Nuclear Physics B228 (1983) 552–566

STATIC PROPERTIES OF NUCLEONS IN THE SKYRME MODEL

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Received 20 June 1983

We compute static properties of baryons in an SU(2) × SU(2) chiral theory (the Skyrme model) whose solitons can be interpreted as the baryons of QCD. Our results are generally within about 30% of experimental values. We also derive some relations that hold generally in soliton models of baryons, and therefore, serve as tests of the $1/N$ expansion.

1. Introduction

Recent developments have provided partial confirmation of Skyrme's old idea [1] that baryons are solitons in the non-linear sigma model. We know that in the large-N limit, QCD becomes equivalent to an effective field theory of mesons [2]. Counting rules suggest [3] that baryons may emerge as solitons in this theory. Although we do not understand in detail the large-N theory of mesons, we know that at low energies this theory reduces to a non-linear sigma model of spontaneously broken chiral symmetry. Moreover, the solitons of the non-linear model have precisely the quantum numbers of QCD baryons [4] provided one includes the effects of the Wess–Zumino coupling [5, 6].

In this paper we will evaluate the static properties of nucleons such as masses, magnetic moments, and charge radii, in a soliton model. For simplicity we will restrict ourselves to the case of two flavors. One simplification in the SU(2) case is that the Wess–Zumino term vanishes. At a pedestrian level, for $U = 1 + iA + O(A^2)$, the Wess–Zumino term is [5, 6]

$$n\Gamma = n\frac{2}{15\pi^2F_\pi^5}\int d^4x\, \varepsilon^{\mu\nu\alpha\beta}\, \mathrm{Tr}\,[A\partial_\mu A\partial_\nu A\partial_\alpha A\partial_\beta A] + \text{higher orders}\,.$$

If $A = a_a\tau_a$, then

$$n\Gamma = n\frac{2}{15\pi^2F_\pi^5}\int d^4x\, \varepsilon^{\mu\nu\alpha\beta} a_a\partial_\mu a_b\partial_\nu a_c\partial_\alpha a_d\partial_\beta a_e\, \mathrm{Tr}\,[\tau_a\tau_b\tau_c\tau_d\tau_e]\,.$$

[1] Supported in part by NSF grant no. PHY80-19754.

This term is completely antisymmetrical in the Lorentz indices, so it needs to be completely antisymmetrical in the isospin indices *b*, *c*, *d* and *e*. But that is impossible because there are only three independent SU(2) generators. More generally, the fifth-rank antisymmetric tensor ω discussed in [6] vanishes on the SU(2) group manifold. Nonetheless, the anomalous baryon-number current, which can be obtained from the WZ term or by the method of Goldstone and Wilczek [7], is still present in the two-flavor case.

Since the proper large-N effective theory is unknown, we will consider here a crude description in which the large-N theory is assumed to be a theory of pions only. In this context, it is necessary to add a non-minimal term to the non-linear sigma model to prevent the solitons from shrinking to zero-size. The simplest reasonable choice is the Skyrme model

$$L = \tfrac{1}{16}F_\pi^2 \operatorname{Tr}(\partial_\mu U \partial_\mu U^\dagger) + \frac{1}{32e^2}\operatorname{Tr}[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2 . \tag{1}$$

Here U is an SU(2) matrix, transforming as $U \to AUB^{-1}$ under chiral SU(2)× SU(2); $F_\pi = 186$ MeV is the pion decay constant; and the last term, which contains the dimensionless parameter e, was introduced by Skyrme to stabilize the solitons. It is the unique term with four derivatives which leads to a positive hamiltonian. (It is also the unique term with four derivatives that leads to a hamiltonian second order in time derivatives.) Although the Skyrme model is only a rough description, since it omits the other mesons and interactions that are present in the large-N limit of QCD, we regard it as a good model for testing the reasonableness of a soliton desciption of nucleons.

1. Kinematics

From the lagrangian (1) we find the soliton solution by using the Skyrme ansatz $U_0(x) = \exp[iF(r)\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}]$, where $F(r) = \pi$ at $r = 0$ and $F(r) \to 0$ as $r \to \infty$. If we substitute this ansatz in (1) we get the expression for the soliton mass:

$$M = 4\pi \int_0^\infty r^2 \left\{ \tfrac{1}{8}F_\pi^2 \left[\left(\frac{\partial F}{\partial r}\right)^2 + 2\frac{\sin^2 F}{r^2} \right] + \frac{1}{2e^2}\frac{\sin^2 F}{r^2}\left[\frac{\sin^2 F}{r^2} + 2\left(\frac{\partial F}{\partial r}\right)^2 \right] \right\} dr . \tag{2}$$

The variational equation from (2) is

$$(\tfrac{1}{4}\tilde{r}^2 + 2\sin^2 F)F'' + \tfrac{1}{2}\tilde{r}F' + \sin 2F F'^2 - \tfrac{1}{4}\sin 2F - \frac{\sin^2 F \sin 2F}{\tilde{r}^2} = 0 .$$

in terms of a dimensionless variable $\tilde{r} = eF_\pi r$. The behaviour of the numerical solution of eq. (3) is shown in fig. 1.

