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S-Matrices of Kink Configurations and Finite-Size Effects *

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Abstract

The S-matrix of the Tricritical Ising Model perturbed by the subleading magnetic operator describes scattering processes of kink excitations which interpolate between two degenerate asymmetric vacua. Exploring this example we describe the use of finite-size techniques in order to compare the spectrum obtained by the Truncated Conformal Space Approach with the information extracted form the S-matrix.

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1 Introduction

It has been pointed out by A. Zamolodchikov that certain deformations of minimal models of conformal field theory (CFT) are described by integrable massive field theories [1]. The corresponding Hamiltonian can be written as

$$H_p = H_{CFT} + \lambda \int \phi(x)\, dx \ . \tag{1.1}$$

Integrability of the perturbed theory is achieved only for few specific operators ϕ of the space of states of the CFT, in general only for the primary fields $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{1,3}$ of the Kac-table [1].

The on-shell behaviour of massive quantum field theories is described by the S-matrix. For integrable massive quantum field theories the S- matrix is factorized, *i.e.* n-particle scattering amplitudes can be decomposed into 2-particle ones.

There is a large variety of methods in order to compute the S matrix (see *e.g.* [2]), but in many cases the latter is just conjectured, on the basis of the physical features of the model under consideration. It is therefore necessary to rely upon different methods in order to link the conjectured S-matrix with the underlying physical model. Two powerful techniques have been developed for this scope: the thermodynamic Bethe ansatz (TBA) [3] and the truncation conformal space approach (TCSA) [4]. Both techniques are applied to a theory in a finite geometry.

The TBA uses the S matrix as an input, in order to determine the thermodynamics of the system. In particular, it enables one to compute the ground state energy $E_0(R)$ for a theory put onto a strip of width R (*i.e*, in the case of periodic boundary conditions, onto a cylinder). One can study the theory in the ultraviolet limit and extract the central charge and the scaling dimensions of the operators of the original CFT. This means that one takes the limit back to the critical point which is described by the CFT which one has perturbed. Comparing the data of the TBA with that of the original CFT gives a powerful method to confirm the validity of a conjectured S-matrix.

The TCSA on the other hand is a non-perturbative method, which gives the spectrum of the hamiltonian H_p in (1.1). The idea is simple: after choosing suitable boundary conditions and a basis in the Hilbert space, diagonalize numerically the hamiltonian operator H_p to get its spectrum. In order to apply this concept successfully, one uses explicitly the conformal structure of the hamiltonian H_p. Defining the theory on a strip of width R, with periodic boundary conditions, the unperturbed part H_{CFT} can be written in terms of the operators $L_0, \bar{L_0}$ of the Virasoro algebra and the central charge c as [5]

$$H_{CFT} = \frac{2\pi}{R}(L_0 + \bar{L_0} - \frac{c}{12}) \ . \tag{1.2}$$

Since in this way H_p is defined in terms of the conformal algebra it is natural to choose as a basis the eigenstates of H_{CFT}, the so-called conformal states. The matrix-elements $\langle \phi_j | H_p | \phi_i \rangle$ can be calculated exactly, since the perturbing piece $\int_0^R \phi(x)\, dx$ of H_p gives a contribution proportional to the structure constants $C_{\phi_j\, \phi\, \phi_i}$ of the CFT. Then the hamiltonian $H_p(R)$ can be diagonalized numerically if one truncates the Hilbert space in order to include only a finite number of conformal states. So a powerful method of checking a conjectured S-matrix under investigation is a) to compare the mass spectrum it predicts with the one obtained by TCSA; b) to extract finite-size (R dependent) quantities from the infinite volume S-matrix and check them against the TCSA data.

Since the subject has grown to be very vast, and our space is rather limited, we will not discuss these developments on a general footing, but limit ourselves to an example, in order to see some of the above techniques at work. We choose for this a rather contradictory subject: the tricritical Ising model (TIM) perturbed by the subleading magnetization operator. As we shall see in the following, there exist two different conjectures for the S - matrix of this model, obtained by completely different reasonings. We will describe the power of finite-size techniques in detail, analyzing this example. For this model the TBA has not been carried out. Therefore one needs to consider the theory of finite-size effects [6] which also relies on the S-matrix as an infinite volume input. We will discuss this problem in detail in section 3.

