the BCS parameters (at \( \delta p_F = 0 \)), the gap at zero temperature \( \Delta_0 \) and the critical temperature \( T_c \), which are related as \( T_c/\Delta_0 = e^{\gamma E}/\pi \approx 0.567 \) where \( \gamma_E \approx 0.577 \) is the Euler’s constant.

Integrating the gap equation Eq. (40) over the gap parameter, we find the condensation energy density at large temperatures

\[
\Omega_{BP} - \Omega_N = 2N(0) \int_0^{\Delta'} \Delta' d\Delta' \left( 2 (-Ei(-1)) \left[ \cosh \left( \frac{\delta p_F(\Delta')}{T} \right) - \cosh \left( \frac{\delta p_F(\Delta)}{T} \right) \right] \right)
\]

\[
+ \frac{7\zeta(3)\Delta'^2 - \Delta^2}{8(\pi T)^2}
\]

Integrating by parts, we obtain

\[
\Omega_{BP} - \Omega_N = -N(0) \frac{7\zeta(3)\Delta^4}{16(\pi T)^2}
\]

\[
- N(0) \int_0^\Delta \Delta^2 d\Delta' \left( -\frac{Ei(-1)}{T} \right) \sinh \left( \frac{\delta p_F(\Delta')}{T} \right) \left( \frac{d\delta p_F(\Delta')}{d\Delta'} \right)
\]

Parametrizing the Fermi momentum mismatch at large temperatures as \( \delta p_F(\Delta) = \delta p_F(1 + \beta^2 \frac{\Delta^2}{2T^2}) \), we find from Eq. (45)

\[
\Omega_{BP} - \Omega_N = -N(0) \frac{\Delta^4}{T^2} \left[ \frac{7\zeta(3)}{16\pi^2} + \frac{\beta^2(-Ei(-1))}{2} \frac{\delta p_F}{T} \sinh \left( \frac{\delta p_F}{T} \right) \right]
\]

\[
= -2p_F^2 T_c^2 \left[ \left( 1 - \frac{T}{T_c} \right)^2 \left( 1 + \beta^2(-Ei(-1)) \right) \frac{8\pi^2}{7\zeta(3)} \frac{\delta p_F}{T_c} \sinh \left( \frac{\delta p_F}{T_c} \right) \right]
\]

\[
+ \left( 1 - \frac{T}{T_c} \right) 4(-Ei(-1)) \left( 1 - \cosh \left( \frac{\delta p_F}{T_c} \right) \right)
\]

where we used the gap equation solution, Eq. (43), and kept the leading terms \( O(\delta p_F^2/T_c^2) \). When \( \delta p_F = 0 \) we reproduce the BCS free energy density, which is the leading term in Eq. (46) since \( \delta p_F \ll T_c \).
Next, we find the difference in the quark number densities in the breached pairing state at finite temperature. For low temperatures, \( \Delta / T \gg 1 \), it is convenient to use the 3-d formula, Eq. (27),

\[
n_u - n_s = 2N(0) \Delta \int_{-\Delta}^{\Delta} \frac{x}{\sqrt{x^2 - 1}} \frac{1}{e(\Delta/T)(x-\delta p_F/\Delta) + 1} - \frac{1}{e(\Delta/T)(x+\delta p_F/\Delta) + 1}
\]

(47)

where \( x = \sqrt{\varepsilon^2 + \Delta^2}/\Delta \), and \( N(0) \) is the density of states. As with the gap equation, expanding the exponentials in the integrand, we find in the leading order

\[
n_u - n_s = 2N(0) \left[ \sqrt{\delta p_F^2 - \Delta^2} + \sinh \left( \frac{\delta p_F}{T} \right) \sqrt{2\pi T \Delta_0} \left( 1 + \frac{3T}{8\Delta_0} \right) e^{-\left(\Delta_0/T\right)} \right]
\]

(48)

when \( \delta p_F > \Delta \). For \( n_u - n_s \) we used integral representation of the appropriate Bessel function, \( K_1(z) = -\frac{d}{dz}K_0(z) = \int_1^\infty te^{-zt}/\sqrt{t^2 - 1} \ dt \) [15], and its asymptotic behavior at large \( z \) (see formula above).

For temperatures near critical, we use the 4-d relation Eq. (28). As with the gap equation, expanding in powers of \( \Delta^2/T^2 \) and performing summation over \( n \) in the leading \( \Delta \)-independent term and interchanging the order of summation over the frequencies and integration over \( \varepsilon \) in the convergent terms, we obtain

\[
n_u - n_s = 2N(0) \left[ \int_0^{\infty} d\varepsilon \left( \frac{1}{e(\varepsilon-\delta p_F)/T + 1} - \frac{1}{e(\varepsilon+\delta p_F)/T + 1} \right) - \frac{\Delta^2}{(\pi T)} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \right] + \ldots
\]

(49)