Now, if $U_0 = \exp[iF(r)\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}]$ is the soliton solution, then $U = AU_0A^{-1}$, where A is an arbitrary constant SU(2) matrix, is a finite-energy solution as well. A solution

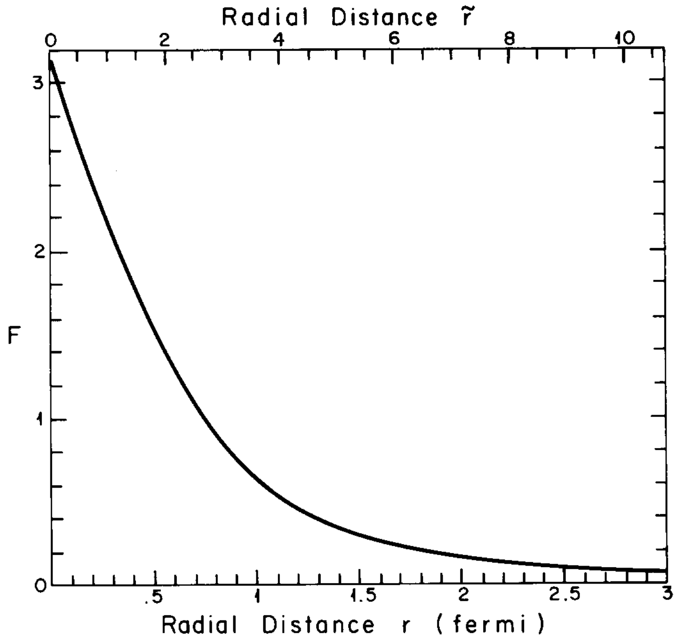

Fig. 1. Plot of F, the numerical solution of eq. (3). F appears in the Skyrme ansatz $U_0(x)=\exp[iF(r)\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}]$. The radial distance is measured in fm, and also in the dimensionless variable $\tilde{r}=eF_\pi r$.

of any given A is not an eigenstate of spin and isospin. We need to treat A as a quantum mechanical variable, as a collective coordinate. The simplest way to do this is to write the lagrangian and all physical observables in terms of a time dependent A. We substitute $U=A(t)U_0A^{-1}(t)$ in the lagrangian, where U_0 is the soliton solution and $A(t)$ is an arbitrary time-dependent SU(2) matrix. This procedure will allow us to write a hamiltonian which we will diagonalize. The eigenstates with the proper spin and isospin will correspond to the nucleon and delta.

So, substituting $U=A(t)U_0A^{-1}(t)$ in (1), after a lengthy calculation, we get

$$L=-M+\lambda\,\mathrm{Tr}\,[\partial_0 A\partial_0 A^{-1}]\,, \tag{4}$$

where M is defined in (2) and $\lambda=\frac{4}{6}\pi(1/e^3F_\pi)\Lambda$ with

$$\Lambda=\int \tilde{r}^2\sin^2 F\left[1+4\left(F'^2+\frac{\sin^2 F}{\tilde{r}^2}\right)\right]\mathrm{d}\tilde{r}\,. \tag{5}$$

Numerically we find $\Lambda=50.9$. The SU(2) matrix A can be written $A=a_0+i\boldsymbol{a}\cdot\boldsymbol{\tau}$, with $a_0^2+\boldsymbol{a}^2=1$. In terms of the a's (4) becomes

$$L=-M+2\lambda\sum_{i=0}^{3}(\dot{a}_i)^2\,.$$

Introducing the conjugate momenta $\pi_i = \partial L/\partial \dot{a}_i = 4\lambda \dot{a}_i$, we can now write the hamiltonian

$$H = \pi_i \dot{a}_i - L = 4\lambda \dot{a}_i \dot{a}_i - L = M + 2\lambda \dot{a}_i \dot{a}_i = M + \frac{1}{8\lambda} \sum_i \pi_i^2 .$$

Performing the usual canonical quantization procedure $\pi_i = -i\partial/\partial a_i$ we get

$$H = M + \frac{1}{8\lambda} \sum_{i=0}^{3} \left(-\frac{\partial^2}{\partial a_i^2} \right) , \tag{6}$$

with the constraint $\sum_{i=0}^{3} a_i^2 = 1$. Because of this constraint, the operator $\sum_{i=0}^{3} \partial^2/\partial a_i^2$ is to be interpreted as the laplacian ∇^2 on the three-sphere. The wave functions (by analogy with usual spherical harmonics) are traceless symmetric polynomials in the a_i. A typical example is $(a_0 + ia_1)^l$, with $-\nabla^2(a_0 + ia_1)^l = l(l+2)(a_0 + ia_1)^l$. Such a wave function has spin and isospin equal to $\frac{1}{2}l$, as one may see by considering the spin and isospin operators

$$I_k = \tfrac{1}{2} i \left(a_0 \frac{\partial}{\partial a_k} - a_k \frac{\partial}{\partial a_0} - \varepsilon_{klm} a_l \frac{\partial}{\partial a_m} \right) ,$$

$$J_k = \tfrac{1}{2} i \left(a_k \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_k} - \varepsilon_{klm} a_l \frac{\partial}{\partial a_m} \right) . \tag{7}$$

An important physical point must be addressed here. Since the nonlinear sigma model field is $U = AU_0A^{-1}$, A and $-A$ correspond to the same U. Naively, one might expect to insist that the wave function $\psi(A)$ obeys $\psi(A) = +\psi(-A)$. Actually, as discussed long ago by Finkelstein and Rubinstein [8], there are two consistent ways to quantize the soliton; one may require $\psi(A) = +\psi(-A)$ for all solitons, or one may require $\psi(A) = -\psi(-A)$ for all solitons. The former choice corresponds to quantizing the soliton as a boson. The latter choice corresponds to quantizing it as a fermion. We wish to follow the second road, of course, so our wave functions will be polynomials of *odd* degree in the a_i. So, the nucleons, of $I = J = \frac{1}{2}$, correspond to wave functions linear in a_i, while the deltas, of $I = J = \frac{3}{2}$, correspond to cubic functions. Wave functions of fifth order and higher correspond to highly excited states (masses ≥ 1730 MeV) which either are lost in the pion-nucleon continuum or else are artifacts of the model. The properly normalized wave functions for proton and neutron states of spin up or spin down along the z axis, and some of the Δ wave functions, are:

$$|p\uparrow\rangle = \frac{1}{\pi}(a_1 + ia_2) , \qquad |p\downarrow\rangle = -\frac{i}{\pi}(a_0 - ia_3) ,$$

$$|n\uparrow\rangle = \frac{i}{\pi}(a_0 + ia_3) , \qquad |n\downarrow\rangle = -\frac{1}{\pi}(a_1 - ia_2) ,$$

$$|\Delta^{++}, s_z = \tfrac{3}{2}\rangle = \frac{\sqrt{2}}{\pi}(a_1 + ia_2)^3 ,$$

$$|\Delta^{+}, s_z = \tfrac{1}{2}\rangle = -\frac{\sqrt{2}}{\pi}(a_1 + ia_2)(1 - 3(a_0^2 + a_3^2)) . \tag{8}$$

