● An overview for string theorists but not with string techniques.
● Problems with lattice.
● Calculations from first principles (real QCD) only at very high densities (baryon chemical potential \( \sim 10^5 \text{ GeV} \)).
● But regions of interest as in CSO’s is for \( \mu \sim 400-500 \text{ MeV} \). Most of the calculations made within NJL models.
● What can be obtained from these model calculations? The results from QCD calculations extrapolated at low \( \mu \) agree with models. The main thrust is in the qualitative properties like phase transitions and their nature depending mainly on the symmetry properties, not very much in the quantitative results.
● The low \( \mu \) region is strongly interacting, it would be nice to be able to investigate it within the ADS/QCD approach (some attempt already present in the literature)
● Main discussion about the ground state.
Summary

- Introduction
- Color Superconductivity: CFL and 2SC phases
- The case of pairing at different Fermi surfaces
- LOFF (or crystalline) phase
- Conclusions
Motivations for the study of high-density QCD

- Study of the QCD phase in a region difficult in lattice calculations
- Understanding the interior of CSO’s

Asymptotic region in $\mu$ fairly well understood: existence of a CS phase.
Real question: does this type of phase persist at relevant densities ($\sim 5-6 \rho_0$)?
CFL and S2C phases


- Only in 1998 (Alford, Rajagopal & Wilczek; Rapp, Schafer, Schuryak & Velkovsky) a real progress.

- Why CS? For an arbitrary attractive interaction it is energetically convenient the formation of Cooper pairs (differmions)

- Due to asymptotic freedom quarks are almost free at high density and we expect diquark condensation in the color attractive channel $3^*$ (the channel 6 is repulsive).
In matter SC only under particular conditions (phonon interaction should overcome the Coulomb force)

\[ \frac{T_c^{(\text{electr.)}}}{E^{(\text{electr.)}}} \approx \frac{1}{10^4} \div \frac{10^5 \, 0 \text{K}}{10^0 \text{K}} \approx 10^{-3} \div 10^{-4} \]

In QCD attractive interaction (antitriplet channel)

\[ \frac{T_c^{(\text{quarks)}}}{E^{(\text{quarks)}}} \approx \frac{50 \text{ MeV}}{100 \text{ MeV}} \approx 1 \]

SC much more efficient in QCD
The symmetry breaking pattern of QCD at high-density depends on the number of flavors with mass \( m < \mu \). Two most interesting cases: \( N_f = 2, 3 \).

Consider the possible pairings at very high density

\[
\langle 0 | \psi^\alpha_{ia} \psi^\beta_{jb} | 0 \rangle \quad \alpha, \beta \text{ color; } \ i, j \text{ flavor; } a,b \text{ spin}
\]

- **Antisymmetry in spin** \((a,b)\) for better use of the Fermi surface
- **Antisymmetry in color** \((a, b)\) for attraction
- **Antisymmetry in flavor** \((i,j)\) for Pauli principle
Only possible pairings

LL and RR

For $\mu >> m_u, m_d, m_s$

Favorite state for $N_f = 3$, CFL (color-flavor locking) (Alford, Rajagopal & Wilczek 1999)

$$\langle 0 | \psi_\alpha^{iL} \psi_\beta^{jL} | 0 \rangle = - \langle 0 | \psi_\alpha^{iR} \psi_\beta^{jR} | 0 \rangle \propto \Delta \varepsilon^{abc}_\varepsilon_{ijC}$$

Symmetry breaking pattern

$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \Rightarrow SU(3)_{c+L+R}$$
Why CFL?

\[
\langle 0 | \psi_\alpha^i \psi_\beta^j | 0 \rangle \propto \Delta \varepsilon_{ijC}^{\alpha\beta} \\
\langle 0 | \psi_\alpha^i \psi_\beta^j | 0 \rangle \propto -\Delta \varepsilon_{ijC}^{\alpha\beta}
\]
What happens going down with $\mu$? If $\mu \ll m_s$, we get 3 colors and 2 flavors (2SC)

$$\langle 0 | \psi_{iL}^\alpha \psi_{jL}^\beta | 0 \rangle = \Delta \varepsilon^{\alpha \beta 3} \varepsilon_{ij}$$

