CONGRESSO NAZIONALE DI FISICA DELLA MATERIA

Brescia, 13-16 June, 1994

Temperature Correlation Functions in the XXO Heisenberg Chain

F. Colomo[1], **A.G. Izergin**[2,3], **V.E. Korepin**[4], **V. Tognetti**[1,5]

[1] *I.N.F.N., Sezione di Firenze*

[2] *Forum - Institute of Condensed Matter Theory of I.N.F.M.*

and *Forum: Project on Condensed Matter Theory of INFM, Firenze-Pisa, Italy*

[3] *Steklov Mathematical Institute, San Petersburg, Russia*

[4] *Institute of Theoretical Physics, Stony Brook University, New York*

[5] *Dipartimento di Fisica, Universita' and Unita' INFM di Firenze*

Abstract

Space and time dependent correlation functions in the Heisenberg XX0 chain (in transverse magnetic field) are expressed in terms of Fredholm determinants of linear integral operators at all temperatures. The obtained expression allows useful computation of spectral shapes.

Moreover, these determinant expressions allow to evaluate the asymptotic behaviour of correlation functions.

1. INTRODUCTION

Recently essential developments have been made in the theory of quantum correlation functions showing that correlators of quantum exactly solvable models satisfy classical completely integrable differential equations [1]-[6] (this program for the example of the nonrelativistic Bose gas is now fulfilled and presented in Ref. [7]). An important preliminary step to obtain these differential equations is to represent correlation functions as the determinants of Fredholm linear integral operators. For the nonrelativistic Bose gas these representations were given in papers [8, 9] in the time independent case and in [10] in the time-dependent case.

Here we present [11]-[14] determinant representations of this kind for the distance, time and temperature dependent two-point correlation functions of the XX0 Heisenberg chain. We further write differential equations for these correlators, and calculate their asymptotics by constructing and solving a matrix Riemann problem, similarly to the case of the nonrelativistic Bose gas [5], [15]-[17] (see also Ref. [7]).

2. XXO HEISENBERG QUANTUM CHAIN

The XX0 chain is the isotropic case of the XY model [18], being also the free fermions point for the XXZ chain. The Hamiltonian describing the nearest neighbour interaction of local 1/2 spins situated at the sites of the one-dimensional periodical lattice in transverse magnetic field, with M (even) sites, is given as

$$H(h) = -\sum_{m=1}^{M} [\sigma_x^{(m)}\sigma_x^{(m+1)} + \sigma_y^{(m)}\sigma_y^{(m+1)} + h\sigma_z^{(m)}] \, . \tag{1}$$

Pauli matrices are normalized as $(\sigma_s^{(m)})^2 = 1$ $(s = x,\ y,\ z)$. Moreover, we define $\sigma_\pm^{(m)} \equiv \frac{1}{2}[\sigma_x^{(m)} \pm i\sigma_y^{(m)}]$.

The ferromagnetic state $\mid 0\rangle \equiv \otimes_{m=1}^{M} \ |\uparrow\rangle_m$ (all spins up) is an eigenstate of the Hamiltonian. All the other eigenstates can be obtained by filling this ferromagnetic state with N quasiparticles ($N = 1,\ 2,\ ...,\ M$) with different quasimomenta p_a, $-\pi < p_a \leq \pi$, ($a = 1,\ ...,\ N$) and energies $\varepsilon(p_a)$,

$$\varepsilon(p) \equiv \varepsilon(p, h) = -4\cos p + 2h \, . \tag{2}$$

Periodical boundary conditions imply:

$$\exp[iMp_a] = (-1)^{N+1}, \qquad a = 1, \ldots, N \tag{3}$$

for the allowed values of quasimomenta. All the momenta of the quasiparticles of a given eigenstate should be different, so that, *e.g.*, for $N = M$ one gets only one eigenstate $\mid 0'\rangle_M = \otimes_{m=1}^{M} \mid\downarrow\rangle_m$ which is the ferromagnetic state with all spins down.