1.1 The Tricritical Ising Model Perturbed with the Subleading Magnetization Operator

The Ising model with vacancies is described at its critical point by the unitary minimal CFT $\mathcal{M}_{4,5}$. Its central charge is $c = \frac{7}{10}$ and it has six primary fields appearing in the Kac-table (table 1), four of them relevant. It represents the universality class ϕ^6 of the Landau-Ginzburg theory

$$\int \mathcal{D}\phi \, e^{-\int (\nabla\phi)^2 + \lambda_6\phi^6 + \lambda_4\phi^4 + \lambda_3\phi^3 + \lambda_2\phi^2 + \lambda_1\phi)} \, d^2r \tag{1.3}$$

at the tricritical point $\lambda_i = 0, \quad i = 1, \ldots, 4$. The primary fields in the Kac-table can be identified with normal ordered Landau-Ginzburg fields. They are collected in table 1.

A perturbation of this critical point with the field $\Phi_{2,1}$ with anomalous dimension $(\Delta, \bar{\Delta}) = (\frac{7}{16}, \frac{7}{16})$, which is identified with the subleading magnetization, drives the TIM into a massive regime. The spectrum using the TCSA was computed first in [7]. The lowest energy levels for the calculation with *periodic* boundary conditions are reproduced in figure 1. One reads off that the ground state is double degenerate. This corresponds to the Landau-Ginzburg picture, in which the potential exhibits two asymmetric degenerate vacua (see fig. 2). This asymmetry can be understood from the fact that the subleading magnetization explicitly breaks the Z_2 symmetry of the theory, since the operator is odd under these transformations (see table 2). For this reason the theory can exhibit the "Φ^3-property", *i.e.*, the absence of a conserved current of spin 3 and therefore the possibility to form a bound state through the process $AA \to A \to AA$. This picture is confirmed by the counting argument, which shows the existence of conserved currents with spins $s = (1, 5, 7, 11, 13)$ [1, 7].

Looking again at fig. 1, we see above the ground state a single excitation at mass m, below the threshold at mass $2m$. This feature is explained qualitatively by the asymmetry of the Landau-Ginzburg potential, since if the potential was symmetric, also the bound state should be double degenerate. Moreover, the fact that this bound state has the same mass m as the fundamental excitations was explained by Zamolodchikov [19] as a consequence of elastic scattering.

This information is sufficient to read off some general features the S-matrix must possess. First of all, the fundamental S matrix must be degenerate, giving rise to two independent kink-configurations. Secondly their S-matrix elements must exhibit a pole for a value of the rapidity $\beta = \frac{2\pi i}{3}$, so as to generate a *single* bound state with the same mass m, and no further bound states below the threshold. In the following we will describe two different conjectures for this S-matrix, both satisfying the above requirements.

2 The Conjectured S-Matrices

It is well known that S-matrices for degenerate particles can be constructed from solutions of the Yang-Baxter equation. In this way the factorization equations are automatically fulfilled, and one only needs to link the spectral parameter to the rapidity variable. This procedure is constrained in such a way that the final expression for the S-matrix fulfills the basic requirements of scattering theory: crossing invariance and unitarity. Both conjectures of the Smatrix of $\mathcal{M}_{4,5} + \phi_{2,1}$ rely on this principle, though their starting points are quite different.

2.1 The Approach of Smirnov

In ref. [8] Smirnov has developed a general scheme in order to describe $\phi_{1,2}$ and $\phi_{2,1}$ perturbations. In order to select the solution of the Yang-Baxter equation (YBE) he uses the equivalence in the classical limit of minimal models and Liouville theory. In this limit the operators $\Phi_{1,n}$ can be identified with the fields $\exp\{-i\frac{n-1}{2}\beta\phi\}$ in Liouville theory. Perturbing the lagrangian with these fields, one finds for $n = 3$ the Sine-Gordon model, and for $n = 2$ the Izergin-Korepin model[1] with the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{\gamma^2} e^{i\gamma\phi} - \frac{\lambda}{\gamma^2} e^{-\frac{i\gamma}{2}\phi} \quad , \tag{2.1}$$

which is also classified as the lagrangian of the $A_2^{(2)}$ Toda field theory with imaginary coupling constant. In order to describe $\phi_{2,1}$ perturbations, which has no classical limit, since the dimension of the corresponding operator diverges, Smirnov uses the fact that in the Coulomb gas description of the minimal models the operators $\phi_{1,2}$ and $\phi_{2,1}$ are dual, in the following sense: generally speaking, in CFT, $\Delta_{r,s}$ for the minimal model $\mathcal{M}_{p,q}$ is equals to $\Delta_{s,r}$ for the model $\mathcal{M}_{q,p}$.