Using the Riemann zeta function, we obtain in the leading order

\[
n_u - n_s = 2N(0) \left[ T \left\{ \frac{1}{2} \frac{\delta p_F}{T} + \frac{2}{e} \sinh \left( \frac{\delta p_F}{T} \right) \right\} - \frac{3\zeta(2)}{4} \frac{\Delta^2}{(\pi T)} \right]
\]

\[
\approx 2N(0) \left[ 0.50 \delta p_F + 0.74 T \sinh \left( \frac{\delta p_F}{T} \right) - 0.39 \frac{\Delta^2}{T} \right]
\]

(50)

where \( \zeta(2) = \pi^2/6 \). Sum of the quark number densities at nonzero temperature coincide with that at zero temperature, Eq. (14),

\[
n_u + n_s = 2N(0) \frac{p_F}{3}
\]

(51)

since the temperature dependent term in Eq. (27) is an odd function of \( \varepsilon \). Particle number density of a free gas gets the temperature correction, Eq. (39), for example

\[
n_u = \frac{p_F^2}{3\pi^2} \left( p_F^2 + \pi^2 T^2 \right)
\]

(52)

In the following section, we use the BP solutions of the gap equation at zero, Eq. (6), and nonzero temperatures, Eqs. (33,43), as well as the quark number densities at \( T = 0 \), Eq. (14), and at \( T \neq 0 \), Eqs. (48,50,51,52), to find the breached pairing states of the quark matter which are electrically neutral.
4 Breached pairing in electrically neutral quark matter

In this section we solve the gap equation(s) imposing condition of the electric neutrality. In addition, when we require the neutrality with respect to the color charges the energy minimum shifts slightly [7]. We therefore put the chemical potentials associated with color charges of the $U(1)_3 \times U(1)_8$ subgroup to zero, $\mu_3 = \mu_8 = 0$. This approximation does not change our results qualitatively.

In the color and flavor antisymmetric channel, the three flavor gap parameter is parametrized by three gaps, $\Delta_{ud}, \Delta_{us}, \Delta_{ds}$, (Appendix A). One minimizes the thermodynamic potential $\Omega$, 

$$\frac{\partial \Omega}{\partial \Delta_i} = 0, \quad \frac{\partial \Omega}{\partial \mu_e} = 0$$

with respect to the gap parameters $\Delta_i$ and the electric chemical potential $\mu_e$ associated with the electric $U(1)$ charge. This is equivalent to looking for the minimum of $\Omega$ in the $(\Delta_i, \mu_e)$ plane only along the electric neutrality line [7]. We however do not solve directly minimization problem for $\Omega$. Numerical minimization was pursued for two flavors in [7] and for three flavors in [8]. Here, we solve analytically the gap equations and the neutrality conditions and find the parameter space where they intersect.

Expressed through the quark number densities, electric neutrality requires

$$\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0,$$

or written explicitly for each pair, 

$$\left(\frac{2}{3}n_u^{(1)} - \frac{1}{3}n_d^{(2)} + \frac{2}{3}n_u^{(1)} - \frac{1}{3}n_s^{(3)} - \frac{1}{3}n_d^{(2)} - \frac{1}{3}n_s^{(3)}\right)$$

$$+ \left(\frac{2}{3}n_u^{(2)} - \frac{1}{3}n_d^{(1)} + \frac{2}{3}n_u^{(3)} - \frac{1}{3}n_s^{(1)} - \frac{1}{3}n_d^{(3)} - \frac{1}{3}n_s^{(2)}\right) - n_e = 0$$

(54)

where the upper index 1, 2, 3 denotes color red, green, blue, correspondingly, the first brackets contains quarks from the $3 \times 3$ block and the second bracket from the $2 \times 2$ blocks (Appendix A). The quark densities are defined by the quark Fermi momenta, given by $p_F^u = \mu - (2/3)\mu_e$, $p_F^d = \mu + (1/3)\mu_e$, $p_F^s = \mu + (1/3)\mu_e - m_s^2/2\mu$ where $\mu$ is the baryon chemical potential and $\mu_e$ is the electric chemical potential. For convenience we introduce the average and the mismatch in Fermi momenta for each pair, for example, for $p_{ud} = (p_F^u + p_F^d)/2$ and $\delta p_{ud} = (p_F^d - p_F^u)/2$ where $\delta p$ is chosen to be positive. Explicitly, we have

$$p_{ud} = \mu - \frac{\mu_e}{6}, \quad \delta p_{ud} = \frac{\mu_e}{2}$$

$$p_{us} = \mu - \frac{\mu_e}{6} - \frac{m_s^2}{4\mu}, \quad \delta p_{us} = \frac{m_s^2}{4\mu} - \frac{\mu_e}{2}$$

$$p_{ds} = \mu + \frac{\mu_e}{3} - \frac{m_s^2}{4\mu}, \quad \delta p_{ds} = \frac{m_s^2}{4\mu}$$

(55)

where the ordering of the Fermi momenta in the non-interacting electrically neutral quark medium is $[17] p_F^u < p_F^d < p_F^s$. In what follows we specify the quark number densities and solve the gap equations under condition of the electric neutrality for two and three flavors.
4.1 Two and three flavor stable quark matter at zero temperature

Three flavor mixed BCS-BP state. At asymptotically high densities three flavor quark matter is in the CFL phase. Going to lower densities, the effective strange quark mass increases making impossible to maintain the BCS pairing for all nine quarks (the CFL phase). The CFL phase becomes unstable when first the down-strange quarks break the BCS pairing at

\[ m_s^2/4\mu \geq \Delta_0, \]  

Eq. (6.55), where \( \Delta_0 \) is the CFL gap.