The model in the thermodynamical limit ($M \to \infty$, h fixed) is the most interesting. For $h \geq h_c \equiv 2$, ferromagnetic state $\mid 0\rangle$ (all spins up) is the ground state of the Hamiltonian. For $0 \leq h < h_c$, the ground state $\mid \Omega\rangle$ is obtained by filling the ferromagnetic state with quasiparticles possessing all the allowed momenta inside the Fermi zone, $-k_F \leq p \leq k_F$, where

$$k_F = \arccos(h/2)\ ; \qquad h \leq h_c = 2\ , \tag{4}$$

is the Fermi momentum. At non zero temperature $T > 0$, the density of quasiparticles in the momentum space is given as $\vartheta(p)/2\pi$, where $\vartheta(p) \equiv \vartheta(p, h, T)$ is the Fermi weight:

$$\vartheta(p) = \frac{1}{1 + \exp[\varepsilon(p)/T]}\ . \tag{5}$$

3. CORRELATION FUNCTIONS

Temperature and time dependent correlators of local spins $\sigma_s^{(m)}(t) \equiv \exp[iHt]\sigma_s^{(m)} \exp[-iHt]$, $\sigma_s^{(m)} \equiv \sigma_s^{(m)}(0)$, $s = x, y, z$, are defined as usual:

$$g_{sr}^{(T)}(m,t) \equiv \langle \sigma_s^{(n_2)}(t_2)\sigma_r^{(n_1)}(t_1)\rangle_T = \frac{\mathrm{Tr}\left\{\exp[-H/T]\sigma_s^{(n_2)}(t_2)\sigma_r^{(n_1)}(t_1)\right\}}{\mathrm{Tr}\left\{\exp[-H/T]\right\}}\ . \tag{6}$$

Due to translation invariance the correlators depend only on the differences,

$$m \equiv n_2 - n_1\ , \qquad t = t_2 - t_1\ . \tag{7}$$

At zero temperature, only the ground state contributes to the traces in (6),

$$g_{sr}^{(0)}(m,t) \equiv \frac{\langle \Omega \mid \sigma_s^{(n_2)}(t_2)\sigma_r^{(n_1)}(t_1) \mid \Omega\rangle}{\langle \Omega \mid \Omega\rangle} \qquad (T = 0)\ . \tag{8}$$

In Ref. [18] the time-independent correlators of XY model were calculated at $h = 0$. The simple answer for the correlator of the third spin components was given; for the XX0 chain it reduces essentially to the square modulus of the Fourier transform of the Fermi weight. The result was generalized to the case of nonzero transverse magnetic field and to the time-dependent correlator [19]; in our notation, for the XX0 model the last result may be written

$$\begin{aligned} g_{zz}^{(T)}(m,t) &= \langle\sigma_z\rangle_T^2 - \frac{1}{\pi^2}\left|\int_{-\pi}^{\pi} dp\ \exp[imp + 4it\cos p]\vartheta(p)\right|^2 + \\ &+ \frac{1}{\pi^2}\left(\int_{-\pi}^{\pi} dp\ \exp[-imp - 4it\cos p]\vartheta(p)\right)\left(\int_{-\pi}^{\pi} dq\ \exp[imq + 4it\cos q]\right) \end{aligned} \tag{9}$$

(for $t = 0$, the last term in the r.h.s. is equal to zero). Here

$$\begin{aligned} \langle\sigma_z\rangle_T &\equiv \langle\sigma_z^{(n)}(t)\rangle_T = 1 - \frac{1}{\pi}\int_{-\pi}^{\pi} dp\,\vartheta(p)\ , \\ \langle\sigma_z\rangle_0 &= 1 - \frac{2k_F}{\pi}\ , \end{aligned} \tag{10}$$

is the magnetization (not depending neither on n nor on t due to translation invariance). Properties of these quantities were considered in much detail [18]-[21]. Real systems for experimental comparisons were found [22].

Correlators of the other local spin components are indeed more complicated. In Ref. [18] these correlators (for the XY model at $t = 0$, $h = 0$) were represented as the determinants of $m \times m$ matrices (m is the distance between correlating spins). This representation was investigated in detail in [20] (see also [23]). In Ref. [24] the structure of the time-dependent correlators was investigated on the basis of an extension of the thermodynamic Wick theorem. In Ref. [2], representation of the autocorrelator ($m = 0$, $t \neq 0$) in the transverse Ising chain in critical magnetic field (closely related to correlators in the XX0 chain at h=0) were given as Fredholm determinants of a linear integral operator.