$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \Rightarrow SU(2)_c \otimes SU(2)_L \otimes SU(2)_R$

However, if $\mu$ is in the intermediate region we face a situation with quarks belonging to different Fermi surfaces (see later). Then other phases could be important (LOFF, etc.)
Pairing fermions with different Fermi momenta

- $M_s$ not zero
- Neutrality with respect to em and color
- Weak equilibrium

All these effects make Fermi momenta of different fermions unequal causing problems to the BCS pairing mechanism.

no free energy cost in neutral singlet, (Amore et al. 2003)
Consider 2 fermions with $m_1 = M$, $m_2 = 0$ at the same chemical potential $\mu$. The Fermi momenta are

$$p_{F_1} = \sqrt{\mu^2 - M^2} \quad p_{F_2} = \mu$$

**Effective chemical potential for the massive quark**

$$\mu_{\text{eff}} = \sqrt{\mu^2 - M^2} \approx \mu - \frac{M^2}{2\mu}$$

**Mismatch:**

$$\delta\mu \approx \frac{M^2}{2\mu} \quad \text{M}^2/2\mu \text{ effective chemical potential}$$
• Weak equilibrium makes chemical potentials of quarks of different charges unequal:

\[ d \rightarrow u e \bar{v} \Rightarrow \mu_d - \mu_u = \mu_e \]

• From this:

\[ \mu_i = \mu + Q_i \mu_Q \]

• N.B. $\mu_e$ is not a free parameter, it is fixed by the neutrality condition:

\[ Q = -\frac{\partial V}{\partial \mu_e} = 0 \]
If the strange quark is massless this equation has solution

\( N_u = N_d = N_s \), \( N_e = 0 \); quark matter electrically neutral with no electrons
● Fermi surfaces for neutral and color singlet unpaired quark matter at the $\beta$ equilibrium and $M_s$ not zero.

● In the normal phase $\mu_3 = \mu_8 = 0$.

By taking into account $M_s$

\[
\mu_e \approx p^d_F - p^u_F \approx p^u_F - p^s_F \approx M_s^2 / 4\mu
\]

\[
p^d_F - p^s_F \approx 2\mu_e, \quad \mu_e = \frac{M_s^2}{4\mu}
\]

The biggest $\delta\mu$ is $d_s$
• Example 2SC: normal BCS pairing when

\[ \mu_u = \mu_d \Rightarrow n_u = n_d \]

• But neutral matter for

\[ n_d \approx 2n_u \Rightarrow \mu_d \approx 2^{1/3}\mu_u \Rightarrow \mu_e = \mu_d - \mu_u \approx \frac{1}{4}\mu_u \neq 0 \]

Mismatch

\[ \delta \mu = \frac{p^d_d - p^u_u}{2} = \frac{\mu_d - \mu_u}{2} = \frac{\mu_e}{2} \approx \frac{\mu_u}{8} \]
As long as $\delta \mu$ is small no effects on BCS pairing, but when increased the BCS pairing is lost and two possibilities arise:

- The system goes back to the normal phase
- Other phases can be formed

Notice that there are also color neutrality conditions

\[
\frac{\partial V}{\partial \mu_3} = T_3 = 0, \quad \frac{\partial V}{\partial \mu_8} = T_8 = 0
\]
The problem of two fermions with different chemical potentials:

\[ \mu_u = \mu + \delta \mu, \quad \mu_d = \mu - \delta \mu \]

\[ \mu = \frac{\mu_u + \mu_d}{2}, \quad \delta \mu = \frac{\mu_u - \mu_d}{2} \]

can be described by an interaction hamiltonian

\[ H_I = -\delta \mu \psi \sigma_3 \psi \]

