# Anisotropic color superconductivity

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# Summary

- Introduction
- Anisotropic phase (LOFF). Critical points
- Crystalline structures in LOFF
- Phonons
- Conclusions

# Introduction

Study of CS back to 1977 (Barrois 1977, Frautschi 1978, Bailin and Love 1984) based on Cooper instability:

At T ~ 0 a degenerate fermion gas is unstable

***Any weak attractive interaction leads to Cooper pair formation***

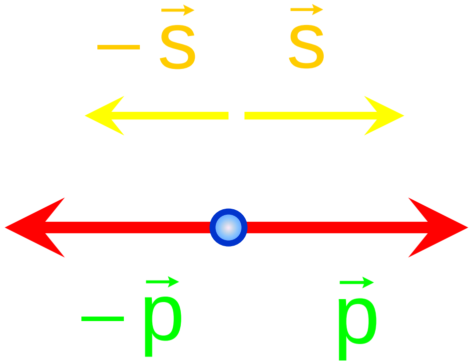
- Hard for electrons (Coulomb vs. phonons)
- Easy in QCD for di-quark formation (attractive channel $\bar{3}$) $(3 \otimes 3 = \bar{3} \oplus 6)$

# Good news!!! CS easy for large $\mu$ due to asymptotic freedom

At high $\mu$, $m_s$, $m_d$, $m_u \sim 0$, 3 colors and 3 flavors

Possible pairings: $\langle 0 | \psi^{\alpha}_{ia} \psi^{\beta}_{jb} | 0 \rangle$

- Antisymmetry in color ($\alpha$, $\beta$) for attraction
- Antisymmetry in spin (a,b) for better use of the Fermi surface
- Antisymmetry in flavor (i, j) for Pauli principle




**Only possible pairings**

**LL and RR**

**Favorite state CFL (color-flavor locking)**

(Alford, Rajagopal & Wilczek 1999)

$$\langle 0 | \psi^{\alpha}_{aL} \psi^{\beta}_{bL} | 0 \rangle = -\langle 0 | \psi^{\alpha}_{aR} \psi^{\beta}_{bR} | 0 \rangle \cong \Delta \varepsilon^{\alpha\beta C} \varepsilon_{abC}$$

**Symmetry breaking pattern**

$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \Rightarrow SU(3)_{c+L+R}$$

What happens going down with $\mu$? If $\mu \ll m_s$ we get

3 colors and 2 flavors (2SC)

$$\langle 0 | \psi^{\alpha}_{aL} \psi^{\beta}_{bL} | 0 \rangle = \Delta \epsilon^{\alpha\beta 3} \epsilon_{ab}$$

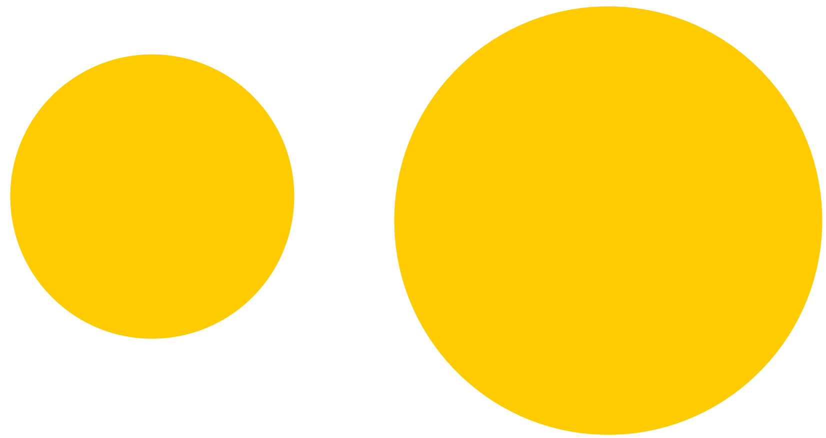
$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \Rightarrow SU(2)_c \otimes SU(2)_L \otimes SU(2)_R$$

In this situation strange quark decouples. But what happens in the intermediate region of $\mu$? The interesting region is for

$$\mu \sim m_s^2/\Delta$$

Consider 2 fermions with $m_1 = M$, $m_2 = 0$ at the same chemical potential $\mu$. The Fermi momenta are

$$p_{F1} = \sqrt{\mu^2 - M^2}$$



$$p_{F2} = \mu$$

To form a BCS condensate one needs common momenta of the pair $p_F^{comm}$

$$p_F^{comm} = \frac{1}{2}\left(\sqrt{\mu^2 - M^2} + \mu\right) \approx \mu - \frac{M^2}{4\mu}$$

With energy cost of $\sim M^2/4\mu$ for bringing the fermions at the same $p_F^{comm}$

To have a stable pair the energy cost must be less than the energy for breaking a pair ~ Δ

$$\frac{M^2}{4\mu} \leq \Delta$$

The problem may be simulated using massless fermions with different chemical potentials (Alford, Bowers & Rajagopal 2000)