From the above it seems natural to choose as the solution of the YBE the R matrix of the $A_2^{(2)}$ quantum group [10] and make the ansatz $S = S_0 R(x, q)$ for the S-matrix, wherein x denotes the spectral parameter and q the parameter of the quantum group. In order to link the above expression to scattering theory one requires crossing symmetry and unitarity. The right identification [8] turns out to be

$$S = S_0(\beta)\; R(e^{\frac{2\pi}{\xi}\beta}, e^{\frac{2\pi^2 i}{\gamma}}) \quad , \tag{2.2}$$

with

$$\xi = \frac{2}{3}\frac{\pi\gamma}{2\pi - \gamma} \quad . \tag{2.3}$$

[1] This model is connected to different names. It was first mentioned by Bullough-Dodd, and analyzed by Zhiber,Mikhailov and Shabat. Finally the R-matrix was found by Izergin-Korepin [9, 10].

The function S_0 is given as

$$S_0(\beta) = \left(\sinh\frac{\pi}{\xi}(\beta - i\pi)\sinh\frac{\pi}{\xi}\left(\beta - \frac{2\pi i}{3}\right)\right)^{-1} \times \exp\left(-2i\int_0^\infty \frac{dx}{x}\frac{\sin\beta x\sinh\frac{\pi x}{3}\cosh\left(\frac{\pi}{6}-\frac{\xi}{2}\right)x}{\cosh\frac{\pi x}{2}\sinh\frac{\xi x}{2}}\right) \quad . \tag{2.4}$$

For rational values of the coupling constant $\gamma = \frac{r}{s}\pi$, corresponding to the perturbation of the theory $\mathcal{M}_{r,s}$, it is useful to perform a basis change to the so-called "shadow-world" [11]. The basis of n-particle states is formed by the RSOS-states

$$\mid \beta_1, j_1, k_1 \mid a_1 \mid \beta_2, j_2, k_2 \mid \quad \ldots \quad \mid a_{n-1} \mid \beta_n, j_n, k_n > \tag{2.5}$$

where β denotes the rapidities, k the type (kink or breather) and j the $U_q(sl(2))$ spin of each particle. The integer numbers a_i, characterizing this dual representation, are constrained by the RSOS conditions

$$a_i \leq \frac{r-2}{2} \quad , \qquad \mid a_k - 1 \mid \;\leq\; a_{k+1} \;\leq\; \min(a_k+1, r-3-a_k) \ . \tag{2.6}$$

The S-matrix of these RSOS states is given as [8]

$$\begin{aligned} &S_{a_{k-1}a_{k+1}}^{a_k a'_k}(\beta_k - \beta_{k+1}) = \\ &\quad \frac{i}{4}\, S_0(\beta_k - \beta_{k+1}) \left[\begin{Bmatrix} 1 & a_{k-1} & a_k \\ 1 & a_{k+1} & a'_k \end{Bmatrix}_q \right. \\ &\quad \times \left(\left(\exp\left(\frac{2\pi}{\xi}(\beta_{k+1}-\beta_k)\right) - 1 \right) q^{(c_{a_{k+1}}+c_{a_{k-1}}-c_{a_k}-c_{a'_k}+3)/2} \right. \\ &\quad \left. - \left(\exp\left(-\frac{2\pi}{\xi}(\beta_{k+1}-\beta_k)\right) - 1 \right) q^{-(c_{a_{k+1}}+c_{a_{k-1}}-c_{a_k}-c_{a'_k}+3)/2} \right) \\ &\quad \left. + q^{-5/2}(q^3+1)(q^2-1)\delta_{a_k,a'_k} \right] . \end{aligned} \tag{2.7}$$

Herein, c_a are the Casimir of the representation a, $c_a = a(a+1)$ and the expression of the $6j$-symbols is that in ref. [11].