4. $\sigma_+\sigma_-$ CORRELATION FUNCTIONS AT $T = 0$

Here the correlators (see (6), (7) for the notations)

$$g_+^{(T)}(m,t) = \langle\sigma_+^{(n_2)}(t_2)\sigma_-^{(n_1)}(t_1)\rangle_T\ , \tag{11}$$

$$g_-^{(T)}(m,t) = \langle \sigma_-^{(n_2)}(t_2)\sigma_+^{(n_1)}(t_1)\rangle_T \ , \tag{12}$$

for the XX0 model in a transverse magnetic field are given as Fredholm determinants of linear integral operators. These representations, quite different from those of paper [18], are instead similar to the representations of two-point correlators previously obtained for the one-dimensional Bose gas [8]-[10].

In order to obtain these representations we proceed as follows [10]. The explicit form for the eigenfunctions of Hamiltonian (1) is well known, being just the simplest case of eigenfunctions of the XXZ model [25] with vanishing of two-particle scattering phases. Using this explicit form one can represent the normalized mean value of, *e.g.*, operator $\sigma_+^{(n_2)}(t_2)\sigma_-^{(n_1)}(t_1)$ on the periodical lattice with finite number M of sites (with respect to any eigenfunction with N quasiparticles over the ferromagnetic vacuum) as the determinant of a $N \times N$ matrix. Then, in the thermodynamical limit, correlator (11) is given by the Fredholm determinant of a linear integral operator.

We start with correlator (11), which, at zero temperature, is represented as follows:

$$g_+^{(0)}(m,t) = \exp[-2iht] \left[G(m,t) + \frac{\partial}{\partial z} \right] \det \left[\hat{I} + \hat{V} - z\hat{R}^{(+)} \right] \Big|_{z=0} \ . \tag{13}$$

In the r.h.s. there is a Fredholm determinant. Linear operators $\hat{V}$ and $\hat{R}^{(+)}$ act on functions $f(p)$ on the interval $-k_F \le p \le k_F$ (k_F is the Fermi momentum (4)) as, *e.g.*,

$$\left(\hat{V} f\right)(p) = \frac{1}{2\pi} \int_{-k_F}^{k_F} dq\, V(p,q) f(q) \ . \tag{14}$$

Operator $\hat{I}$ is the identity operator (with kernel $\delta(p-q)$). The kernels of operators $\hat{V}$, $\hat{R}^{(+)}$, are

$$V(p,q) = \frac{E_+(p)E_-(q) - E_-(p)E_+(q)}{\tan\frac{1}{2}(p-q)} - G(m,t)E_-(p)E_-(q) \ , \tag{15}$$

$$R^{(+)}(p,q) = E_+(p)E_+(q) \ , \tag{16}$$

where functions E_+, E_-, are given as

$$\begin{aligned} E_-(p) &\equiv E_-(m,t,p) = \exp[-\frac{i}{2}mp - 2it\cos p] \ , \\ E_+(p) &\equiv E_+(m,t,p) = E_-(p)E(m,t,p) \end{aligned} \tag{17}$$

Functions $G(m,t)$ and $E(m,t,p)$ are defined as follow:

$$G(m,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\ \exp[imq + 4it\cos q] = i^m J_m(4t)\ , \tag{18}$$

(J_m is the Bessel function) and

$$\begin{aligned} E(m,t,p) &= \frac{1}{2\pi}\mathcal{P}\int_{-\pi}^{\pi} dq\ \frac{\exp[imq + 4it\cos q]}{\tan\frac{1}{2}(q-p)} \equiv \\ &\equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\ \frac{\exp[imq + 4it\cos q] - \exp[imp + 4it\cos p]}{\tan\frac{1}{2}(q-p)}\ ; \end{aligned} \tag{19}$$

here $\mathcal{P}$ means the principal value. It should be mentioned that $k_F = 0$ for $h \geq h_c \equiv 2$. In this case the ground state is the ferromagnetic state $\mid 0\rangle$ and the correlator is just the "wave packet":

$$g_+^{(0)}(m,t) = \exp[-2iht]G(m,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\ \exp[imq - it\varepsilon(q)]\ , \tag{20}$$

5. $\sigma_+\sigma_-$ CORRELATION FUNCTIONS AT $T \neq 0$

In the case of non zero temperature ($T > 0$) the representations are similar:

$$g_+^{(T)}(m,t) = \exp[-2iht][G(m,t) + \frac{\partial}{\partial z}] \det\left[\hat{I} + \hat{V}_T - z\hat{R}_T^{(+)}\right]\Big|_{z=0}\ , \qquad m \geq 0\ , \tag{21}$$