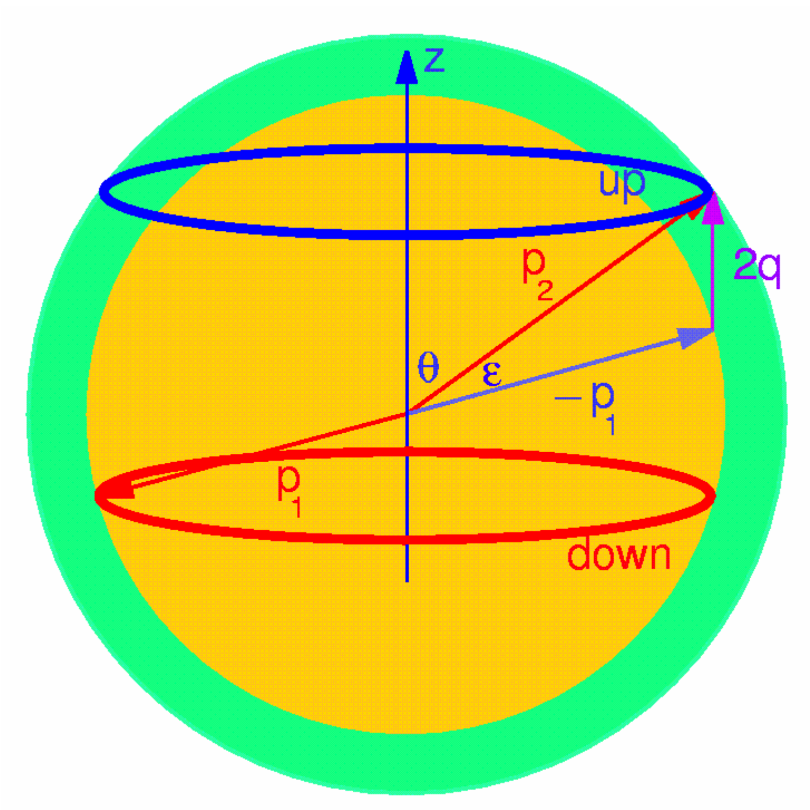
Analogous problem studied by Larkin & Ovchinnikov, Fulde & Ferrel 1964. Proposal of a new way of pairing. LOFF phase

LOFF: ferromagnetic alloy with paramagnetic impurities. The impurities produce a constant exchange field acting upon the electron spins giving rise to an effective difference in the chemical potentials of the opposite spins. Very difficult experimentally but claims of observations in heavy fermion superconductors (Gloos & al 1993) and in quasi-two dimensional layered organic superconductors (Nam & al. 1999, Manalo & Klein 2000)

# LOFF phase

**The LOFF pairing breaks translational and rotational invariance**

$$\langle \mathrm{LOFF} | \psi(x)\psi(x) | \mathrm{LOFF} \rangle \approx \Delta e^{2i\vec{q}\cdot\vec{x}}$$




$$\vec{p}_1 + \vec{p}_2 = 2\vec{q}$$

$|\vec{q}|$ fixed variationally

$\vec{q}/|\vec{q}|$ chosen spontaneously

# Strategy of calculations at large $\mu$

Microscopic description → $\mathscr{L}_{QCD}$

↓

Quasi-particles (dressed fermions as electrons in metals). Decoupling of antiparticles (Hong 2000) → $\mathscr{L}_{HDET}$

$p - p_F >> \Delta$

↓

Decoupling of gapped quasi-particles. Only light modes as Goldstones, etc. (R.C. & Gatto; Hong, Rho & Zahed 1999) → $\mathscr{L}_{Gold}$

$p - p_F << \Delta$

$\mathscr{L}_{\mathrm{HDET}}$ may be used for evaluating the gap and for matching the parameters of $\mathscr{L}_{\mathrm{Gold}}$

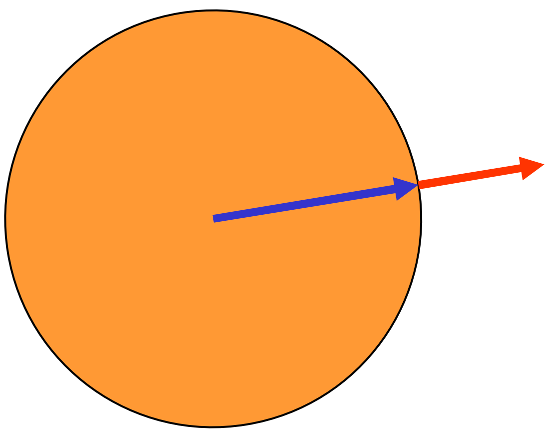
# Gap equation for BCS

Interactions gap the fermions → Quasi-particles

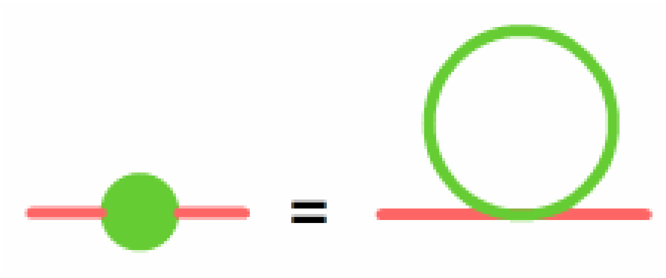
$$\varepsilon(\vec{p},\Delta) = \sqrt{\xi^2 + \Delta_{BCS}^2}$$

$$\xi = E(\vec{p}) - \mu \cong \left.\frac{\partial E(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_F} \cdot (\vec{p} - \vec{p}_F) = \vec{v}_F \cdot (\vec{p} - \vec{p}_F)$$

Fermi velocity — residual momentum



# Start from euclidean gap equation for 4-fermion interaction




$$1 = -g\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p_4 - i\mu)^2 + |\vec{p}|^2 + \Delta_{BCS}^2}$$

$$1 = gT\int \frac{d^3p}{(2\pi)^3} \sum_{n=-\infty}^{+\infty} \frac{1}{((2n+1)\pi T)^2 + \varepsilon(\vec{p},\Delta)^2}$$

$$1 = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1 - n_u - n_d}{\varepsilon(\vec{p},\Delta)}$$

$$n_u = n_d = \frac{1}{e^{\varepsilon(\vec{p},\Delta)/T} + 1}$$

## For T $\mathcal{T}$ 0

$$1 = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\xi^2(\vec{p}) + \Delta_{BCS}^2}}$$

## At weak coupling

$$1 \cong \frac{g}{2\pi^2}\frac{p_F^2}{v_F}\log\frac{2\bar{\xi}}{\Delta_{BCS}} \qquad (\bar{\xi} = \text{cutoff})$$

$$\Delta_{BCS} \approx 2\bar{\xi}e^{-2/g\rho}$$

$$\rho = \frac{p_F^2}{\pi^2 v_F}$$ density of states

# Anisotropic superconductivity

$\mu_1 \neq \mu_2$ or paramagnetic impurities ($\delta\mu \sim H$) give rise to an energy additive term

$$H_I = -\delta\mu\sigma_3$$

According LOFF this favours pair formation with momenta

$$\vec{p}_1 = \vec{k} + \vec{q} \qquad \vec{p}_2 = -\vec{k} + \vec{q}$$

Simplest case (single plane wave) → $\langle\psi(x)\psi(x)\rangle = \Delta e^{2i\vec{q}\cdot\vec{x}}$