We find that, due to the small Fermi momentum of the strange quark, \( p_s^F \ll p_u^F < p_d^F \), up blue-strange red and down blue-strange green quarks participate in the breached pairing while up green-down red quarks pair according to the BCS mechanism. The remaining up red, down green and strange blue quarks form the BCS condensate. Out of total nine quasiparticle excitations, there are five BCS and four BP modes, leading to seven gapped and two gapless excitations (Appendix A). The system of gap equations for the BCS \( \Delta_{ud} \) and the BP \( \Delta_{us} \sim \Delta_{ds} \) gap parameters decouples in the \( \bar{3} \) channel and reduces to the two two-flavor gap equations (Appendix A). We therefore use the quark number densities and the stability criterion obtained for the two-flavor case in the previous sections.

When quarks participate in the BCS pairing, their Fermi momenta and corresponding densities are modified by the strong interactions in a way to equalize the densities of two species which form the Cooper pairs [6]. In our case, \( n_u^{(1)} = n_d^{(2)} = n_s^{(3)} \), meaning that up red, down green and strange blue quarks form an electrically neutral quark matter, \( \frac{2}{3}n_u^{(1)} - \frac{1}{3}n_d^{(2)} + \frac{2}{3}n_u^{(1)} - \frac{1}{3}n_s^{(3)} - \frac{1}{3}n_d^{(2)} - \frac{1}{3}n_s^{(3)} = 0 \). Therefore the electric neutrality condition, Eq. (54), including quarks of other colors is given by

\[
\left( \frac{2}{3}n_u - \frac{1}{3}n_d \right)_{BCS} + \left( \frac{2}{3}n_u - \frac{1}{3}n_s - \frac{1}{3}n_d - \frac{1}{3}n_s \right)_{BP} - n_e = 0 \quad (56)
\]

where \( n_e = \mu_e^3/(3\pi^2) \). Using Eq. (14) for the quark number densities, we obtain

\[
\frac{1}{3}(\mu - \mu_e)^3 + (\mu - \mu_e - m_s^2/4\mu)^2 \left[ \frac{1}{3}(\mu - \mu_e - m_s^2/4\mu)^2 + 3\sqrt{(m_s^2/4\mu - \mu_e/2)^2 - \Delta^2} \right]
\]

\[
-\frac{2}{3}(\mu + \mu_e - m_s^2/4\mu)^3 - \mu_e^3 = 0 \quad (57)
\]

where \( \Delta \equiv \Delta_{us} \). Since down and strange quarks carry the same electric charge, the breached region (with no pairing) from the \( \langle ds \rangle \) condensate does not contribute to the electric neutrality. The breached region from the \( \langle us \rangle \) condensate contribute to the excess of the positive electric charge, that already exists in the non-interacting quark matter where strange quarks are less abandon than up and down quarks at \( m_s \neq 0 \). Therefore a finite density of electrons in needed, \( \mu_e \neq 0 \), to satisfy the electric neutrality.

Since Eq. (57) includes the gap parameter \( \Delta_{us} \), we solve the neutrality condition with respect to \( \delta p_{us} = (m_s^2/4\mu - \mu_e/2) \) before \( (\Delta_{us} = 0) \) and after \( (\Delta_{us} \neq 0) \) pairing. Using the parametrization \( \delta p_{us}(\Delta_{us}) = \delta p_{us}(1 + \alpha^2 \Delta_{us}^2) \), we obtain in the leading order

\[
\alpha^2 = \frac{3(3 - 4R + x)^2}{45 + 271R^2 + 24x + 231x^2 - 2R(45 + 239x)} \quad (58)
\]
where \( x = \frac{\delta p_{us}}{\mu} \) is the numerical solution of the neutrality condition Eq. (57) before pairing at some fixed strange quark mass and \( R = \frac{m_s^2}{4\mu^2} \). According to the stability criteria, Eqs. (8,??), the BP solution is stable when \( \alpha^2 > 1/2 \). Fixing the baryon chemical potential \( \mu = 400 \text{MeV} \) and increasing the strange quark mass \( m_s = 150 - 327 \text{MeV} \), we obtain \( |p^u_F - p^d_F| = 14 - 56 \text{MeV} \) which grows and the coefficient in the stability criteria for the BP \( \Delta_{us} \) solution \( \alpha^2 = 0.59 - 0.50 \) which drops. Satisfying the condition of breaking the CFL state, \( \Delta_0 < m_s^2/4\mu \) with \( \Delta_0 = 20 \text{MeV} \), we have \( m_s > 179 \text{MeV} \). This means that in this range of parameters, 179 < \( m_s < 327 \text{MeV} \) and under the neutrality condition Eq. (57) the mixed BCS-BP phase for three flavors is stable, while for larger masses, \( m_s > 327 \text{MeV} \) it becomes unstable.