**In normal SC:**
Gap equation:
\[ 1 = \frac{g}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\varepsilon(p,\Delta)} (1 - n_u - n_d) \]

\[ n_{u,d} = \frac{1}{e^{(\varepsilon(p,\Delta)\pm\delta\mu)/T} + 1} \]
\[ \varepsilon(p,\Delta) = \sqrt{(|p| - \mu)^2 + \Delta^2} \]

For \( T \to 0 \)

The blocking region reduces the gap:
\[ \Delta \ll \Delta_{BCS} \]
In a simple model (no supplementary conditions), with two fermions at chemical potentials $\mu_1$ and $\mu_2$, the system goes normal through a 1st order transition at the Chandrasekhar - Clogston point. Another unstable phase exists.

$$\delta \mu = \Delta_{BCS}$$

Sarma phase, tangential to the BCS phase at $\delta \mu = \Delta$

$$\delta \mu_1 = \frac{\Delta_{BCS}}{\sqrt{2}}$$
The point $|\delta \mu| = \Delta$ is special. In the presence of a mismatch new features are present. The spectrum of quasiparticles is

$$E(p) = | \pm \delta \mu + \sqrt{(p - \mu)^2 + \Delta^2}|$$

For $|\delta \mu| < \Delta$, the gaps are $\Delta - \delta \mu$ and $\Delta + \delta \mu$.

For $|\delta \mu| = \Delta$, an unpairing (blocking) region opens up and gapless modes are present.

$$E(p) = 0 \iff p = \mu \pm \sqrt{\delta \mu^2 - \Delta^2}$$

Energy cost for pairing

$$2\delta \mu$$

Energy gained in pairing

$$2\Delta$$

$2\delta \mu > 2\Delta$
If neutrality constraints are present the situation is different.

For $|\delta \mu| > \Delta$ ($\delta \mu = \mu_e/2$) 2 gapped quarks become gapless. The gapped quarks begin to unpair destroying the BCS solution. But a new stable phase exists, the gapless 2SC (g2SC) phase.

It is the unstable phase (Sarma phase) which becomes stable in this case (and in gCFL, see later) when charge neutrality is required.

(Huang & Shovkovy, 2003)
\[ \delta \mu = \frac{\mu_e}{2} \]

Solutions of the gap equation

\[ \mu_e = \text{const.} \]

Neutrality line

Normal state

\[ \delta \mu = \frac{\mu_e}{2} = \Delta \]

\[ \mu_e = 148 \text{ MeV} \]
The case of 3 flavors

(Alford, Kouvaris & Rajagopal, 2005)

$$\langle 0 | \psi^\alpha_{aL} \psi^\beta_{bL} | 0 \rangle = \Delta_1 \epsilon^{\alpha\beta 1} \epsilon_{ab1} + \Delta_2 \epsilon^{\alpha\beta 2} \epsilon_{ab2} + \Delta_3 \epsilon^{\alpha\beta 3} \epsilon_{ab3}$$

Different phases are characterized by different values for the gaps. For instance (but many other possibilities exist)

CFL :  $\Delta_1 = \Delta_2 = \Delta_3 = \Delta$

g2SC :  $\Delta_3 \neq 0$, $\Delta_1 = \Delta_2 = 0$

gCFL :  $\Delta_3 > \Delta_2 > \Delta_1$
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**Gaps in gCFL**

$\Delta_1$: ds – pairing
$\Delta_2$: us – pairing
$\Delta_3$: ud – pairing
Strange quark mass effects:

- Shift of the chemical potential for the strange quarks:
  \[ \mu_{\alpha s} \Rightarrow \mu_{\alpha s} - \frac{M_s^2}{2\mu} \]

- Color and electric neutrality in CFL requires
  \[ \mu_8 = -\frac{M_s^2}{2\mu}, \quad \mu_3 = \mu_e = 0 \]