Returning to eq. (6), the eigenvalues of the hamiltonian are $E = M + (1/8\lambda)l(l+2)$ where $l = 2J$. So, the nucleon and delta masses are given by

$$M_N = M + \frac{1}{2\lambda}\frac{3}{4}, \qquad M_\Delta = M + \frac{1}{2\lambda}\frac{15}{4}, \tag{9}$$

where M, obtained by evaluting (2) numerically, is given by $M = 36.5F_\pi/e$ and $\lambda = \frac{4}{6}\pi(1/e^3F_\pi)50.9$, as already said. We have found that the best procedure in dealing with this model is to adjust e and F_π to fit the nucleon and delta masses. The results are $e = 5.45$ and $F_\pi = 129$ MeV. Thus, on the basis of the values of the baryon masses, we require (or predict) in this model a value of F_π that is 30% lower than the experimental value of 186 MeV.

2. Currents, charge radii and magnetic moments

In order to compute weak and electromagnetic couplings of baryons, we need first to evaluate the currents in terms of collective coordinates. The Noether current associated with the V−A transformation $\delta U = iQU$ is

$$J^\mu_{V-A} = \tfrac{1}{8}iF_\pi^2 \,\mathrm{Tr}\,[(\partial^\mu U)U^\dagger Q] + \frac{i}{8e^2}\mathrm{Tr}\,\{[(\partial_\nu U)U^\dagger, Q][(\partial^\mu U)U^\dagger, (\partial^\nu U)U^\dagger]\}\,. \tag{10}$$

The V+A current is obtained by exchanging U with $U^\dagger$.

The anomalous baryon current is instead [7, 6]

$$B^\mu = \frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{Tr}\,[(U^\dagger\partial_\nu U)(U^\dagger\partial_\alpha U)(U^\dagger\partial_\beta U)]\,, \tag{11}$$

where our notation is $\varepsilon_{0123} = -\varepsilon^{0123} = 1$.

If we substitute $U = A(t)U_0A^{-1}(t)$ in (10), we get rather complicated expressions for the vector and axial currents V and A. The following angular integrals, which are much simpler, are adequate for our purposes:

$$\int \mathrm{d}\Omega\, V^{a,0} = \tfrac{1}{3}i4\pi\Lambda'\,\mathrm{Tr}\,[(\partial_0 A)A^{-1}\tau_a]\,, \tag{12}$$

$$\int \mathrm{d}\Omega\, \boldsymbol{q}\cdot\boldsymbol{x}V^{a,i} = \tfrac{1}{3}i\pi\Lambda'\,\mathrm{Tr}\,(\boldsymbol{\tau}\cdot\boldsymbol{q}\tau_i A^{-1}\tau_a A)\,. \tag{13}$$

$$\int \mathrm{d}\Omega\, A^{a,i} = \tfrac{1}{3}\pi D'\,\mathrm{Tr}\,(\tau_i A^{-1}\tau_a A)\,, \tag{14}$$

where Λ' and D' are respectively

$$\Lambda' = \sin^2 F\left[F_\pi^2 + \frac{4}{e^2}\left(F'^2 + \frac{\sin^2 F}{r^2}\right)\right], \tag{15}$$

$$D' = F_\pi^2\left(F' + \frac{\sin 2F}{r}\right) + \frac{4}{e^2}\left(\frac{\sin 2F}{r}F'^2 + 2\frac{\sin^2 F}{r^2}F' + \frac{\sin^2 F\sin 2F}{r^3}\right). \tag{16}$$

In the computation of the above formulas from (10) we have neglected terms which are quadratic in time derivatives. In the semiclassical limit the solitons rotate slowly, so terms quadratic in time derivatives are higher order in the semiclassical approximation.

The expression in (7) for the isospin generator I_k can be derived from (12) by integrating over r, and replacing $\dot{a}_i$ by the canonical momentum.

From (11) we derive the baryon current and charge density

$$B^0 = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F' , \tag{17}$$

$$B^i = i\frac{\varepsilon^{ijk}}{2\pi^2}\frac{\sin^2 F}{r}F'\hat{x}_k \operatorname{Tr}[(\partial_0 A^{-1})A\tau_j] . \tag{18}$$

The baryon charge per unit r is therefore

$$\rho_B(r) = 4\pi r^2 B^0(r) = -\frac{2}{\pi}\sin^2 F F' ,$$

and its integral $\int_0^\infty \rho_B(r)\,dr = 1$ gives the baryonic charge.

The isoscalar mean square radius is given by

$$\langle r^2\rangle_{I=0} = \int_0^\infty r^2\rho_B(r)\,dr = \frac{4.47}{e^2F_\pi^2} = 4.47\ (0.28)^2\ \text{fm}^2 ,$$

and we get $\langle r^2\rangle_{I=0}^{1/2} = 0.59$ fm, while the corresponding experimental value is 0.72 fm.

From (12) and (15) we can compute the isovector charge density per unit r

$$\rho_{I=1}(r) = \frac{r^2\sin^2 F\,(F_\pi^2 + (4/e^2)(F'^2 + \sin^2 F/r^2))}{\int_0^\infty r^2\sin^2 F\,(F_\pi^2 + (4/e^2)(F'^2 + \sin^2 F/r^2))\,dr} ,$$

and finally derive the proton and neutron charge distributions which are plotted in fig. 2.