We apply now this general procedure to the TIM [12, 13], or better to say, to $\mathcal{M}_{4,5}$ perturbed by the subleading magnetic operator $\phi_{2,1}$. As discussed above we have to set the parameter $\gamma = \frac{5}{4}\pi$, getting from (2.3) $\xi = \frac{10}{9}\pi$. From eq. (2.6), the only possible values of a_i are 0 and 1 and the one-particle states are the vectors: $\mid K_{01}\rangle$, $\mid K_{10}\rangle$ and

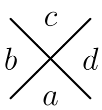
$| K_{11}\rangle$. All of them have the same mass m. Notice that the state $| K_{00}\rangle$ is not allowed. A basis for the two-particle asymptotic states is

$$| K_{01}K_{10}\rangle,\ \ | K_{01}K_{11}\rangle,\ \ | K_{11}K_{11}\rangle,\ \ | K_{11}K_{10}\rangle,\ \ | K_{10}K_{01}\rangle\ . \tag{2.8}$$

The two-particle scattering-processes

$$|K_{bc}K_{cd}\rangle = \sum_a S_{bd}^{ac}|K_{ba}K_{ad}\rangle \quad , \tag{2.9}$$

computed in [12] can be written as



$$= \ S_{bd}^{ac}(\beta) = \frac{i}{2}(-1)^{a-c}\, S_0(\beta)\ \times$$

$$\left\{ \delta_{ac}\sinh\left(\frac{2\pi}{\xi}(i\pi-\beta)\right) + \delta_{bd}\sinh(\frac{2\beta}{\xi})\left(\frac{[2a+1][2c+1]}{[2b+1][2d+1]}\right)^{\frac{1}{2}} \right\} \tag{2.10}$$

where

$$[n] \equiv \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \qquad .$$

The function $S_0(\beta)$ is given by

$$\begin{aligned} S_0(\beta) \ &= \ -\left(\sinh\frac{9}{10}(\beta - i\pi)\sinh\frac{9}{10}\left(\beta - \frac{2\pi i}{3}\right)\right)^{-1} \\ &\quad \times w\left(\beta,-\frac{1}{5}\right) w\left(\beta,+\frac{1}{10}\right) w\left(\beta,\frac{3}{10}\right) \\ &\quad \times t\left(\beta,\frac{2}{9}\right) t\left(\beta,-\frac{8}{9}\right) t\left(\beta,\frac{7}{9}\right) t\left(\beta,-\frac{1}{9}\right)\ , \end{aligned} \tag{2.11}$$

with the abbreviations

$$w(\beta,x) = \frac{\sinh\left(\frac{9}{10}\beta + i\pi x\right)}{\sinh\left(\frac{9}{10}\beta - i\pi x\right)}\ ;$$

$$t(\beta,x) = \frac{\sinh\ \frac{1}{2}(\beta + i\pi x)}{\sinh\ \frac{1}{2}(\beta - i\pi x)}\ .$$

This S-matrix satisfies the unitarity relations

$$\sum_e S_{bc}^{ae}(\beta)S_{bc}^{ed}(-\beta) = \delta^{ad}\ \ , \tag{2.12}$$

and also the factorization equations

$$\sum_k S_{bk}^{ac}(\beta)S_{ae}^{kf}(\beta+\beta')S_{kd}^{ce}(\beta') = \sum_k S_{ke}^{df}(\beta)S_{bd}^{ck}(\beta+\beta')S_{ak}^{bf}(\beta')\ \ . \tag{2.13}$$

Notice that in this original formulation the crossing symmetry is realized in a non trivial way,

$$S_{bd}^{ac}(i\pi-\beta)=(-1)^{a-c}\left(\frac{[2a+1][2c+1]}{[2b+1][2d+1]}\right)^{\frac{1}{2}}S_{ac}^{db}(\beta)\quad . \tag{2.14}$$

Let us discuss now some features of this S-matrix. The amplitudes (2.10) are periodic along the imaginary axis of β with period $10\,\pi i$. In figure 3 we show the analytic structure of the S-matrix. Note that the only poles giving rise to a bound state are located at $\beta=\frac{2\pi i}{3}$, and at $\beta=\frac{\pi i}{3}$, since all other poles are overlapped by zeros, and do not give rise to a singularity of the S-matrix. The direct channel corresponds to the pole at $\beta=\frac{2\pi i}{3}$. Since we require the model to describe a unitary field theory, we expect the residues at this pole to be imaginary positive. They are given by