- The transition CFL to gCFL starts with the unpairing of the pair \( ds \) with (close to the transition)
  \[ \delta \mu_{ds} = \frac{M_s^2}{2\mu} \]
It follows:

\[
\frac{M_s^2}{2\Delta} \quad \text{Energy cost for pairing}
\]

\[
\frac{\mu}{2\Delta} \quad \text{Energy gained in pairing}
\] begins to unpair \[
\frac{M_s^2}{\mu} > 2\Delta
\]

Calculations within a NJL model (modelled on one-gluon exchange):

- Write the free energy: \[ V(\mu, \mu_3, \mu_8, \mu_e, \Delta_i) \]
- Solve:

Neutralitiy \[
\frac{\partial V}{\partial \mu_e} = \frac{\partial V}{\partial \mu_3} = \frac{\partial V}{\partial \mu_8} = 0
\]

Gap equations \[
\frac{\partial V}{\partial \Delta_i} = 0
\]
\* CFL \Leftrightarrow gCFL 2^{\text{nd}} \text{ order transition at } M_s^2/\mu \sim 2\Delta, \text{ when the pairing } ds \text{ starts breaking}

(Alford, Kouvaris & Rajagopal, 2005)

\( (\Delta_0 = 25 \text{ MeV}, \mu = 500 \text{ MeV}) \)
● gCFL has gapless quasiparticles, and there are gluon imaginary masses (RC et al. 2004, Fukushima 2005).

● Instability present also in g2SC (Huang & Shovkovy 2004; Alford & Wang 2005)

\[ m_M^2(M_g) \over m_M^2(0) \]

\[ m_g^2 = \frac{\mu^2 g^2}{3\pi^2} \]
Proposals for solving the chromomagnetic instability

- Gluon condensation. Assuming artificially $<A_{\mu 3}>$ or $<A_{\mu 8}>$ not zero (of order 10 MeV) this can be done (RC et al. 2004). In g2SC the chromomagnetic instability can be cured by a chromo-magnetic condensate (Gorbar, Hashimoto, Miransky, 2005 & 2006; Kiriyama, Rischke, Shovkovy, 2006). Rotational symmetry is broken and this makes a connection with the inhomogeneous LOFF phase (see later). At the moment no extension to the three flavor case.
• **CFL-\(K^0\) phase.** When the stress is not too large (high density) the CFL pattern might be modified by a flavor rotation of the condensate equivalent to a condensate of \(K^0\) mesons (Bedaque, Schafer 2002). This occurs for \(m_s > m^{1/3} \Delta^{2/3}\). Also in this phase gapless modes are present and the gluonic instability arises (Kryjevski, Schafer 2005, Kryjevski, Yamada 2005). With a space dependent condensate a current can be generated which resolves the instability. Again some relations with the LOFF phase.
• **Single flavor pairing.** If the stress is too big single flavor pairing could occur but the gap is generally too small. It could be important at low $\mu$ before the nuclear phase (see for instance Alford 2006)

• **Secondary pairing.** The gapless modes could pair forming a secondary gap, but the gap is far too small (Huang, Shovkovy, 2003; Hong 2005; Alford, Wang, 2005)

• **Mixed phases of nuclear and quark matter** (Alford, Rajagopal, Reddy, Wilczek, 2001) as well as mixed phases between different CS phases, have been found either unstable or energetically disfavored (Neumann, Buballa, Oertel, 2002; Alford, Kouvaris, Rajagopal, 2004).
Chromomagnetic instability of g2SC makes the crystalline phase (LOFF) with two flavors energetically favored (Giannakis & Ren 2004), also there are no chromomagnetic instability although it has gapless modes (Giannakis & Ren 2005).

This makes the LOFF phase very interesting
LOFF phase

- **LOFF** (Larkin, Ovchinnikov; Fulde & Ferrel, 1964): ferromagnetic alloy with paramagnetic impurities.