The similar phase, containing seven gapped and two gapless excitations, was obtained numerically by Alford, Kouvaris, Rajagopal [8], and was called by the authors the gapless CFL.

When the ordering of the Fermi momenta in the strongly interacting neutral quark matter follows the pattern \( p^s_F \ll p^u_F < p^d_F \), there is a hierarchy of scales between the gap parameters \( \Delta_{ds} < \Delta_{us} < \Delta_{ud} \). We might find phases containing six (when down and strange quarks do not pair) and eight BP modes. The phase with the maximum number (eight) of the BP modes requires the comparable Fermi momenta mismatches between different pairs and is the BP analog of the CFL phase. It seems, however, that these phases are not stable under condition of the electric neutrality at zero temperature.

**Two flavor BP state.** At large strange quark mass, only up and down quarks (up red with down green, and up green with down red) participate in the breached pairing, while strange quarks of all three colors and blue up and down quarks form the free gas. The electric neutrality condition, Eq. (54), is written

\[
2 \left( \frac{2}{3} n_u - \frac{1}{3} n_d \right)_{BP} + 2 \left( \frac{2}{3} n_u - \frac{1}{3} n_d - \frac{1}{3} n_s - n_e \right) = 0
\]

Using the quark number densities, we obtain

\[
2(\mu - \frac{\mu_e}{6})^2 \left[ \frac{1}{3}(\mu - \frac{\mu_e}{6}) - 3 \sqrt{\frac{\mu_e}{2}} - \Delta_{ud}^2 \right] + \frac{2}{3}(\mu - \frac{2\mu_e}{3})^3 - \frac{1}{3}(\mu + \frac{\mu_e}{3})^3
\]

\[
-(\mu + \frac{\mu_e}{3} - \frac{m_e^2}{2\mu})^3 - \mu_e^3 = 0
\]

where \( \Delta \equiv \Delta_{ud} \). We solve the neutrality condition, Eq. (60), with respect to \( \delta p_{ud} = \mu_e/2 \) before \( (\Delta_{ud} = 0) \) and after \( (\Delta_{ud} \neq 0) \) pairing. Using the parametrization \( \delta p_{ud}(\Delta_{ud}) = \delta p_{ud}(1 + \alpha^2 \frac{\Delta_{ud}^2}{2\delta p_{ud}^2}) \), we obtain in the leading order

\[
\alpha^2 = \frac{9 - 6x + x^2}{18 - 18x + 48x^2 + 12R^2 - 4R(3 + 2x)}
\]

and for the infinitely large \( m_s \) when the free gas of the strange quarks does not participate in the neutrality balance,

\[
\alpha^2 = \frac{27 - 18x + 3x^2}{45 - 66x + 140x^2}
\]
Increasing the strange quark mass $m_s = 200 - 400 \text{MeV}$, we obtain the electric chemical potential $\mu_e = 23 - 70 \text{MeV}$ which grows and the coefficient for the stability criterion is $\alpha^2 = 0.53 - 0.59$ which also grows. Satisfying $\Delta_0 < m_s^2/4\mu$ with $\Delta_0 = 50 \text{MeV}$, we have $m_s > 283 \text{MeV}$. For two flavors we need higher coupling ($\Delta_0$) than for three flavor case in order to satisfy the electric neutrality condition, i.e. the electric neutrality point $\mu_e(\Delta = 0)$ should be to the left from the gap equation point $\mu_e(\Delta = 0) = \Delta_0$ (see Fig. (1)). For $m_s \to \infty$, we obtain $\mu_e = 86 \text{MeV}$ and $\alpha^2 = 0.64$. Therefore the BP two-flavor phase is stable for $m_s > 283 \text{MeV}$, and there is no restriction on $m_s$ from above.

At $m_s \to \infty$, this phase was obtained numerically by Shovkovy, Huang at $T = 0$ [7] and $T \neq 0$ [11], [12] and was called by the authors the gapless 2SC.

### 4.2 Stability of the quark matter at nonzero temperature

Here we find the solution of the gap equation under the condition of the electric neutrality at nonzero temperature graphically. We use the gap equation solutions, Eqs. (33,43), and the quark number densities, Eqs. (48,50,51,52), at nonzero temperature.

#### Figure 1: Gap as a function of the Fermi momenta mismatch, $2\delta p_F$, at small temperatures.

The BCS and BP are the two branches of the gap equation solution, and N denotes the electric neutrality line. Left: Three flavor mixed BCS-BP state, $T = 2 \text{MeV}$, $\Delta_0 = 20 \text{MeV}$, $m_s = 260 \text{MeV}$. Right: Two flavor BP state, $T = 5 \text{MeV}$, $\Delta_0 = 50 \text{MeV}$, $m_s = 300 \text{MeV}$. In both cases $\mu = 400 \text{MeV}$.