The isovector mean square charge radius $\int_0^\infty r^2\rho_{I=0}(r)\,dr$ is divergent, as expected in the chiral limit [9]. The introduction of quark masses in this model [10] will cure this problem, as it does in nature.

The definitions of isoscalar and isovector magnetic moments are respectively

$$\boldsymbol{\mu}_{I=0} = \tfrac{1}{2}\int \boldsymbol{r}\times\boldsymbol{B}\,d^3x , \tag{19}$$

$$\boldsymbol{\mu}_{I=1} = \tfrac{1}{2}\int \boldsymbol{r}\times\boldsymbol{V}^3\,d^3x . \tag{20}$$

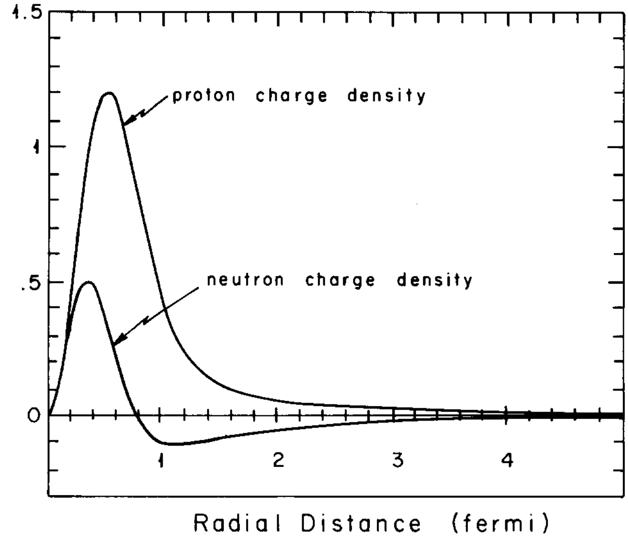

Fig. 2. Plot of the proton and neutron charge densities. These charge densities are given as functions of the radial distance r, and include a factor of $4\pi r^2$.

Therefore, from (18) the isoscalar magnetic moment density is

$$\rho_M^{I=0}(r)=\frac{r^2F'\sin^2 F}{\int_0^\infty r^2F'\sin^2 F\,\mathrm{d}r}\,.$$

The isoscalar magnetic mean radius is defined by

$$\langle r^2\rangle_{M,I=0}=\int_0^\infty r^2\rho_M^{I=0}(r)\,\mathrm{d}r\,.$$

We get $\langle r^2\rangle_{M,I=0}^{1/2}=0.92\,\mathrm{fm}$, against the experimental value of 0.81 fm.

The simplest way to extract the g factors is to calculate the expectation value of the magnetic moment operators in a proton state of spin up, using the forms given earlier for the wave functions. From (18) and (19) the isoscalar magnetic moment is

$$\begin{aligned}(\mu_{I=0})_i&=\tfrac{1}{2}\int \mathrm{d}^3x\,\varepsilon_{lmi}x_l\langle \mathrm{p}\uparrow|B_m|\mathrm{p}\uparrow\rangle\\ &=-\frac{1}{2}\frac{i}{2\pi^2}\int \mathrm{d}^3x\,\sin^2 F F'\hat{x}_l\hat{x}_k\varepsilon_{lmi}\varepsilon_{mjk}\langle \mathrm{p}\uparrow|\mathrm{Tr}\,[(\partial_0A^{-1})A\tau_j]|\mathrm{p}\uparrow\rangle\,.\end{aligned}$$

It is easy to check that $\langle \mathrm{p}\uparrow|\,\mathrm{Tr}\,[(\partial_0A^{-1})A\tau_j]|\mathrm{p}\uparrow\rangle=-\delta_{j3}i/2\lambda$. It follows that

$$(\boldsymbol{\mu}_{I=0})_3=\frac{\langle \tilde{r}^2\rangle_{I=0}}{\Lambda}\frac{e}{F_\pi}\frac{1}{4\pi}\,. \tag{21}$$

The g factor is defined by writing $\boldsymbol{\mu}=(g/4M)\boldsymbol{\sigma}$. The isoscalar g factor $g_{I=0}=g_p+g_n$ is 1.11 in this model (the experimental value is 1.76, instead).

In order to compute the isovector magnetic moment, we start from (13) and integrate in the radial variable. We get

$$\int d^3x\, \boldsymbol{q}\cdot \boldsymbol{x} V^{3,i} = \tfrac{1}{3} i\pi \frac{\Lambda}{F_\pi e^3} \operatorname{Tr}(\boldsymbol{\tau}\cdot \boldsymbol{q}\tau_i A^{-1}\tau_3 A)\,,$$

with Λ given in (5). Now

$$\operatorname{Tr}(\boldsymbol{\tau}\cdot \boldsymbol{q}\tau_i A^{-1}\tau_j A) = i q_l \varepsilon_{lim} \operatorname{Tr}(\tau_m A^{-1}\tau_j A)\,.$$

A detailed calculation using the nucleon wave function given in (8) shows that for any nucleon states N and N′

$$\langle N'|\operatorname{Tr}[\tau_i A^{-1}\tau_j A]|N\rangle = -\tfrac{2}{3}\langle N'|\sigma_i\tau_j|N\rangle\,. \tag{22}$$

Therefore

$$\langle p\uparrow|\int d^3x\, \boldsymbol{q}\cdot\boldsymbol{x} V_i^3|p\uparrow\rangle = -\tfrac{1}{3} q_l \pi \frac{\Lambda}{F_\pi e^3}\varepsilon_{li3}(-\tfrac{2}{3})\,,$$

$$\langle p\uparrow|\int d^3x\, x_l V_i^3|p\uparrow\rangle = \tfrac{2}{9}\pi \frac{\Lambda}{F_\pi e^3}\varepsilon_{li3}\,.$$

In conclusion, from (20) we get

$$(\boldsymbol{\mu}_{I=1})_3 = \tfrac{2}{9}\pi \frac{\Lambda}{F_\pi e^3}\,. \tag{23}$$

The isovector g factor $g_{I=1}=g_p-g_n$ turns out to be 6.38 against the experimental value of 9.4. The magnetic moments for the proton and neutron, measured in terms of Bohr magneton, are $\mu_p=\frac{1}{2}g_p=1.87$ and $\mu_n=\frac{1}{2}g_n=-1.31$ respectively. The ratio $|\mu_p/\mu_n|$ turns out to be 1.43 (see table 1), as opposed to 1.5 in the quark model and 1.46 experimentally.