- The impurities produce a constant exchange field acting upon the electron spin giving rise to an effective difference in the chemical potentials of the electrons producing a mismatch of the Fermi momenta.

- Studied also in the QCD context (Alford, Bowers & Rajagopal, 2000, for a review R.C. & Nardulli, 2003).
According to LOFF, close to first order point (CC point), possible condensation with non zero total momentum:

\[ \vec{p}_1 = \vec{k} + \vec{q}, \quad \vec{p}_2 = -\vec{k} + \vec{q} \]

More generally:

\[ \langle \psi(x)\psi(x) \rangle = \Delta e^{2i\vec{q} \cdot \vec{x}} \]

\[ \langle \psi(x)\psi(x) \rangle = \sum_m \Delta_m e^{2i\vec{q}_m \cdot \vec{x}} \]

\[ \vec{p}_1 + \vec{p}_2 = 2\vec{q} \]

\[ |\vec{q}| \quad \text{fixed variationally} \]

\[ \frac{\vec{q}}{|\vec{q}|} \quad \text{chosen spontaneously} \]
Single plane wave:

\[ E(\vec{k}) - \mu \Rightarrow E(\pm \vec{k} + \vec{q}) - \mu \mp \delta \mu \approx \sqrt{(|\vec{k}| - \mu)^2 + \Delta^2} \mp \bar{\mu} \]

\[ \bar{\mu} = \delta \mu - \vec{v}_F \cdot \vec{q} \]

Also in this case, for \( \bar{\mu} = \delta \mu - \vec{v}_F \cdot \vec{q} > \Delta \)

an unpairing (blocking) region opens up and gapless modes are present

More general possibilities include a crystalline structure \((\text{Larkin & Ovchinnikov 1964, Bowers & Rajagopal 2002})\)

\[ \langle \psi(x) \psi(x) \rangle = \Delta \sum e^{2i\vec{q}_i \cdot \vec{x}} \]

The \( q_i \)'s define the crystal pointing at its vertices.
The LOFF phase has been studied via a Ginzburg-Landau expansion of the grand potential

\[ \Omega = \alpha \Delta^2 + \frac{\beta}{2} \Delta^4 + \frac{\gamma}{3} \Delta^6 + \cdots \]

(for regular crystalline structures all the \( \Delta_q \) are equal)

The coefficients can be determined microscopically for the different structures

(Bowers and Rajagopal (2002))
General strategy

✶ Gap equation

✶ Propagator expansion

✶ Insert in the gap equation
We get the equation

\[ \alpha \Delta + \beta \Delta^3 + \gamma \Delta^5 + \cdots = 0 \]

Which is the same as

\[ \frac{\partial \Omega}{\partial \Delta} = 0 \]

with

\[ \alpha \Delta = \quad + \quad \]
\[ \beta \Delta^3 = \quad \]
\[ \gamma \Delta^5 = \quad \]

The first coefficient has universal structure, independent on the crystal. From its analysis one draws the following results.
LOFF and BCS

\[ \Omega_{\text{BCS}} - \Omega_{\text{normal}} = -\frac{\rho}{4} (\Delta_{\text{BCS}}^2 - 2\delta\mu^2) \]

\[ \Omega_{\text{LOFF}} - \Omega_{\text{normal}} = -0.44\rho(\delta\mu - \delta\mu_2)^2 \]

\[ \Delta_{\text{LOFF}} \approx 1.15\sqrt{(\delta\mu_2 - \delta\mu)} \]

\[ \delta\mu_1 = \Delta_{\text{BCS}} / \sqrt{2} \quad \delta\mu_2 \approx 0.754\Delta_{\text{BCS}} \]