Operators $\hat{V}_T$, $\hat{R}_T^{(+)}$, act over the interval $[-\pi,\ \pi]$,

$$(\hat{V}_T f)(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\, V_T(p,q) f(q)\ , \tag{22}$$

etc., their kernels being equal to

$$\begin{aligned} V_T(p,q) &= \sqrt{\vartheta(p)}\, V(p,q)\, \sqrt{\vartheta(q)}\ , \\ R_T^{(+)}(p,q) &= \sqrt{\vartheta(p)}\, R^{(+)}(p,q)\, \sqrt{\vartheta(q)}\ , \end{aligned} \tag{23}$$

where $\vartheta(p)$ is the Fermi weight (5) and functions $V(p,q)$, $R^{(+)}(p,q)$, are defined in eqs. (15), (16).

Analogous representations are valid also for correlator (12):

$$g_-^{(0)}(m,t) = \exp[2iht]\,\frac{\partial}{\partial z}\ \det\left[\hat{I} + \hat{V} + z\hat{R}^{(-)}\right]\Big|_{z=0}\ , \tag{24}$$

$$g_-^{(T)}(m,t) = \exp[2iht]\,\frac{\partial}{\partial z}\ \det\left[(\hat{I} + \hat{V}_T + z\hat{R}_T^{(-)}\right]\Big|_{z=0}\ , \tag{25}$$

where $\hat{V}$ and $\hat{V}_T$ are the same operators as in (13), (21) and the kernels of operators $\hat{R}^{(-)}$ and $\hat{R}_T^{(-)}$ (acting over the interval $[-k_F,\ k_F]$, see (14), and $[-\pi,\ \pi]$, see (22), respectively) are

$$R^{(-)}(p,q) \quad = \quad E_-(p)E_-(q)\ , \tag{26}$$

$$R_T^{(-)}(p,q) \quad = \quad \sqrt{\vartheta(p)}\ R^{(-)}(p,q)\ \sqrt{\vartheta(q)}\ , \tag{27}$$

with functions $E_-(q)$ defined in (17). It is worth mentioning that the zero-temperature correlator (24) is equal to zero for magnetic field $h \geq h_c = 2$.

The previous results allows the computation of spectral shapes, as done in [26] for the high-excitation-emission spectra of one-dimensional Frenkel excitons.

6. ASYMPTOTICS OF CORRELATION FUNCTIONS AT $T \neq 0$

It was already mentioned that representations similar to those obtained above gave an opportunity to obtain differential equations for correlation functions in the case of impenetrable bosons (the V Painlevé transcendent in the equal time zero temperature case [1] and integrable partial differential equations for time and temperature dependent correlators [4, 5]). This allowed to construct exact asymptotics for the correlators [1, 16, 17]. Corresponding results have indeed be obtained for the XX0 chain [27]-[28].

The strategy is the following: after having expressed the correlation function as a determinant of an integral operator (of Freedholm type), it is possible to show that it satisfies a set of differential equations of Ablowitz and Ladik [29] type (integrable discretization of the non-linear Schrodinger equation). This means that the correlation function of the XXO model is the τ function of Ablowitz-Ladik's differential-difference equation.

It is then possible to solve Ablowitz-Ladik's equation in the asymptotic limit of large space and time separations, through the use of Matrix Riemann-Hilbert problem [7],[17].

The result reads as follow. We consider finite temperature $0 < T < \infty$ and moderate magnetic field $0 \leq h < 2$. The asymptotics are evaluated in those cases where both space and time separation go to infinity $m \to \infty$, $t \to \infty$, in some direction ϕ:

$$\frac{m}{4t} = \cot\phi, \qquad 0 \leq \phi \leq \frac{\pi}{2}\ . \tag{28}$$

Correlation function $g_+^{(T)}(m,t)$ decays exponentially in any direction, but the rate of decay depends on the direction in space-time. In the space-like direction $0 \leq \phi \leq \frac{\pi}{4}$ we have the following asymptotic behaviour:

$$g_+^{(T)}(m,t) \longrightarrow C \exp\left\{ \frac{m}{2\pi} \int_{-\pi}^{\pi} dp \ln \left| \tanh \left(\frac{h - 2\cos p}{T} \right) \right| \right\} . \tag{29}$$

In the time-like direction $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$ we get instead:

$$g_+^{(T)}(m,t) \longrightarrow C\, t^{2\nu_+^2 + 2\nu_-^2} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} dp\, |m - 4t \sin p| \cdot \ln \left| \tanh \left(\frac{h - 2\cos p}{T} \right) \right| \right\} , \tag{30}$$

where the values $\nu_\pm$ which define the pre-exponent are:

$$\nu_\pm = \frac{1}{2\pi} \ln \left| \tanh \left(\frac{h \mp 2\cos p_0}{T} \right) \right| , \tag{31}$$

and p_0 is defined through $\frac{m}{4t} = \sin p_0$. It should be mentioned that the constant factor C in (29) does not depend on the direction ϕ, but it does depend on ϕ in (30).