TABLE 1

Quantity	Prediction	Experiment
M_N	input	939 MeV
M_Δ	input	1232 MeV
F_π	129 MeV	186 MeV
$\langle r^2\rangle^{1/2}_{I=0}$	0.59 fm	0.72 fm
$\langle r^2\rangle^{1/2}_{M,I=0}$	0.92 fm	0.81 fm
μ_p	1.87	2.79
μ_n	−1.31	−1.91
$\left\lvert\frac{\mu_p}{\mu_n}\right\rvert$	1.43	1.46
g_A	0.61	1.23
$g_{\pi NN}$	8.9	13.5
$g_{\pi N\Delta}$	13.2	20.3
$\mu_{N\Delta}$	2.3	3.3

3. Mass relations

It is interesting to form certain combinations of experimentally measured quantities from which the parameters of the Skyrme model cancel out. Combining our various formulas, one finds the following formula for the isoscalar g factor in terms of experimentally measured quantities

$$g_{I=0}=\tfrac{4}{9}\langle r^2\rangle_{I=0}M_{\mathrm{N}}(M_\Delta-M_{\mathrm{N}})\,. \tag{24}$$

This formula is very well satisfied experimentally. The left-hand side is 1.76 and the right-hand side is 1.66. We also find a formula for the isovector g factor from which the Skyrme model parameters cancel out:

$$g_{I=1}=\frac{2M_{\mathrm{N}}}{M_\Delta-M_{\mathrm{N}}}\,. \tag{25}$$

This relation is not so well satisfied experimentally, the left-hand side being 9.4 and the right-hand side 6.38.

Relations (24) and (25) are clearly much more general than the rest of our formulas. For instance, it is easy to see that they continue to hold if an arbitrary isospin conserving potential energy $V(U)$ is included in the model, the most obvious candidate being a term Tr U to simulate the effects of quark masses. It is natural to wonder exactly how broad is the range of validity of these formulas.

Consider the soliton before it begins to rotate as a spherically symmetric classical body with an energy density $T_{00}(r)$. (We will treat the soliton as a non-relativistic object and ignore the pressure T_{ij} relative to T_{00}. Actually the proper inclusion of T_{ij} does not modify the formulas.) If such a body begins to rotate with angular frequency $\boldsymbol{\omega}$, the velocity at position $\boldsymbol{x}$ is $\boldsymbol{v}(r)=\boldsymbol{\omega}\times\boldsymbol{x}$, and the momentum density is $T_{0i}(\boldsymbol{x})=T_{00}(r)\varepsilon_{ijk}\omega_j x_k$. The angular momentum of the spinning body is

$$\begin{aligned}J_i &= \int \mathrm{d}^3x\, \varepsilon_{ijk}x_j T_{0k}(x)\\ &= \int \mathrm{d}^3x\,(\omega_i r^2 - x_i\boldsymbol{x}\cdot\boldsymbol{\omega})T_{00}\\ &= \tfrac{2}{3}\omega_i\int \mathrm{d}^3x\, T_{00}r^2\,.\end{aligned}$$

We have simply obtained the formula $\boldsymbol{J}=I\boldsymbol{\omega}$, where the moment of inertia is $I=\frac{2}{3}\int \mathrm{d}^3x\,T_{00}r^2$. If the body begins to rotate its kinetic energy will be

$$\begin{aligned}T &= \tfrac{1}{2}\int \mathrm{d}^3x\, T_{00}\boldsymbol{v}^2\\ &= \tfrac{1}{2}\int \mathrm{d}^3x\, T_{00}(\boldsymbol{\omega}\times\boldsymbol{x})^2\\ &= \tfrac{1}{3}\boldsymbol{\omega}^2\int \mathrm{d}^3x\, T_{00}r^2=\frac{\boldsymbol{J}^2}{2I}\,.\end{aligned}$$

For the nucleon, $\boldsymbol{J}^2=\frac{3}{4}\hbar^2$; for the Δ, $\boldsymbol{J}^2=\frac{15}{4}\hbar^2$. Interpreting the mass difference between the delta and nucleon as a consequence of the rotational kinetic energy, we find for the moment of inertia $I=\frac{3}{2}(M_\Delta-M_N)^{-1}$. The rotational frequency of the nucleon is hence $\boldsymbol{\omega}=\boldsymbol{J}/I=\frac{2}{3}(M_\Delta-M_N)\boldsymbol{J}$.

The soliton before it begins to spin has some isoscalar charge density $\rho_{I=0}(r)$, but the isoscalar current density vanishes for a soliton at rest because of spherical symmetry and current conservation (or because of time reversal invariance). A rotating soliton has the current density $\boldsymbol{B}=\rho_{I=0}\boldsymbol{v}=\rho_{I=0}(\boldsymbol{\omega}\times\boldsymbol{x})$. So the magnetic moment of the rotating soliton is

$$\boldsymbol{\mu}_{I=0}=\frac{1}{2}\int d^3x\,\boldsymbol{x}\times\boldsymbol{B}=\frac{1}{2}\int d^3x\,\rho_{I=0}\boldsymbol{x}\times(\boldsymbol{\omega}\times\boldsymbol{x})$$

$$=\frac{1}{3}\boldsymbol{\omega}\int d^3x\,\rho_{I=0}r^2=\frac{1}{3}\omega\langle r^2\rangle_{I=0}\,. \tag{26}$$

Combining this with $\boldsymbol{\omega}=\frac{2}{3}(M_\Delta-M_N)\boldsymbol{J}$ and with the definition $\boldsymbol{\mu}=(g/2M)\boldsymbol{J}$ of the g factor, we find the result (24) for the isoscalar magnetic moment of the nucleon.