*Three flavor mixed BCS-BP state at $T \neq 0$. As at zero temperature, we analyze the BP gaps $\Delta_{us} \sim \Delta_{ds}$ without reference to the BCS gaps formed by the other quarks (Appendix A). We specify the electric neutrality for the BCS-BP state, Eq. (56), at small.
temperatures

\[
\frac{1}{3} \left( \mu - \frac{\mu_e}{6} \right)^3 + \left( \mu - \frac{\mu_e}{6} - \frac{m_s^2}{4\mu} \right)^2 \left[ \frac{1}{3} \left( \mu - \frac{\mu_e}{6} - \frac{m_s^2}{4\mu} \right) + 3 \left( \frac{m_s^2}{4\mu} - \frac{\mu_e}{2} \right)^2 - \Delta^2 \right]
+ 3 \sinh \left( \frac{1}{T} \left( \frac{m_s^2}{4\mu} - \frac{\mu_e}{2} \right) \right) \sqrt{2\pi T \Delta_0 \, e^{-\left( \Delta_0/T \right)}} \right] - \frac{2}{3} \left( \mu + \frac{\mu_e}{3} - \frac{m_s^2}{4\mu} \right)^3 - \mu_e \left( \mu_e^2 + \pi^2 T^2 \right) = 0
\] (63)

and large temperatures,

\[
\frac{1}{3} \left( \mu - \frac{\mu_e}{6} \right)^3 + \left( \mu - \frac{\mu_e}{6} - \frac{m_s^2}{4\mu} \right)^2 \left[ \frac{1}{3} \left( \mu - \frac{\mu_e}{6} - \frac{m_s^2}{4\mu} \right) \right]
+ 3T \left\{ \frac{1}{2T} \left( \frac{m_s^2}{4\mu} - \frac{\mu_e}{2} \right) + \frac{2}{e} \sinh \left( \frac{1}{T} \left( \frac{m_s^2}{4\mu} - \frac{\mu_e}{2} \right) \right) \right\} - \frac{3\pi \Delta^2}{8} = 0
\] (64)

We solve the neutrality condition with respect to \( \Delta \) as a function of the Fermi momenta mismatch. In Fig.(1), the neutrality line first intersects the gap equation solution and then the border between the BP and the BCS states, given by \( \Delta = \delta p_F \). Quark matter is positively charged to the left from the neutrality line, and it is negatively charged to the right. In order to get an intersection between the gap equation solution and the neutrality line, the Fermi momenta mismatch in the neutrality condition for a free gas should satisfy \( \delta p_F(\Delta = 0) < \Delta_0 \) where \( \Delta_0 \) is the BCS gap. We therefore have stronger coupling for the two flavors (\( \Delta_0 = 50 \, MeV \)) than for the three flavors (\( \Delta_0 = 20 \, MeV \)). This agrees with the renormalization group scaling of the coupling constant as we go from high to lower densities. We obtain the electrically neutral solution in the range of strange quark masses which agrees with our stability analyses at \( T = 0 \).

Two flavor BP state at \( T \neq 0 \). As at zero temperature, we consider the BP gap \( \Delta_{ud} \). The electric neutrality for the two flavor BP state, Eq. (59), is written at small temperatures

\[
2 \left( \mu - \frac{\mu_e}{6} \right)^2 \left[ \frac{1}{3} \left( \mu - \frac{\mu_e}{6} \right) - 3 \left( \frac{\mu_e}{2} \right)^2 - \Delta^2 - 3 \sinh \left( \frac{\mu_e}{2T} \right) \sqrt{2\pi T \Delta_0 \, e^{-\left( \Delta_0/T \right)}} \right]
+ \frac{2}{3} \left( \mu - \frac{2\mu_e}{3} \right) \left( \left( \mu - \frac{2\mu_e}{3} \right)^2 + \pi^2 T^2 \right) - \frac{1}{3} \left( \mu + \frac{\mu_e}{3} \right) \left( \left( \mu + \frac{\mu_e}{3} \right)^2 + \pi^2 T^2 \right)
-(\mu + \frac{\mu_e}{3} - \frac{m_s^2}{2\mu}) \left( \left( \mu + \frac{\mu_e}{3} - \frac{m_s^2}{2\mu} \right)^2 + \pi^2 T^2 \right) - \mu_e \left( \mu_e^2 + \pi^2 T^2 \right) = 0
\] (65)

and at large temperatures,

\[
2 \left( \mu - \frac{\mu_e}{6} \right)^2 \left[ \frac{1}{3} \left( \mu - \frac{\mu_e}{6} \right) - 3T \left\{ \frac{\mu_e}{4T} + \frac{2}{e} \sinh \left( \frac{\mu_e}{2T} \right) \right\} \right] + \frac{3\pi \Delta^2}{8} = 0
\]
Figure 2: Gap as a function of the Fermi momenta mismatch, $2\delta p_F$, at large temperatures. Left: Three flavor mixed BCS-BP state, $T = 9 \, \text{MeV}$, $T_c = 11 \, \text{MeV}$, $m_s = 260 \, \text{MeV}$. Right: Two flavor BP state, $T = 24 \, \text{MeV}$, $T_c = 28 \, \text{MeV}$, $m_s = 300 \, \text{MeV}$. In both cases $\mu = 400 \, \text{MeV}$.