More generally → $\langle\psi(x)\psi(x)\rangle = \Delta \sum_m c_m e^{2i\vec{q}_m\cdot\vec{x}}$

## **Simple plane wave:** energy shift

$$E(\vec{p}) - \mu \rightarrow E(\pm\vec{k} + \vec{q}) - \mu \mp \delta\mu \approx \xi \mp \bar{\mu}$$

$$\bar{\mu} = \delta\mu - \vec{v}_F \cdot \vec{q}$$

Gap equation:

$$1 = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1 - n_u - n_d}{\varepsilon(\vec{p},\Delta)}$$

$$n_u \neq n_d \quad \longrightarrow \quad n_{u,d} = \frac{1}{e^{(\varepsilon(\vec{p},\Delta)\pm\bar{\mu})/T} + 1}$$

For T 𝒯 0

blocking region $\varepsilon < |\bar{\mu}|$

$$1 = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1}{\varepsilon(\vec{p},\Delta)}(1 - \theta(-\varepsilon - \bar{\mu}) - \theta(-\varepsilon + \bar{\mu}))$$

The blocking region reduces the gap:

$$\Delta_{LOFF} << \Delta_{BCS}$$

Possibility of a crystalline structure (Larkin & Ovchinnikov 1964, Bowers & Rajagopal 2002)

$$\langle\psi(x)\psi(x)\rangle = \sum_{|\vec{q}_i|=1.2\delta\mu} \Delta_{\vec{q}_i} e^{2i\vec{q}_i\cdot\vec{x}}$$

→ see later

The $q_i$'s define the crystal pointing at its vertices.

The LOFF phase is studied (except for the single plane wave) via a Ginzburg-Landau expansion of the grand potential

$$\Omega = \alpha\Delta^2 + \frac{\beta}{2}\Delta^4 + \frac{\gamma}{3}\Delta^6 + \cdots$$

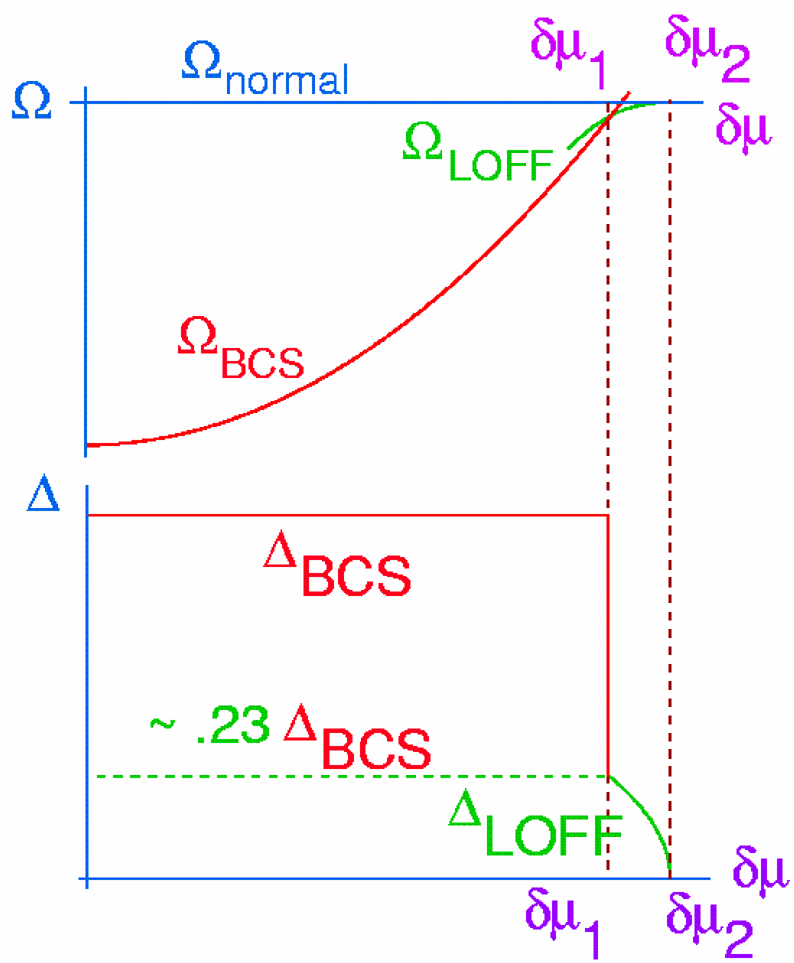
(for regular crystalline structures all the $\Delta_q$ are equal)

The coefficients can be determined microscopically for the different structures. The first coefficient has universal structure, independent on the crystal. From its analysis one draws the following results

# Two critical values in $\delta\mu$:

$$\delta\mu = \frac{1}{2}(\mu_1 - \mu_2)$$

# LOFF and BCS




$$\Omega_{BCS} - \Omega_{normal} = -\frac{\rho}{4}(\Delta^2_{BCS} - 2\delta\mu^2)$$

$$\Omega_{LOFF} - \Omega_{normal} = -0.44\rho(\delta\mu - \delta\mu_2)^2$$

$$\Delta_{LOFF} \approx 1.15\sqrt{(\delta\mu_2 - \delta\mu)}$$

$$\delta\mu_1 = \Delta_{BCS} / \sqrt{2}$$

$$\delta\mu_2 \approx 0.754\Delta_{BCS}$$

Small window. Opens up in QCD? (Leibovich, Rajagopal & Shuster 2001; Giannakis, Liu & Ren 2002)