Now, to what extent is this result general? The relations $\boldsymbol{J}=I\boldsymbol{\omega}$, $T=\boldsymbol{J}^2/2I$ are completely general formulas for the angular momentum and kinetic energy of a slowly rotating body. (These formulas hold even when the hamiltonian after elimination of non-propagating degrees of freedom is non-local.) The nucleon and delta are slowly rotating bodies in the large-N limit, with I of order N and $\boldsymbol{\omega}$ of order $1/N$. The formula $\boldsymbol{\omega}=\frac{2}{3}(M_\Delta-M_N)\boldsymbol{J}$ is a rigorous formula for the rotational frequency of the nucleon or delta in the large-N limit or in any semi-classical soliton description.

Unfortunately, the formula $\boldsymbol{B}=\rho_{I=0}(\omega\times\boldsymbol{x})$ is not a completely general formula for the current density induced in a static object when it begins to rotate. This formula holds for a *macroscopic* body, but whether it holds for a *microscopic* body such as a soliton depends on how the current and charge densities are constructed from the elementary fields. Likewise the formula (26) is not a completely general formula for the magnetic moment of a rotating sphere. In general there may be a non-locality in the relation between the charge density and the induced current; this non-locality spoils the relation $\boldsymbol{\mu}_{I=0}=\frac{1}{3}\boldsymbol{\omega}\langle r^2\rangle_{I=0}$.

If the baryon current is given by Skyrme's formula (11) then we will have (26), and the successful relation (24) will hold regardless of the choice of the chiral model lagrangian. However, a realistic description of nature requires additions to the Skyrme current. For instance, the $J=1$, $I=0$ ω meson is observed to couple to the isoscalar current. This suggests the addition to the current of an extra term $\Delta B_\mu=\alpha\,\partial_\nu\omega_{\mu\nu}$, where $\omega_{\mu\nu}=\partial_\mu\omega_\nu-\partial_\nu\omega_\mu$. With this addition to the current the relation (26) no longer holds for a rotating soliton, and with it eq. (24) is lost.

Thus, the successful formula (24) depends on the definition of the baryon current but not on the choice of the lagrangian. It can likewise be shown that eq. (25) holds as long as the lagrangian only involves spinless fields and their first derivatives, but can be modified by including higher derivatives or fields of higher spin.

4. Axial coupling and the Goldberger-Treiman relation

To evaluate the axial coupling g_A we calculate the integral $\int d^3x\, A_i^a(x)$ in a soliton state. The relation of this integral with the axial coupling is slightly subtle. The standard definition of the axial current matrix element is

$$\langle N'(p_2)|A_\mu^a(0)|N(p_1)\rangle = \bar{u}(p_2)\tau^a(\gamma_\mu\gamma_5 g_A(q^2) + q_\mu\gamma_5 h_A(q^2))u(p_1)\,. \tag{27}$$

Current conservation implies $2mg_A(q^2)+q^2h_A(q^2)=0$. In the non-relativistic limit, for the spatial components of the current, (27) becomes

$$\langle N'(p_2)|A_i^a(0)|N(p_1)\rangle = g_A(q^2)\left(\delta_{ij}-\frac{q_iq_j}{|q|^2}\right)\langle N'|\sigma_j\tau^a|N\rangle\,. \tag{28}$$

The $1/|q|^2$ singularity in (28) reflects, of course, the pion pole. The $\boldsymbol{q}\to 0$ limit of (28) is ambiguous. Taking the limit in a symmetric way, replacing q_iq_j by $\frac{1}{3}\delta_{ij}|q|^2$, the right-hand side of (28) becomes $\frac{2}{3}g_A\langle N'|\sigma_i\tau^a|N\rangle$ in the limit $q\to 0$; here $g_A = g_A(0)$ is the usual axial coupling constant.

Corresponding to this subtlety, the integral $\int d^3x\, A_i^a(x)$ in a soliton state is not absolutely convergent. Performing first the angular integral and then the radial integral corresponds to the symmetric limit just described. With this prescription for the integral we find

$$\int d^3x\, A_i^a(x) = \frac{\pi}{3e^2} D\, \mathrm{Tr}\,[\tau_i A^{-1}\tau_a A]\,, \tag{29}$$

where

$$D = \int_0^\infty d\tilde{r}\,\tilde{r}^2\left[\left(F'+\frac{\sin 2F}{\tilde{r}}\right)+4\left(\frac{\sin 2F}{\tilde{r}}(F')^2+\frac{2\sin^2 F}{\tilde{r}^2}F'+\frac{\sin^2 F\sin 2F}{\tilde{r}^3}\right)\right].$$

Numerically we find $D=-17.2$. As we have discussed before (22) $\mathrm{Tr}\,[\tau_i A^{-1}\tau_a A]$, evaluated in a nucleon state, equals $-\frac{2}{3}\langle\sigma_i\tau_a\rangle$. Setting (29) equal to $\frac{2}{3}g_A$ (corresponding to the symmetric $\boldsymbol{q}\to 0$ limit of (28)) we get

$$g_A = \tfrac{3}{2}(-\tfrac{2}{3})\frac{\pi}{3e^2}D = 0.61\,, \tag{30}$$

which unfortunately is not in good agreement with the experimental value $g_A = 1.23$. Although the Adler–Weisberger sum rule, which is a consequence of chiral symmetry, is surely obeyed in the Skyrme model, we do not know how it works out.