$$\begin{aligned}
r_1 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\;S_{00}^{11}(\beta)=0\,;\\
r_2 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\;S_{01}^{11}(\beta)=i\,\left(\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)}\right)^2\,\omega\,;\\
r_3 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\;S_{11}^{11}(\beta)=i\,\omega\,;\\
r_4 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\;S_{11}^{01}(\beta)=i\,\left(\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)}\right)^{\frac{1}{2}}\,\omega\,;\\
r_5 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\;S_{11}^{00}(\beta)=i\,\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)}\,\omega\,;
\end{aligned} \tag{2.15}$$

where we use the abbreviation $s(x)=\sin(\pi x)$ and

$$\omega=\frac{5}{9}\,\frac{s\left(\frac{1}{5}\right)s\left(\frac{1}{10}\right)s\left(\frac{4}{9}\right)s\left(\frac{1}{9}\right)s^2\left(\frac{5}{18}\right)}{s\left(\frac{3}{10}\right)s\left(\frac{1}{18}\right)s\left(\frac{7}{18}\right)s^2\left(\frac{2}{9}\right)}\quad . \tag{2.16}$$

Their numerical values are collected in Table 1. Indeed the residues are positive, apart the one of the amplitude S_{00}^{11} in which an additional zero cancels the pole, and therefore the residue becomes zero.

Let us discuss other properties of the scattering theory under consideration. For real values of β, the amplitudes $S_{00}^{11}(\beta)$ and $S_{01}^{11}(\beta)$ are numbers of modulus 1. It is therefore convenient to define the following phase shifts

$$\begin{aligned}
S_{00}^{11}(\beta) &\equiv e^{2i\delta_0(\beta)}\,;\\
S_{01}^{11}(\beta) &\equiv e^{2i\delta_1(\beta)}\,.
\end{aligned} \tag{2.17}$$

The non-diagonal sector of the scattering processes is characterized by the 2×2 symmetric S-matrix

$$\left(\begin{array}{cc} S_{11}^{11}(\beta) & S_{11}^{01}(\beta) \\ S_{11}^{01}(\beta) & S_{11}^{00}(\beta) \end{array}\right) . \tag{2.18}$$

We can define the corresponding phase shifts by diagonalizing the matrix (2.18). The eigenvalues turn out to be [12, 13] the same functions as in (2.17)

$$\left(\begin{array}{cc} e^{2i\delta_0(\beta)} & 0 \\ 0 & e^{2i\delta_1(\beta)} \end{array}\right) . \tag{2.19}$$

A basis of eigenvectors is given by

$$\mid \phi_i(\beta_1)\phi_i(\beta_2)\rangle \;=\; \sum_{j=0}^{1} U_{ij} \mid K_{1j}(\beta_1)K_{j1}(\beta_2)\rangle \;, \qquad i=0,1 \;, \tag{2.20}$$

where U is an unitary matrix which does not depend on β

$$U \;=\; \frac{1}{\sqrt{1+a^2}} \left(\begin{array}{cc} 1 & a \\ -a & 1 \end{array}\right) . \tag{2.21}$$

The asymptotic behaviour of the phase shifts is the following:

$$\begin{aligned} \lim_{\beta\to\infty} e^{2i\delta_0(\beta)} &= e^{\frac{6\pi i}{5}} ; \\ \lim_{\beta\to\infty} e^{2i\delta_1(\beta)} &= e^{\frac{3\pi i}{5}} . \end{aligned} \tag{2.22}$$

We can use this nontrivial asymptotic values of the phase-shifts in order to define generalized bilinear commutation relations for the "kinks" ϕ_0 and ϕ_1 [14, 15, 16]

$$\phi_i(t,x)\phi_j(t,y) = \phi_j(t,y)\phi_i(t,x)\, e^{2\pi i s_{ij}\epsilon(x-y)} . \tag{2.23}$$

The generalized "spin" s_{ij} is a parameter related to the asymptotic behaviour of the S-matrix. A consistent assignment is given by

$$\begin{aligned} s_{00} &= \frac{3}{5} = \frac{\delta_0(\infty)}{\pi} ; \\ s_{01} &= 0 ; \\ s_{11} &= \frac{3}{10} = \frac{\delta_1(\infty)}{\pi} . \end{aligned} \tag{2.24}$$