The LOFF gap equation around zero LOFF gap gives

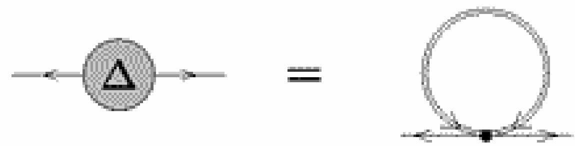
$$\log\frac{\Delta_{BCS}}{2\delta\mu} \cong \frac{1}{2} f(z)$$

$$f(z) = \int_{-1}^{+1} \log(1+uz)du \qquad z = \frac{\delta\mu}{|\vec{q}|}$$

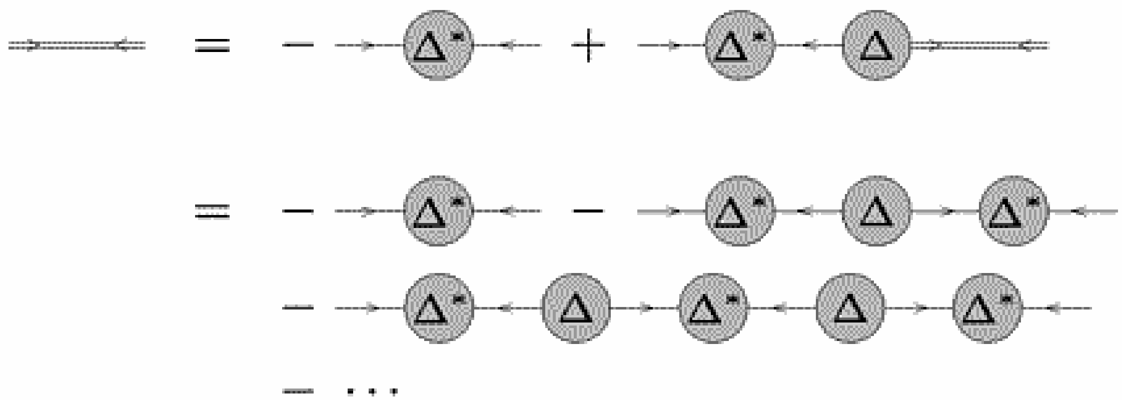
For $\delta\mu \to \delta\mu_2$, f(z) must reach a minimum

$$z = \coth(z) \Rightarrow \frac{\delta\mu}{|\vec{q}|} \cong 1.2 \Rightarrow \delta\mu_2 \cong 0.754\Delta_{BCS}$$

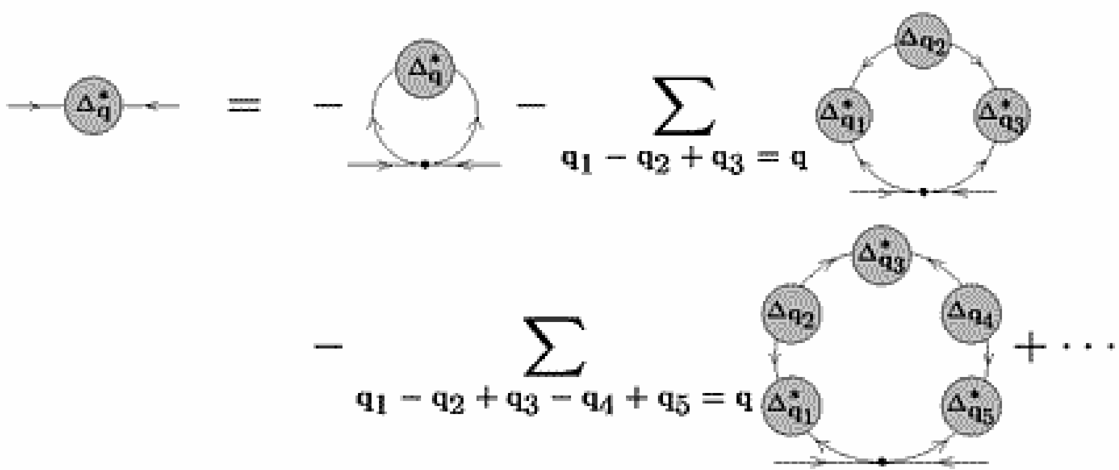
- Dyson equation for the gaps $\{\Delta_{\mathbf{q}}\}$:



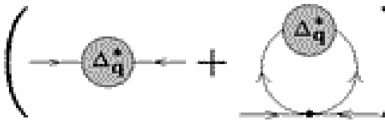

- Expand anomalous propagator in powers of $\Delta$:



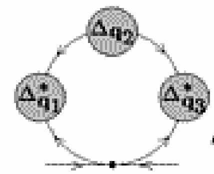

- Substitute into Dyson equation:



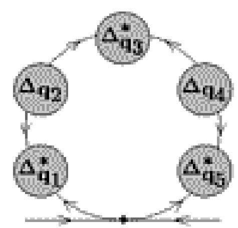

The expansion and the results as given by Bowers & Rajagopal 2002

- $\alpha(\mathbf{q}) = \Big($  $\Big) \Big/ \Delta^*_{\mathbf{q}}$

$$\propto \quad 1 - 2i\lambda \int \frac{d^4p}{(2\pi)^4} \gamma_\mu (\not p - \not\mu_d)^{-1} (\not p + 2\not q + \not\mu_u)^{-1} \gamma^\mu$$

- $J(\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\mathbf{q}_4) =$  $\Big/ \left( \Delta^*_{\mathbf{q}_1} \Delta_{\mathbf{q}_2} \Delta^*_{\mathbf{q}_3} \right)$

$$\begin{aligned} \propto \quad & \int \frac{d^4p}{(2\pi)^4} \gamma_\mu (\not p - \not\mu_d)^{-1} (\not p + 2\not q_1 + \not\mu_u)^{-1} \\ & \times (\not p + 2\not q_1 - 2\not q_2 - \not\mu_d)^{-1} \\ & \times (\not p + 2\not q_1 - 2\not q_2 + 2\not q_3 + \not\mu_u)^{-1} \gamma^\mu \end{aligned}$$