Small window. Opens up in QCD (Leibovich, Rajagopal & Shuster 2001)
Along the critical line

\[(at \ T = 0, \ q = 1.2 \delta \mu_2)\]
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<th>$\overline{y}$</th>
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<td>-4.637</td>
<td>1.867</td>
</tr>
<tr>
<td>square antipris</td>
<td>10</td>
<td>$D_{4d}(1441)$</td>
<td>-14.298</td>
<td>7318.885</td>
<td>$-9.1 \times 10^{-6}$</td>
<td>0.755</td>
</tr>
<tr>
<td>bicapped</td>
<td>10</td>
<td>$D_{4d}(1441)$</td>
<td>-14.298</td>
<td>7318.885</td>
<td>$-9.1 \times 10^{-6}$</td>
<td>0.755</td>
</tr>
<tr>
<td>icosahedron</td>
<td>12</td>
<td>$I_h(1551)$</td>
<td>204.873</td>
<td>145076.754</td>
<td>0</td>
<td>0.754</td>
</tr>
<tr>
<td>cuboctahedron</td>
<td>12</td>
<td>$O_h(444)$</td>
<td>-5.296</td>
<td>97086.514</td>
<td>$-2.6 \times 10^{-9}$</td>
<td>0.754</td>
</tr>
<tr>
<td>dodecahedron</td>
<td>20</td>
<td>$I_h(5555)$</td>
<td>-527.357</td>
<td>114166.566</td>
<td>-0.0019</td>
<td>0.772</td>
</tr>
</tbody>
</table>

Preferred structure: face-centered cube
Results about LOFF with three flavors

Study of LOFF with 3 flavors within the following simplifying hypothesis (RC, Gatto, Ippolito, Nardulli & Ruggieri, 2005)

- Study within the Landau-Ginzburg approximation.
- Only electrical neutrality imposed (chemical potentials $\mu_3$ and $\mu_8$ taken equal to zero, OK if close to the transition to the normal state).
- $M_s$ treated as in gCFL. Pairing similar to gCFL with inhomogeneity in terms of simple plane waves, as for the simplest LOFF phase.

\[
\langle \psi^\alpha_{aL} \psi^\beta_{bL} \rangle = \sum_{I=1}^{3} \Delta_I(\vec{x}) e^{\alpha\beta I} \epsilon_{abI}, \quad \Delta_I(\vec{x}) = \Delta_I e^{2i\vec{q}_I \cdot \vec{x}}
\]
• A further simplifications is to assume only the following geometrical configurations for the vectors $q_i$, $i = 1,2,3$

\[
1 \quad \uparrow \quad \uparrow \quad \uparrow \\
2 \quad \downarrow \quad \uparrow \quad \uparrow \\
3 \quad \uparrow \quad \uparrow \quad \uparrow \\
4 \quad \uparrow \quad \uparrow \quad \downarrow 
\]

• The free energy, in the GL expansion, has the form

\[
\Omega - \Omega_{\text{normal}} = \sum_{I=1}^{3} \left( \frac{\alpha_I}{2} \Delta_I^2 + \frac{\beta_I}{4} \Delta_I^4 + \sum_{I \neq J} \frac{\beta_{IJ}}{4} \Delta_I^2 \Delta_J^2 \right) + O(\Delta^6)
\]

\[
\Omega_{\text{normal}} = -\frac{3}{12} \pi^2 (\mu_u^4 + \mu_d^4 + \mu_s^4) - \frac{1}{12} \pi^2 \mu_e^4
\]

• with coefficients $\alpha_I$, $\beta_I$ and $\beta_{IJ}$ calculable from an effective NJL four-fermi interaction simulating one-gluon exchange
\[ \Delta_0 \equiv \Delta_{BCS}, \quad \mu_u = \mu - \frac{2}{3}\mu_e, \quad \mu_d = \mu + \frac{1}{3}\mu_e, \quad \mu_s = \mu + \frac{1}{3}\mu_e - \frac{M_s^2}{2\mu} \]

\[ \alpha_I(q, \delta \mu) = -\frac{4\mu^2}{\pi^2} \left( 1 - \frac{\delta \mu}{2q} \log \left| \frac{q + \delta \mu}{q - \delta \mu} \right| - \frac{1}{2} \log \left| \frac{4(q^2 - \delta \mu_I^2)}{\Delta_0^2} \right| \right) \]

\[ \beta_I(q, \delta \mu) = \frac{\mu^2}{\pi^2 q^2 - \delta \mu_I^2} \]

\[ \beta_{12} = -\frac{3\mu^2}{\pi^2} \int \frac{dn}{4\pi} \frac{1}{(2q_1 \cdot n + \mu_s - \mu_d)(2q_2 \cdot n + \mu_s - \mu_u)} \]