There is another useful way to compute g_A, which links it to the long-distance behaviour of the soliton solution $F(r)$, and turns out to be particularly useful for proving the Goldberger-Treiman relation.

The requirement of current conservation $\partial_\mu A^\mu = 0$ reduces to $\partial_i A^i = 0$ in the static approximation. Therefore the volume integral of the axial current can be computed as a surface integral by using the divergence theorem, as follows:

$$\int d^3x\, A_i^a = \int d^3x\, \partial_j(x_i A_j^a) = \int_S x_i A_j^a \hat{x}_j\, dS\,. \tag{31}$$

The definition of the axial current from (10) is

$$A_i^a = \tfrac{1}{8} i F_\pi^2 \operatorname{Tr}[(\partial_i U_0) U_0^\dagger + U_0^\dagger \partial_i U_0) A^{-1} \tau^a A] + \text{higher derivatives}\,, \tag{32}$$

where $U_0 = \cos F + i \sin(F) \boldsymbol{\tau} \cdot \hat{\boldsymbol{x}}$ is the soliton solution. At large distances $F(r)$ goes like B/r^2 where B can be extracted from the computer solution and is $B = B'/e^2 F_\pi^2$ with $B' = 8.6$ Therefore at large distances

$$U_0 = 1 + i\frac{B}{r^2} \boldsymbol{\tau} \cdot \hat{\boldsymbol{x}}\,,$$

$$\partial^i U_0 = -i\frac{B}{r^3}(\tau_i - 3\boldsymbol{\tau} \cdot \hat{\boldsymbol{x}} \hat{x}^i)\,.$$

It follows from (32) that the current to be used in formula (31) is

$$A_i^a = \tfrac{1}{4} F_\pi^2 \frac{B}{r^3} [(\tau_i - 3\boldsymbol{\tau} \cdot \hat{\boldsymbol{x}} \hat{x}_i) A^{-1} \tau_a A] + \cdots\,. \tag{33}$$

Therefore from (31) we obtain

$$\int d^3x\, A_i^a = -F_\pi^3 B \tfrac{2}{3} \pi \operatorname{Tr}[\tau_i A^{-1} \tau^a A]\,.$$

From (22) and the definition of g_A we get therefore

$$g_A = \tfrac{3}{2} F_\pi^2 B \tfrac{2}{3} \pi \tfrac{2}{3} = 2B' \frac{\pi}{3e^2} = 0.61\,, \tag{34}$$

as before. Eqs. (30) and (34) imply a relation between D and B'. Indeed, by using (3) one can integrate D with the result $D = -2B'$.

Finally, let us check the Goldberger-Treiman relation in this model. The old fashioned lagrangian for pions π coupled to nucleons ψ is

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \pi^a)^2 + i g_{\pi NN} \pi^a \bar{\psi} \gamma_5 \tau^a \psi\,.$$

The non-relativistic reduction of the coupling term is $(g_{\pi NN}/2M_N) \partial_i \pi^a \bar{\psi} \sigma_i \tau^a \psi$. From this form one can find the large-distance behaviour of the expectation value of the pion field in a nucleon state

$$\langle \pi^a(x) \rangle = -\frac{g_{\pi NN}}{8\pi M_N} \frac{x_i}{r^3} \langle \sigma_i \tau^a \rangle\,. \tag{35}$$

On the other hand, we can find the expectation value of the pion field at great distances from a soliton by studying the asymptotic behaviour of the soliton solution. The small fluctuations of U around its vacuum expectation value are related to the pion field by

$$U = 1 + 2i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{F_\pi} + \cdots .$$

With $U = AU_0A^{-1}$ and $U_0 = 1 + i(B/r^2)\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\ldots$, we find the large-distance behaviour of the pion field:

$$\pi^a = \tfrac{1}{4}BF_\pi\frac{x_i}{r^3}\,\mathrm{Tr}\,[\tau_i A^{-1}\tau^a A] .$$

By using (22) and (34)

$$\langle\pi^a\rangle = -\tfrac{1}{6}BF_\pi\frac{x_i}{r^3}\langle\sigma_i\tau^a\rangle = -\frac{g_A}{F_\pi}\frac{1}{4\pi}\langle\sigma_i\tau^a\rangle . \qquad (36)$$

So comparing (35) and (36) we finally get the Goldberger–Treiman relation

$$g_A = \frac{F_\pi g_{\pi NN}}{2M_N} .$$

The predicted value of $g_{\pi NN}$ is 8.9 compared with the experimental value of 13.5.

5. Decays of the Δ

In this section, we will calculate the amplitudes for the decay processes $\Delta\to N\pi$ and $\Delta\to N\gamma$. The decay $\Delta\to N\gamma$ is related by a simple quark model argument [11] to the nucleon magnetic moment. A similar quark model argument [12] relates the amplitude $\Delta\to N\pi$ to the pion-nucleon coupling. For a review of the quark model relations, see [13]. We will see that the $1/N$ expansion makes predictions for Δ decays analogous to the predictions of the quark model. These predictions are model-independent in the sense that they hold for any soliton model of baryons and serve as quantitative tests of the $1/N$ expansion. The Skyrme model will not enter in this section except in the concluding paragraph.

In the large-N limit the Δ and the nucleon are nearly degenerate, so the decays $\Delta\to N\pi$ and $\Delta\to N\gamma$ involve soft pions and photons. Also, the nucleon and the Δ are described by the same classical soliton solution with different but known wave functions for the collective coordinates (8). Hence the coupling of the soft pion or photon in Δ decay can be computed in terms of the static coupling of pions or photons to nucleons.