- $K(\mathbf{q}_1 \cdots \mathbf{q}_6) =$  $\Big/ \left( \Delta^*_{\mathbf{q}_1} \Delta_{\mathbf{q}_2} \Delta^*_{\mathbf{q}_3} \Delta_{\mathbf{q}_4} \Delta^*_{\mathbf{q}_5} \right)$

$$\begin{aligned} \propto \quad & \int \frac{d^4p}{(2\pi)^4} \gamma_\mu (\not p - \not\mu_d)^{-1} (\not p + 2\not q_1 + \not\mu_u)^{-1} \\ & \times (\not p + 2\not q_1 - 2\not q_2 - \not\mu_d)^{-1} \\ & \times (\not p + 2\not q_1 - 2\not q_2 + 2\not q_3 + \not\mu_u)^{-1} \\ & \times (\not p + 2\not q_1 - 2\not q_2 + 2\not q_3 - 2\not q_4 - \not\mu_d)^{-1} \\ & \times (\not p + 2\not q_1 - 2\not q_2 + 2\not q_3 - 2\not q_4 + 2\not q_5 + \not\mu_u)^{-1} \gamma^\mu \end{aligned}$$

| Structure | P | $\mathcal{G}$(Föppl) | $\bar{\beta}$ | $\bar{\gamma}$ | $\bar{\Omega}_{\min}$ | $\delta\mu_*/\Delta_0$ |
|---|---|---|---|---|---|---|
| point | 1 | $C_{\infty v}(1)$ | 0.569 | 1.637 | 0 | 0.754 |
| antipodal pair | 2 | $D_{\infty v}(11)$ | 0.138 | 1.952 | 0 | 0.754 |
| triangle | 3 | $D_{3h}(3)$ | -1.976 | 1.687 | -0.452 | 0.872 |
| tetrahedron | 4 | $T_d(13)$ | -5.727 | 4.350 | -1.655 | 1.074 |
| square | 4 | $D_{4h}(4)$ | -10.350 | -1.538 | – | – |
| pentagon | 5 | $D_{5h}(5)$ | -13.004 | 8.386 | -5.211 | 1.607 |
| trigonal bipyramid | 5 | $D_{3h}(131)$ | -11.613 | 13.913 | -1.348 | 1.085 |
| square pyramid | 5 | $C_{4v}(14)$ | -22.014 | -70.442 | – | – |
| octahedron | 6 | $O_h(141)$ | -31.466 | 19.711 | -13.365 | 3.625 |
| trigonal prism | 6 | $D_{3h}(33)$ | -35.018 | -35.202 | – | – |
| hexagon | 6 | $D_{6h}(6)$ | 23.669 | 6009.225 | 0 | 0.754 |
| pentagonal bipyramid | 7 | $D_{5h}(151)$ | -29.158 | 54.822 | -1.375 | 1.143 |
| capped trigonal antiprism | 7 | $C_{3v}(13\bar{3})$ | -65.112 | -195.592 | – | – |
| cube | 8 | $O_h(44)$ | -110.757 | -459.242 | – | – |
| square antiprism | 8 | $D_{4d}(4\bar{4})$ | -57.363 | -6.866 | – | – |
| hexagonal bipyramid | 8 | $D_{6h}(161)$ | -8.074 | 5595.528 | $-2.8 \times 10^{-6}$ | 0.755 |
| augmented trigonal prism | 9 | $D_{3h}(3\bar{3}\bar{3})$ | -69.857 | 129.259 | -3.401 | 1.656 |
| capped square prism | 9 | $C_{4v}(144)$ | -95.529 | 7771.152 | -0.0024 | 0.773 |
| capped square antiprism | 9 | $C_{4v}(14\bar{4})$ | -68.025 | 106.362 | -4.637 | 1.867 |
| bicapped square antiprism | 10 | $D_{4d}(14\bar{4}1)$ | -14.298 | 7318.885 | $-9.1 \times 10^{-6}$ | 0.755 |
| icosahedron | 12 | $I_h(15\bar{5}1)$ | 204.873 | 145076.754 | 0 | 0.754 |
| cuboctahedron | 12 | $O_h(4\bar{4}\bar{4})$ | -5.296 | 97086.514 | $-2.6 \times 10^{-9}$ | 0.754 |
| dodecahedron | 20 | $I_h(5555)$ | -527.357 | 114166.566 | -0.0019 | 0.772 |

Preferred structure:

face-centered cube

# Phonons

In the LOFF phase translations and rotations are broken

↓

phonons

Phonon field through the phase of the condensate (R.C., Gatto, Mannarelli & Nardulli 2002):

$$\langle \psi(x)\psi(x) \rangle = \Delta e^{2i\vec{q}\cdot\vec{x}} \rightarrow \Delta e^{i\Phi(x)} \qquad \langle \Phi(x) \rangle = 2\vec{q}\cdot\vec{x}$$

introducing

$$\frac{1}{f}\phi(x) = \Phi(x) - 2\vec{q}\cdot\vec{x}$$

⇒

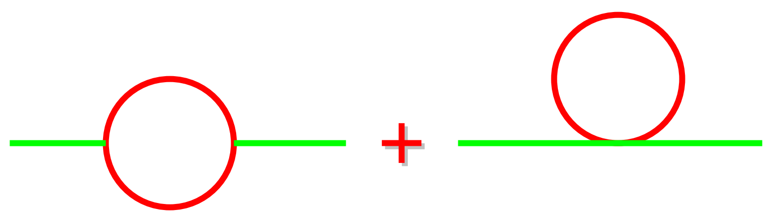
$$\vec{q} = (0,0,1)$$

$$\mathcal{L}_{phonon} = \left[\frac{1}{2}\dot{\phi}^2 - v_\perp^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) - v_{\parallel}^2\frac{\partial^2\phi}{\partial z^2}\right]$$