Others by the exchange:

\[ 12 \to 23, \quad \mu_s \leftrightarrow \mu_d \]

\[ 12 \to 13, \quad \mu_s \leftrightarrow \mu_u \]
We require:

\[
\frac{\partial \Omega}{\partial \Delta_I} = \frac{\partial \Omega}{\partial q^I} = \frac{\partial \Omega}{\partial \mu_e} = 0
\]

At the lowest order in \(\Delta_I\)

\[
\frac{\partial \Omega}{\partial q^I} = 0 \Rightarrow \frac{\partial \alpha_I}{\partial q^I} = 0
\]

since \(\alpha_I\) depends only on \(q_i\) and \(\delta \mu_i\)

we get the same result as in the simplest LOFF case:

\[
|\vec{q}_I| = 1.2 \delta \mu_I
\]

In the GL approximation we expect to be pretty close to the normal phase, therefore we will assume \(\mu_3 = \mu_8 = 0\). At the same order we expect \(\Delta_2 = \Delta_3\) (equal mismatch) and \(\Delta_1 = 0\) (ds mismatch is twice the ud and us).
Once assumed $\Delta_1 = 0$, only two configurations for $q_2$ and $q_3$, parallel or antiparallel. The antiparallel is disfavored due to the lack of configurations space for the up fermions.
We have assumed the same parameters as in Alford et al. in gCFL, $\Delta_0 = 25$ MeV, $\mu = 500$ MeV.

$\Delta_1 : ds - pairing$

$\Delta_2 : us - pairing$

$\Delta_3 : ud - pairing$

$\Delta_1 = 0, \quad \Delta_2 = \Delta_3$
Comparison with other phases

- LOFF phase takes over gCFL at about 128 MeV and goes over to the normal phase at about 150 MeV

confirmed by an exact solution of the gap equation (Mannarelli, Rajagopal, Sharma, 2006)
No chromo-magnetic instability in the LOFF phase with three flavors (Ciminale, Gatto, Nardulli, Ruggieri, 2006)

\[ M_1 = M_2 = M_4 = M_5 \]

\[ M_6 = M_7 \]
Extension to a crystalline structure (Rajagopal, Sharma 2006), always within the simplifying assumption $\Delta_1 = 0$ and $\Delta_2 = \Delta_3$

$$\langle ud \rangle \approx \Delta_3 \sum_a \exp(2i\vec{q}_3^a \cdot \vec{r}), \quad \langle us \rangle \approx \Delta_2 \sum_a \exp(2i\vec{q}_2^a \cdot \vec{r})$$

The sum over the index $a$ goes up to 8 $q_i^a$. Assuming also $\Delta_2 = \Delta_3$ the favored structures (always in the GL approximation up to $\Delta^6$) among 11 structures analyzed are

CubeX

2Cube45z
Conclusions

● Various phases are competing, many of them having gapless modes. However, when such modes are present a chromomagnetic instability arises.

● Also the LOFF phase is gapless but the gluon instability does not seem to appear.

● Recent studies of the LOFF phase with three flavors seem to suggest that this should be the favored phase after CFL, although this study is very much simplified and more careful investigations should be performed.

● The problem of the QCD phases at moderate densities and low temperature is still open.
In the LOFF phase translations and rotations are broken.