\[
\begin{align*}
+ &\frac{2}{3}(\mu - \frac{2\mu_e}{3})\left((\mu - \frac{2\mu_e}{3})^2 + \pi^2 T^2\right) - \frac{1}{3}(\mu + \frac{\mu_e}{3})\left((\mu + \frac{\mu_e}{3})^2 + \pi^2 T^2\right) \\
- &\left(\mu + \frac{\mu_e}{3} - \frac{m_s^2}{2\mu}\right)\left((\mu + \frac{\mu_e}{3} - \frac{m_s^2}{2\mu})^2 + \pi^2 T^2\right) - \mu_e\left(\mu_e^2 + \pi^2 T^2\right) = 0
\end{align*}
\]

In Fig.(2) we used the same set of parameters as in Fig.(1) (including $T_c = 0.567 \Delta_0$), but only increased the temperature. At temperatures near the critical one, the BCS and the BP solutions form one curve. We find that it is much simpler to satisfy the neutrality condition at high temperatures, that leads to a larger parameter space where the neutral BP phase is possible. High temperatures also open an opportunity for neutral phases containing more than four BP modes.

## 5 Conclusions

We analyzed the breached pairing superconductivity at zero and finite temperatures. As in the previous work [5], we found that the additional constraints and physical conditions are crucial for the stability of the BP phase. We found analytical expressions for the stability criteria, showing the parameter space where the BP phase is stable under a general constraint.

Imposing the condition of the electric neutrality, we found the two-flavor BP and the three flavor mixed BCS-BP phases, which are stable over a wide range of parameters. Both phases contain four BP modes in their quasiparticle spectrum. Fixing the chemical potential at $\mu = 400 \, \text{MeV}$ and increasing the effective strange quark mass, we found that the CFL breaks at $m_s \sim 179 \, \text{MeV}$, followed by BP phases including the mixed BCS-BP phase for $179 < m_s < 327 \, \text{MeV}$ and the two flavor BP phase for $283 < m_s$. Phases
containing more than four BP modes which preclude the two-flavor BP phase might be also possible. In the region of \( m_s \) where different BP phases overlap, a phase which is more energetically favorable wins.

At nonzero temperature we found solutions of the gap equation, which are consistent with the numerically found solutions in [9]. At low temperatures, there are two distinct branches, the BCS and the BP solutions, while there is only one curve for both solutions at high temperatures. It is much simpler to satisfy the neutrality condition at high temperatures, leading to a larger parameter space where the neutral BP phases are possible. High temperatures also open an opportunity for neutral phases containing more than four BP modes.

**Acknowledgements**

The author would like to thank Frank Wilczek for his insight and useful suggestions over the course of this work and for reading the manuscript; and Michael Forbes, Chris Kouvaris and Krishna Rajagopal for many helpful discussions. This work is supported in part by funds provided by the U.S. Department of Energy under cooperative research agreement DF-FC02-94ER40818.

**Appendix. Gap equations for two and three flavors**

In this appendix, we obtain the gap equations for diquark condensates containing two and three quark flavors. We use a toy model where the full interaction between quarks is replaced by a four-fermion interaction with the quantum numbers of a single-gluon exchange, \( L_{\text{int}} = G \int d^4x (\bar{\psi}(x) \gamma^\mu \lambda^A \gamma^\mu \psi(x) \gamma_A \gamma^\mu \lambda^B \gamma^\mu \psi(y)) \), where \( \alpha, \beta, \text{etc.} \) are color indices and \( i, j \) are the flavor indices, \( \lambda^A \) are the color \( SU(3) \) generators satisfying \( \lambda^A \lambda^A_{\alpha\beta} = \frac{2}{3} (\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\gamma} \delta_{\beta\delta}) \). We allow condensation in the channel \( \Delta_{ij}^\alpha = G \langle \psi_i^\alpha \gamma_5 \psi_j^\beta \rangle \) with the simplest color-flavor structure, suggested in [18], that interpolates between the color-flavor locking phase at \( M_s = 0 \) and the 2SC at large \( M_s \),

\[
\Delta_{ij}^{\alpha\gamma} = \begin{pmatrix}
 b + e & b & c \\
 b & b + e & c \\
 c & c & d \\
\end{pmatrix},
\]