Coupling phonons to fermions (quasi-particles) trough the gap term

$$\Delta(x)\psi^T C\psi \to \Delta e^{i\Phi(x)}\psi^T C\psi$$

It is possible to evaluate the parameters of $L_{phonon}$ (R.C., Gatto, Mannarelli & Nardulli 2002)



$$v_\perp^2 = \frac{1}{2}\left(1 - \left(\frac{\delta\mu}{|\vec{q}|}\right)^2\right) \approx 0.153 \qquad v_{\parallel}^2 = \left(\frac{\delta\mu}{|\vec{q}|}\right)^2 \approx 0.694$$

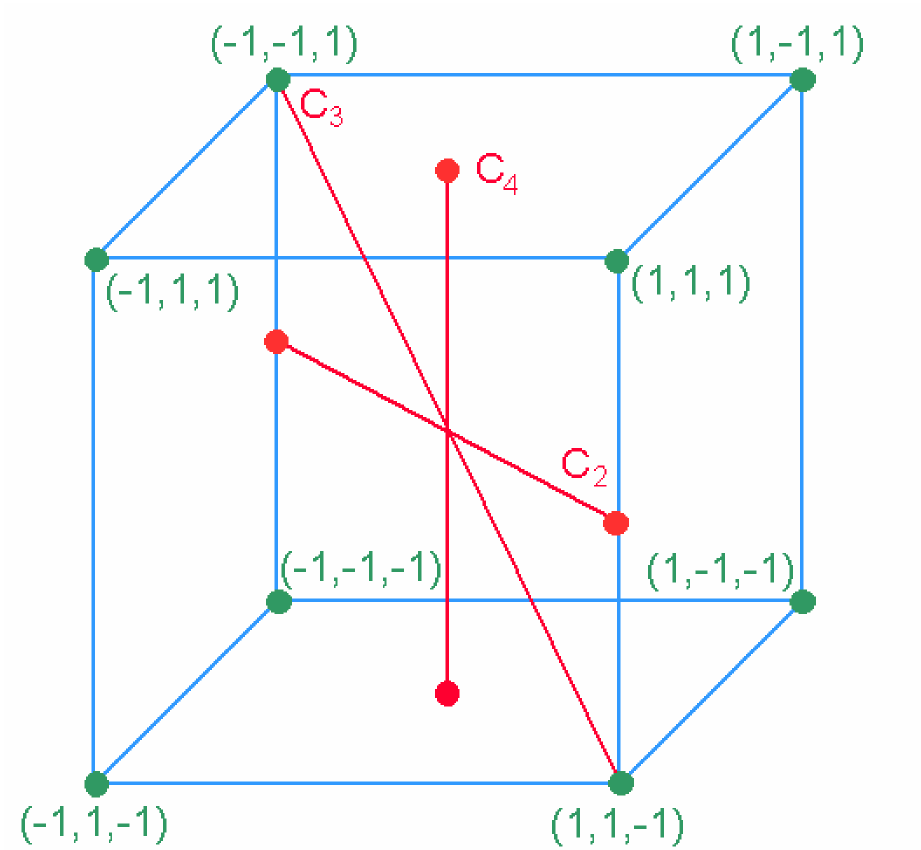
# Cubic structure

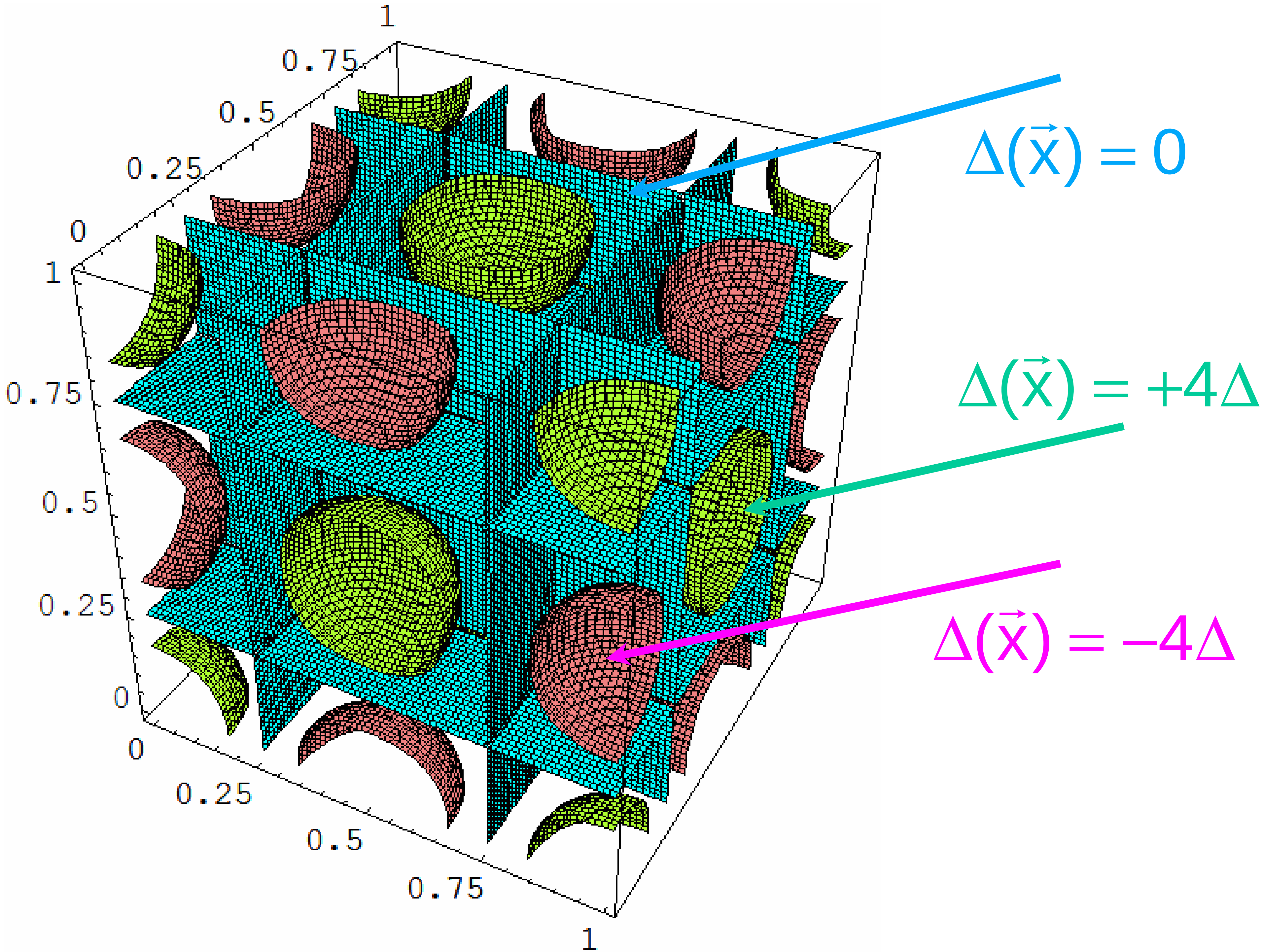
$$\Delta(x) = \Delta \sum_{k=1}^{8} e^{2i\vec{q}_k \cdot \vec{x}} = \Delta \sum_{i=1,2,3;\varepsilon_i=\pm} e^{2i|\vec{q}|\varepsilon_i x_i} \Rightarrow \Delta \sum_{i=1,2,3;\varepsilon_i=\pm} e^{i\varepsilon_i \Phi^{(i)}(x)}$$