References

[1] M. Jimbo, T. Miwa, Y. Mori, M. Sato, *Physica* **D1** (1980) 80;

[2] B.M. McCoy, J.H.H. Perk, R.E. Schrock. *Nucl. Phys.* **B220** **[FS 8]** (1983) 35;

[3] A.R. Its, A.G. Izergin, V.E. Korepin. *Phys.Lett.* **A141** (1989) 121;

[4] A.R. Its, A.G. Izergin, V.E. Korepin, *Comm. Math. Phys.* **129** (1990) 205;

[5] A.R. Its, A.G. Izergin, V.E. Korepin, N.A. Slavnov, *Int. Jour. Mod. Phys.* **B4** (1990) 1003;

[6] A.R. Its, A.G. Izergin, V.E. Korepin, N.Ju. Novokshenov, *Nucl. Phys.* **B340** (1990) 752;

[7] V.E. Korepin, A.G.Izergin, N.M. Bogoliubov, *"Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz"* Cambridge University Press (1992);

[8] A. Lenard, *Jour. Math. Phys.* **5** (1964) 930;

[9] A. Lenard, *Jour. Math. Phys.* **7** (1966) 1268;

[10] V.E. Korepin, N.A. Slavnov, *Comm. Math. Phys.* **136** (1991) 633;

[11] F.Colomo, A.V.Izergin, V.E.Korepin, V.Tognetti, *Correlators in the Heisenberg XXO Chain as Fredholm Determinants*, Physics Letters **A169** (1992) 243;

[12] F.Colomo, A.V.Izergin, V.E.Korepin, V.Tognetti, *Determinant Representation for Two-Points Correlators of the Heisenberg XX0 Chain in Transverse Magnetic Field*, Teor. Mat. Fiz. **94** (1993) 19, or Theor. Math. Phys. **94** (1993) 11.

[13] A.R. Its, A.G. Izergin, V.E. Korepin, N.A. Slavnov, *Phys. Rev. Lett.* **70** (1993) 1704;

[14] F. Colomo, A.R. Its, A.G. Izergin, V.E. Korepin, *"Asymptotics of Correlators of Quantum Spins"*, in preparation;

[15] A.R. Its, A.G. Izergin, V.E. Korepin, *Commun. Math. Phys.* **130** (1990) 471;

[16] A.R. Its, A.G. Izergin, V.E. Korepin, *Physica* **D53** (1991) 187;

[17] A.R. Its, A.G. Izergin, V.E. Korepin, G.G. Varzugin, *Physica* **D54** (1992) 351;

[18] E. Lieb, T. Schultz, D. Mattis, *Ann. Phys.*, **16** (1961) 407;

[19] T. Niemeijer, *Physica* **36** (1967) 377;

[20] B.M. McCoy, *Phys. Rev.* **173** (1968) 531;

[21] S. Katsura, T. Horiguchi, M.Suzuki, *Physica* **46** (1970) 67;

[22] M. D'Iorio, R.L. Armstrong, D.R. Taylor, *Phys. Rev.* **B27** (1983) 1664;

[23] T. Tonegawa, *Solid State Comm.* **40** (1981) 983;

[24] J.H.H. Perk, H.W. Capel, *Physica* **A89** (1977) 265;

[25] C.N. Yang, C.P. Yang, *Phys. Rev.* **150** (1966) 321.

[26] H. Suzuura, T. Tokihiro, Y. Ohta, *Phys. Rev.* **B49** (1994) 4344.

[27] A.R. Its, A.G. Izergin, V.E. Korepin, N.A. Slavnov, *Phys. Rev. Lett.* **70** (1993) 1704;

[28] F. Colomo, A.R. Its, A.G. Izergin, V.E. Korepin, *"Asymptotics of Correlators of Quantum Spins"*, in preparation.

[29] M.J. Ablowitz, J.F. Ladik, *Stud. Appl. Math.* **55** (1976) 213.