(67)

where the basis vectors are \( (\alpha, i) = (1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2) \) with colors \((1, 2, 3)\) correspond to red, green, blue and flavors \((1, 2, 3)\) correspond to up, down, strange [18]. Detailed properties of the condensate ansatz are discussed in [18], in particular this condensate locks color and flavor. In the mean field, the four-fermion
interaction $L_{\text{int}}$ leads to an effective action

$$
(\psi^\dagger \psi) \left( \begin{array}{cc} p_0 I - E & Q \\ Q & p_0 I + E \end{array} \right) \left( \begin{array}{c} \psi \\ \psi^\dagger \end{array} \right) \tag{68}
$$

where $\psi$ is a 9 component color-flavor spinor; $I = \delta^{\alpha\beta}\delta_{ij}$, $E$ and $Q$ are color-flavor matrices, $E_{ij}^{\alpha\beta} = \delta^\alpha\beta(\varepsilon_{u}\delta_{ij1} + \varepsilon_{d}\delta_{ij2} + \varepsilon_{F}\delta_{ij3})$ and $Q_{ij}^{\alpha\gamma} = 3\Delta_{ij}^{\gamma\alpha} - \Delta_{ij}^{\alpha\gamma}$. $L_{\text{int}}$ generates also the condensate $(1/G)Q_{ij}^{\alpha\beta}\Delta_{ij}^{\alpha\beta}$, equal to

$$
-\frac{1}{G^2}\left( b^2 + 10be + e^2 + d^2 - 2(c^2 - 6cf + f^2) \right) \tag{69}
$$

Diagonalizing the effective action matrix, Eq. (68), we obtain the free energy as a sum of the quasiparticle energies and the condensate term. Variations of the free energy with respect to the gap parameters give the gap equations. Matrix $Q_{ij}^{\alpha\beta}$ is block-diagonal in the color-flavor space, that permits to diagonalize corresponding parts of the effective action separately. Unitary transforming part of the effective action for the indices $(1, 1), (2, 2), (3, 3)$, we obtain the $3 \times 3$ part of the condensate $Q_{ij}^{\alpha\beta}$ and the kinetic term $E_{ij}^{\alpha\gamma}$

$$
\begin{pmatrix}
3b - e & 0 & 0 \\
0 & b + 5e & -\sqrt{2}(c - 3f) \\
0 & -\sqrt{2}(c - 3f) & 2d
\end{pmatrix}, \begin{pmatrix}
\bar{\varepsilon} & -\delta\varepsilon & 0 \\
-\delta\varepsilon & \bar{\varepsilon} & 0 \\
0 & 0 & \varepsilon_s
\end{pmatrix} \tag{70}
$$

correspondingly, where $\bar{\varepsilon} = \frac{1}{2}(\varepsilon_u + \varepsilon_d)$ and $\delta\varepsilon = \frac{1}{2}(\varepsilon_u - \varepsilon_d)$. The 2 × 2 parts of the $Q_{ij}^{\alpha\beta}$ are off-diagonal containing matrix elements $(3b - e)$ for $(1, 2), (2, 1)$ indices, and $(3c - f)$ for $(1, 3), (3, 1)$ and $(2, 3), (3, 2)$ indices.

Mixed BCS-BP three flavor state. When $|\varepsilon_u - \varepsilon_d| \ll |\varepsilon_u - \varepsilon_s| \sim |\varepsilon_d - \varepsilon_s|$, i.e. $\delta\varepsilon \sim 0$, system of the eigenvalue equations for the $3 \times 3$-block, Eq. (68), decouples, producing two quasiparticle energies with the gap $(3b - e)$ and four quasiparticle energies satisfying

$$
\lambda^4 - \lambda^2 \left[ \bar{\varepsilon}^2 + \varepsilon_s^2 + (b + 5e)^2 + 4(c - 3f)^2 + 4d^2 \right] + \left( \bar{\varepsilon}^2 + (b + 5e)^2 \right) \left( \varepsilon_s^2 + 4d^2 \right) + 4(c - 3f)^2 \left( \bar{\varepsilon}\varepsilon_s - 2d(b + 5e) \right) + 4(c - 3f)^4 = 0 \tag{71}
$$

where the eigenvalue energy is $\lambda + p_0$. All quasiparticle energies with their degeneracies are

$$
p_0 \pm \sqrt{\varepsilon_u^2 + (3b - e)^2} \tag{1}
$$
$$
p_0 \pm \sqrt{\varepsilon_u^2 + \varepsilon_s^2 + (b + 5e)^2 + 4(c - 3f)^2 + 4d^2 \pm \sqrt{D}} \tag{2}
$$
$$
p_0 \pm \sqrt{\varepsilon_u^2 + (3b - e)^2} \tag{2}
$$
$$
p_0 + \frac{1}{2}(\varepsilon_u - \varepsilon_s) \pm \sqrt{\left( \frac{1}{2}(\varepsilon_u + \varepsilon_s) \right)^2 + (3c - f)^2} \tag{2}
$$
$$
p_0 - \frac{1}{2}(\varepsilon_u - \varepsilon_s) \pm \sqrt{\left( \frac{1}{2}(\varepsilon_u + \varepsilon_s) \right)^2 + (3c - f)^2} \tag{2}
$$

where

$$
D = \left[ \varepsilon_u^2 + \varepsilon_s^2 + (b + 5e)^2 + 4(c - 3f)^2 + 4d^2 \right]^2 \tag{73}
$$
$$
- 4 \left[ (\varepsilon_u^2 + (b + 5e)^2)(\varepsilon_s^2 + 4d^2) + 4(c - 3f)^2(\varepsilon_u\varepsilon_s - 2d(b + 5e)) + 4(c - 3f)^4 \right]
$$