Notice that the previous monodromy properties are those of the chiral field $\Psi = \Phi_{\frac{6}{10},0}$ of the original CFT of the TIM. This field occupies the position $(1,3)$ in the Kac-table of the model. The operator product expansion of Ψ with itself reads

$$\Psi(z)\Psi(0) = \frac{1}{z^{\frac{6}{5}}}\,\mathbf{1} + \frac{C_{\Psi,\Psi,\Psi}}{z^{\frac{3}{5}}}\,\Psi(0) + \ldots \tag{2.25}$$

where $C_{\Psi,\Psi,\Psi}$ is the structure constant of the OPE algebra. Moving z around the origin, $z \to e^{2\pi i} z$, the phase acquired from the first term on the right hand side of (2.25) comes from the conformal dimension of the operator Ψ itself. In contrast, the phase obtained from the second term is due to the insertion of an additional operator Ψ. A similar structure appears in the scattering processes of the "kinks" ϕ_i: in the amplitude of the kink ϕ_0 there is no bound state in the s-channel (corresponding to the "identity term" in (2.25)) whereas in the amplitude of ϕ_1 a kink can be created as a bound state for $\beta = \frac{2\pi i}{3}$ (corresponding to the "Ψ term" in (2.25)). In the ultraviolet limit, the fields ϕ_i should give rise to the operator $\Psi(z)$, similarly to the case analyzed in [16]. Note that a rigorous proof of this statement would require the analysis of the form factors.

The previously discussed fact happens to be a particular case of a general situation of the RSOS S-matrices coming from Smirnov's reduction. If we study Smirnov's formula [8] for generic values of $r \geq 5$, *i.e.* we consider the $\Phi_{2,1}$ deformation of $\mathcal{M}_{r,r+1}$, we see that we have $0 \leq a_i \leq \left[\frac{r-1}{2}\right]$ and therefore we get many more amplitudes. In general we have three independent diagonal amplitudes, $S_{00}^{11}(\beta) = e^{2i\delta_0(\beta)}$, $S_{01}^{11}(\beta) = e^{2i\delta_1(\beta)}$ and $S_{02}^{11}(\beta) = e^{2i\delta_2(\beta)}$ which define three independent phase-shifts. Their asymptotic behaviour is

$$
\begin{aligned}
\lim_{\beta\to\infty} S_{00}^{11}(\beta) &= e^{2i\pi\Delta_{1,3}} \,, \\
\lim_{\beta\to\infty} S_{01}^{11}(\beta) &= e^{i\pi\Delta_{1,3}} \,, \\
\lim_{\beta\to\infty} S_{00}^{11}(\beta) &= e^{i\pi(2\Delta_{1,3}-\Delta_{1,5})} \,,
\end{aligned}
\tag{2.26}
$$

where $\Delta_{1,3}$ and $\Delta_{1,5}$ are the anomalous dimensions of the corresponding fields of the original CFT. The non-diagonal sector of the scattering processes is a block-diagonal matrix, with 3×3 and 2×2 non-diagonal blocks. They may be diagonalized as well [13], and we get as eigenvalues of all the 3×3 blocks the functions $e^{2i\delta_0(\beta)}$, $e^{2i\delta_1(\beta)}$ and $e^{2i\delta_2(\beta)}$, whereas as eigenvalues of all the 2×2 matrices we get the functions $e^{2i\delta_0(\beta)}$ and $e^{2i\delta_1(\beta)}$. No other independent functions appear. Therefore, also the asymptotic behaviour in the non-diagonal sector is given by (2.26). In the context of the previously discussed monodromy properties, this appears to be related to the OPE of $\Psi \equiv \Phi_{1,3}$ of the original CFT $\mathcal{M}_{r,r+1}$:

$$
\Psi(z)\Psi(0) = \frac{1}{z^{2\Delta_{1,3}}}\, \mathbf{1} + \frac{C_{\Psi,\Psi,\Psi}}{z^{\Delta_{1,3}}}\, \Psi(0) + \frac{C_{\Psi,\Psi,\Phi_{1,5}}}{z^{2\Delta_{1,3}-\Delta_{1,5}}}\, \Phi_{1,5}(0) + \ldots \tag{2.27}
$$

In the previous case of TIM, with $r = 4$, the last channel could not be open because $\Phi_{1,5}$ does not appear in the Kac-table of the primary fields of the original CFT, and the singular part of the OPE stops after the two first terms. The opening of the new channel $\Phi_{1,5}$ for $r \geq 5$ corresponds to the appearance of states (forbidden by the RSOS selection rules, eq. (2.6)) with spin $j = 2$ in the s-channel of $S^{i,i}_{i-1,i+1}(\beta)$, $i = 1, ..., \left[\frac{r-1}{2}\right] - 1$.