In view of chiral symmetry, the pion couplings to baryons can be expressed as derivative couplings. For soft pions, the coupling will involve mainly the first derivative of the pion field $\partial_i\pi^a$, multiplied by some operator $\mathcal{O}_i^a$ acting on the collective coordinates. In $\mathcal{O}_i^a$ time derivatives of A can be neglected (since the nucleon rotates slowly in the large-N limit) so $\mathcal{O}_i^a$ must be a function of A only. The only function of A that transforms properly under spin and isospin ($\mathcal{O}_i^a$ must have $I=J=1$) is $\mathrm{Tr}\,[\tau_i A^{-1}\tau_a A]$. So in the large-$N$ limit, irrespective of other details, the coupling of soft pions to baryons is of the form

$$\mathscr{L}_\pi = \delta \partial_i \pi^a \,\mathrm{Tr}\,[\tau_i A^{-1}\tau_a A]\,, \tag{37}$$

for some δ.

The pion-nucleon coupling is related to δ by evaluating the matrix element of (37) between initial and final nucleon states. We have already done this, in effect, in computing $g_{\pi NN}$ in the Skyrme model, and the relation is $\delta = \frac{3}{4}g_{\pi NN}/M_N$, M_N being the nucleon mass. On the other hand, we can describe the hadronic decay of the Δ by taking the matrix element of (37) between an initial Δ and a final nucleon.

Let us define a coupling $g_{\pi N\Delta}$ as follows (it is called $\mathcal{M}_{\uparrow\uparrow}$ in [12]). For a decay $\Delta^{++}(s_z=\frac{3}{2}) \to \mathrm{p}(s_z=\frac{1}{2})+\pi^+$, we define the amplitude to be $g_{\pi N\Delta}(k_x+ik_y)/2M_N$, where $\boldsymbol{k}$ is the c.m. momentum of the pion. Evaluating the matrix element of (37), we find $g_{\pi N\Delta}=\frac{3}{2}g_{\pi NN}$. The quark model relation of [12] is instead $g_{\pi N\Delta}=\frac{6}{5}g_{\pi NN}$. The relation $g_{\pi N\Delta}=\frac{3}{2}g_{\pi NN}$, which follows from the $1/N$ expansion without other assumptions, is in excellent agreement with experiment. With the experimental value $g_{\pi NN}=13.5$, it gives a value of 125 MeV for the width of the Δ; the experimental value is about 120 MeV.

A similar analysis can be made for the electromagnetic decay of the Δ. The decay $\Delta \to N\gamma$ violates isospin, so it involves only the isovector part of the electromagnetic current. The isovector coupling of the magnetic field $\boldsymbol{B}$ to baryons must be of the form $\boldsymbol{B}\cdot\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is an operator acting on the collective coordinates of baryons. $\boldsymbol{\mu}$, the isovector magnetic moment operator of baryons, must be the third component of an isovector. Neglecting time derivatives, the only possibility is $\mu_i = \alpha\,\mathrm{Tr}\,[\tau_i A^{-1}\tau_3 A]$ where α is some constant. So the magnetic coupling to baryons is

$$\mathscr{L}_{\mathrm{mag}} = \boldsymbol{B}\cdot\boldsymbol{\mu} = \alpha B_i\,\mathrm{Tr}\,[\tau_i A^{-1}\tau_3 A]\,. \tag{38}$$

A relation of this form holds in any soliton description of baryons; only the value of α is model dependent. The value of α determines the isovector part of the nucleon magnetic moment. The relation is obtained by calculating the matrix element of (38) between initial and final nucleon states; the calculation is essentially the one we have already performed in deriving eq. (23). Writing the proton and neutron magnetic moments as $\boldsymbol{\mu}_p=\mu_p\boldsymbol{\sigma}$, $\boldsymbol{\mu}_n=\mu_n\boldsymbol{\sigma}$, the relation is $\alpha=\frac{3}{4}(\mu_p-\mu_n)$.

We can now calculate the amplitude for $\Delta\to N\gamma$ by evaluating the matrix element of (38) between the initial Δ and final nucleon. Let us define a transition moment $\mu_{N\Delta}$ by the formula $\mu_{N\Delta}=\langle \mathrm{p}, s_z=\frac{1}{2}|\mu_z|\Delta^+, s_z=\frac{1}{2}\rangle$ where $\mu_z = \frac{3}{4}(\mu_p-\mu_n)\,\mathrm{Tr}\,[\tau_3 A^{-1}\tau_3 A]$ is the z component of the baryon magnetic moment operator. Using wave functions in (8) we find $\mu_{N\Delta}=\sqrt{\frac{1}{2}}\,(\mu_p-\mu_n)$. This agrees very well with the experimental value $\mu_{N\Delta}=(0.70\pm0.01)\,(\mu_p-\mu_n)$. The quark model

(11) gives $\mu_{N\Delta}=\frac{2}{5}\sqrt{2}(\mu_p-\mu_n)=0.57(\mu_p-\mu_n)$ (this relation is often written $\mu_{N\Delta}=\frac{2}{3}\sqrt{2}\,\mu_p$; we are using here the quark model prediction $\mu_n=-\frac{2}{3}\mu_p$).

The model-independent tests of the $1/N$ expansion $g_{\pi N\Delta}=\frac{3}{2}g_{\pi NN}$ and $\mu_{N\Delta}=\sqrt{\frac{1}{2}}\,(\mu_p-\mu_n)$ work very well (perhaps fortuitously so) if one takes $g_{\pi NN}$ and $\mu_p-\mu_n$ from experiment. The Skyrme model, however, is less successful. Since the Skyrme model values of $g_{\pi NN}$, μ_p, and μ_n are all about 30% too small, the predictions for $\mu_{N\Delta}$ and $g_{\pi N\Delta}$ are too low (see table 1) by a similar margin.

C.R.N. acknowledges useful conversations with S. Gupta.

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