3 scalar fields $\Phi^{(i)}(x)$

$$\left\langle \Phi^{(i)}(x) \right\rangle = 2|\vec{q}|x_i$$

$$\frac{1}{f}\varphi^{(i)}(x) = \Phi^{(i)}(x) - 2|\vec{q}|x_i$$





1
0.75
0.5
0.25
0
1
0.75
0.5
0.25
0
0
0.25
0.5
0.75
1
$\Delta(\vec{x}) = 0$
$\Delta(\vec{x}) = +4\Delta$
$\Delta(\vec{x}) = -4\Delta$

$\Phi^{(i)}(x)$ transforms under the group $O_h$ of the cube.

Its e.v. ~ $x^i$ breaks $O(3)xO_h$ $\mathcal{T}$ $O_h^{diag}$. Therefore we get

$$\mathcal{L}_{phonon} = \frac{1}{2}\sum_{i=1,2,3}\left(\frac{\partial\phi^{(i)}}{\partial t}\right)^2 - \frac{a}{2}\sum_{i=1,2,3}|\vec{\nabla}\phi^{(i)}|^2$$

$$-\frac{b}{2}\sum_{i=1,2,3}\left(\partial_i\phi^{(i)}\right)^2 - c\sum_{i<j=1,2,3}\left(\partial_i\phi^{(i)}\partial_j\phi^{(j)}\right)$$

Coupling phonons to fermions (quasi-particles) trough the gap term

$$\Delta(x)\psi^T C\psi \rightarrow \Delta \sum_{i=1,2,3;\varepsilon_i=\pm} e^{i\varepsilon_i\Phi^{(i)}(x)}\psi^T C\psi$$

we get for the coefficients

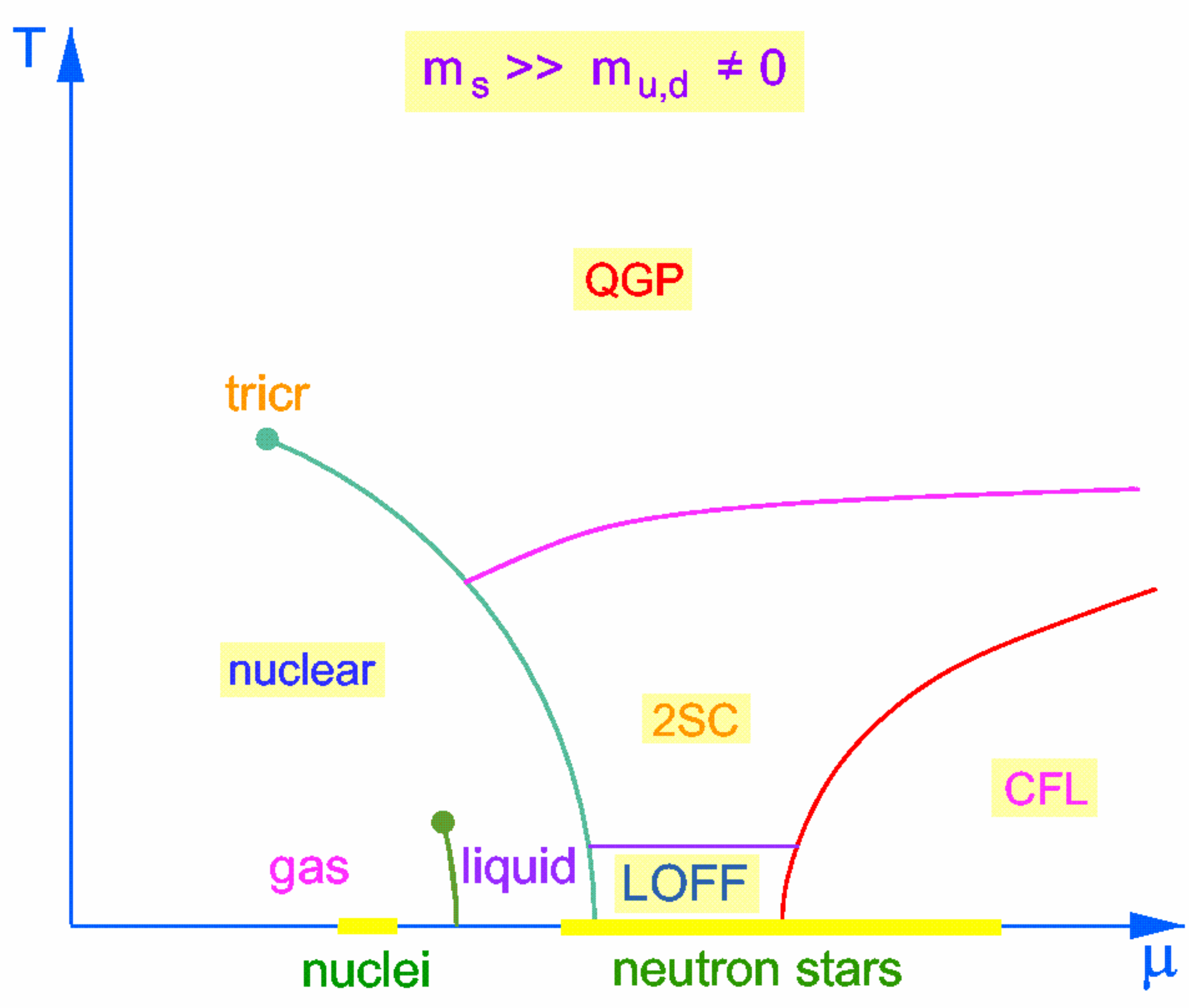
$$a = \frac{1}{12} \qquad b = 0 \qquad c = \frac{1}{12}\left(3\left(\frac{\delta\mu}{|\vec{q}\,|}\right)^2 - 1\right)$$