Phonons

Phonon field through the phase of the condensate (R.C., Gatto, Mannarelli & Nardulli 2002):

\[
\langle \psi(x)\psi(x) \rangle = \Delta e^{2i\mathbf{q} \cdot \mathbf{x}} \rightarrow \Delta e^{i\Phi(x)} \quad \langle \Phi(x) \rangle = 2\mathbf{q} \cdot \mathbf{x}
\]

Introducing

\[
\frac{1}{f} \varphi(x) = \Phi(x) - 2\mathbf{q} \cdot \mathbf{x}
\]
\[ \hat{\mathbf{q}} = (0,0,1) \]

\[ L_{\text{phonon}} = \left[ \frac{1}{2} \phi^{2} - \nu^{2} \left( \frac{\partial^{2} \varphi}{\partial x^{2}} + \frac{\partial^{2} \varphi}{\partial y^{2}} \right) - \nu^{2} \frac{\partial^{2} \varphi}{\partial z^{2}} \right] \]

Coupling phonons to fermions (quasi-particles) through the gap term

\[ \Delta(x) \psi^\dagger C \psi \rightarrow \Delta e^{i\Phi(x)} \psi^\dagger C \psi \]

It is possible to evaluate the parameters of \( L_{\text{phonon}} \) (R.C., Gatto, Mannarelli & Nardulli 2002)

\[ v_{\perp}^{2} = \frac{1}{2} \left( 1 - \left( \frac{\delta \mu}{|\hat{\mathbf{q}}|} \right)^{2} \right) \approx 0.153 \]

\[ v_{||}^{2} = \left( \frac{\delta \mu}{|\hat{\mathbf{q}}|} \right)^{2} \approx 0.694 \]
Cubic structure

\[ \Delta(x) = \Delta \sum_{k=1}^{8} e^{2i\bar{q} \cdot \bar{x}} = \Delta \sum_{i=1,2,3; \varepsilon_i = \pm} e^{2i|\bar{q}|\varepsilon_i x_i} \Rightarrow \Delta \sum_{i=1,2,3; \varepsilon_i = \pm} e^{i\varepsilon_i \Phi^{(i)}(x)} \]

3 scalar fields \( \Phi^{(i)}(x) \)

\[ \langle \Phi^{(i)}(x) \rangle = 2 |\bar{q}| x_i \]

\[ \frac{1}{f} \varphi^{(i)}(x) = \Phi^{(i)}(x) - 2 |\bar{q}| x_i \]
\[ \Phi^{(i)}(x) \text{ transforms under the group } O_h \text{ of the cube.} \]

Its e.v. \( \sim x^i \) breaks \( O(3) \times O_h \rightarrow O_h^{\text{diag}} \). Therefore we get

\[ L_{\text{phonon}} = \frac{1}{2} \sum_{i=1,2,3} \left( \frac{\partial \Phi^{(i)}}{\partial t} \right)^2 - \frac{a}{2} \sum_{i=1,2,3} | \vec{\nabla} \Phi^{(i)} |^2 \]

\[ -\frac{b}{2} \sum_{i=1,2,3} \left( \partial_i \Phi^{(i)} \right)^2 - c \sum_{i<j=1,2,3} \left( \partial_i \Phi^{(i)} \partial_j \Phi^{(j)} \right) \]

Coupling phonons to fermions (quasi-particles) through the gap term

\[ \Delta(x) \psi^T C \psi \rightarrow \Delta \sum_{i=1,2,3; \varepsilon_i = \pm} e^{i \varepsilon_i \Phi^{(i)}(x)} \psi^T C \psi \]
we get for the coefficients

\[ a = \frac{1}{12} \quad b = 0 \quad c = \frac{1}{12} \left( 3 \left( \frac{\delta \mu}{|q|} \right)^2 - 1 \right) \]

One can evaluate the effective lagrangian for the gluons in anisotropic medium. For the cube one finds

**Isotropic propagation**

This because the second order invariant for the cube and for the rotation group are the same!