We define $S_{\pm} = \frac{1}{\sqrt{2}} \sqrt{\varepsilon_u^2 + \varepsilon_s^2 + (b + 5e)^2 + 4(c - 3f)^2 + 4d^2 \pm \sqrt{D}}$. There are $(9) \times 2$ eigenvalues. In the CFL limit ($b = c$, $e = f$, $d = b + e$, $\varepsilon_u = \varepsilon_s$), these eigenvalues reduce to $(8)$ modes with the gap $\Delta_8 = 3b - e$ and $(1)$ mode with the gap $\Delta_1 = 8e$, manifesting the $SU(3)_V$ symmetry where $V = color + flavor$. Strange quark mass breaks the $SU(3)_V$ and $8$ modes split into isomultiplets, $8 = 3 + 2 + 2 + 1$. In the $\bar{8}$ channel these quasiparticle energies were obtained in [16]. Varying the free energy, given by the sum of quasiparticle and $S$ modes split into isomultiplets, $S = 3 + 2 + 2 + 1$. The $\bar{8}$ block reads

$$\frac{1}{G} b + 2 \int \frac{d^3p}{(2\pi)^3} \frac{b}{\sqrt{\varepsilon_u^2 + (4b)^2}} = 0$$

$$\frac{1}{G} c + 2 \int \frac{d^3p}{(2\pi)^3} \frac{c}{\sqrt{\left(\frac{1}{2}(\varepsilon_u + \varepsilon_s)\right)^2 + (4c)^2}} = 0$$  \hspace{1cm} (74)

describing the BCS gap $\Delta_{ud} = 4b$ and the breached pairing gap $\Delta_{us} \sim \Delta_{ds} = 4c$, where $Q = \{ |\frac{1}{2}(\varepsilon_u + \varepsilon_s)| > \sqrt{\left(\frac{1}{2}(\varepsilon_u - \varepsilon_s)\right)^2 - (4c)^2} \}$ is the BP momentum integration area. Eq. (74) contains the two-flavor gap equations, Eq. (1), with $G = g/4$.

**BP two flavor state.** When $\delta \varepsilon \neq 0$ and $\delta \varepsilon \ll |\bar{\varepsilon} - \varepsilon_s|$, the gap parameters $c = f = d = 0$, and the eigenvalue equation for the $3 \times 3$ block reads

$$\lambda^4 - 2\lambda^2 \left( \varepsilon^2 + \frac{1}{2}((3b - e)^2 + (b + 5e)^2) + \delta \varepsilon^2 \right) + \left( \varepsilon^2 + \frac{1}{2}((3b - e)^2 + (b + 5e)^2) - \delta \varepsilon^2 \right)^2$$

$$+ \left[ \delta \varepsilon^2 ((3b - e) + (b + 5e))^2 - \frac{1}{4} ((3b - e)^2 - (b + 5e)^2)^2 \right] = 0$$  \hspace{1cm} (75)

Solving this equation we omit terms in the square bracket, since they do not contribute to the gap equation in the $3$ channel ($e = -b$). The quasiparticle energies are

$$p_0 + \delta \varepsilon \pm \sqrt{\varepsilon^2 + \frac{1}{4}((3b - e)^2 + (b + 5e)^2)}$$

$$p_0 - \delta \varepsilon \pm \sqrt{\varepsilon^2 + \frac{1}{4}((3b - e)^2 + (b + 5e)^2)}$$

$$p_0 + \delta \varepsilon \pm \sqrt{\varepsilon^2 + (3b - e)^2}$$

$$p_0 - \delta \varepsilon \pm \sqrt{\varepsilon^2 + (3b - e)^2}$$  \hspace{1cm} (76)

which are $(4) \times 2$ modes. Condensate is $\frac{1}{G} 2(b^2 + 10be + e^2)$. Combining, $3 \frac{\partial H}{\partial b} + \frac{\partial H}{\partial c} = 0$, and taking the $3$ channel, we obtain

$$\frac{1}{G} b + 2 \int Q \frac{d^3p}{(2\pi)^3} \frac{b}{\sqrt{\varepsilon^2 + (4b)^2}} = 0$$  \hspace{1cm} (77)

describing the breached pairing gap $\Delta_{ud} = 4b$, where $Q = \{ |\bar{\varepsilon}| > \sqrt{\delta \varepsilon^2 - (4b)^2} \}$ is the BP momentum integration area.
References