One can evaluate the effective lagrangian for the gluons in tha anisotropic medium. For the cube one finds

**Isotropic propagation**

This because the second order invariant for the cube and for the rotation group are the same!

# Outlook



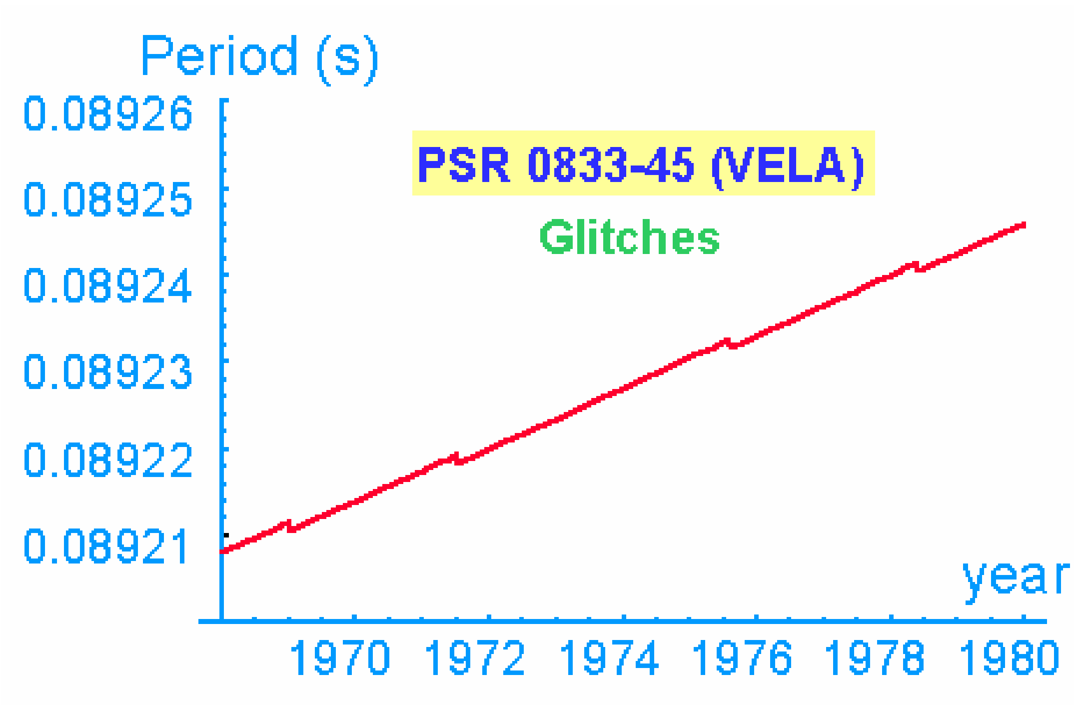

Why the interest in the LOFF phase in QCD?

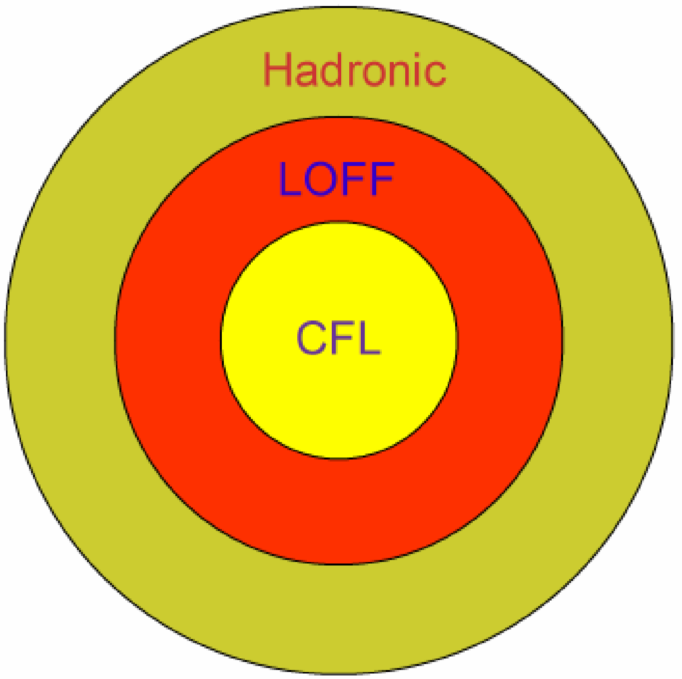
# Neutron stars

Glitches: discontinuity in the period of the pulsars.

$(\Delta\Omega/\Omega \approx 10^{-6})$

Possible explanation: LOFF region inside the star



Recent achieving of degenerate ultracold Fermi gases opens up new fascinating possibilities of reaching the onset of Cooper pairing of hyperfine doublets. However reaching equal populations is a big technical problem. (Combescot 2001)

New possibility for the LOFF state?