Next let us comment on the crossing symmetry. We found that the crossing symmetry is implemented with a non-trivial charge conjugation. This led to the above correspondence of the phase-shifts in the ultraviolet limit with the operator product expansion of the field $\Phi_{1,3}$. But in scattering theory one usually requires strict crossing symmetry, *i.e.*,

$$S^{bd}_{ac}(i\pi - \beta) = S^{ac}_{bd}(\beta) \quad . \tag{2.28}$$

The only way to introduce a non-trivial charge conjugation is to consider an asymmetric basis in the Hilbert-space. In this way it must be possible to find a representation in which the charge conjugation trivially realized. The new amplitude can differ from (2.10 only by a gauge-transformation. This is indeed possible [13], and one can define a new S-matrix by

$$\hat{S}^{ac}_{bd} \equiv (-1)^{c-a} \left(\frac{[2a+1][2c+1]}{[2b+1][2d+1]} \right)^{-\frac{\beta}{2\pi i}} S^{ac}_{bd} \tag{2.29}$$

which satisfies (2.28). However, it introduces the unpleasant feature of an oscillating asymptotic behaviour into the S-matrix, and therefore does not represent a convenient basis to relate the massive theory with an underlying CFT reached in the ultraviolet limit.

Now let us finally turn to the spectrum. We first analyze the bound state structure. In the amplitude S^{11}_{00} there is no bound state in the direct channel but only the singularity coming from the state $\mid K_{11}\rangle$ exchanged in the t-channel. This is easily seen from fig. 4, where we stretch the original amplitudes along the vertical direction (s-channel) and along the horizontal one (t-channel). Since the state $\mid K_{00}\rangle$ is not physical, the residue in the direct channel is zero. In the amplitude S^{11}_{01} we have the bound state $\mid K_{01}\rangle$ in the direct channel and the singularity due to $\mid K_{11}\rangle$ in the crossed channel. In S^{11}_{11}, the state $\mid K_{11}\rangle$ appears as a bound state in both channels. In S^{01}_{11} the situation is reversed with respect to that of S^{11}_{01}, as it should be from the crossing symmetry property (2.14): the state $\mid K_{11}\rangle$ appears in the t-channel and $\mid K_{01}\rangle$ in the direct channel. Finally, in S^{00}_{11} there is the bound state $\mid K_{11}\rangle$ in the direct channel but the residue on the t-channel pole is zero, again because $| K_{00}\rangle$ is unphysical. This situation is, of course, that obtained by applying crossing to S_{00}^{11}.

Let us compare this picture with the data coming from the TCSA. The one-particle line a of fig. 1 corresponds to the state $| K_{11}\rangle$. This energy level is not doubly degenerate because the state $| K_{00}\rangle$ is forbidden by the RSOS selection rules, eq. (2.6). With periodic boundary conditions, the kink states $| K_{01}\rangle$ and $| K_{10}\rangle$ are projected out and $| K_{11}\rangle$ is the only one-particle state that can appear in the spectrum.

In order to determine the pattern of the energy levels obtained from TCSA and to relate the scattering processes to the data of the original unperturbed CFT (along the line suggested in [17]), we would need a higher-level Bethe ansatz technique. This is because our actual situation deals with kink-like excitations in contrast to that of ref. [17] which considers only diagonal, breather-like S matrices. The Bethe-ansatz technique gets quite complicated in the case of a S-matrix with kink excitations, and has been carried out only for particular examples [18], all of them describing $\phi_{1,3}$ perturbations of minimal models.

2.2 The Approach of Zamolodchikov

Zamolodchikov [19] has started his analysis from a less formal approach. Having the TCSA spectrum as an input he conjectured that the S matrix is described by the geometry of the Hard Square Lattice Model (HSLM) (as, by the way, is the S-matrix resulting from the Smirnov approach). Let us repeat here his reasoning, though in a some what different formulation: the Boltzmann weights of the HSLM in the critical regime can be written as [20]

$$\omega(a,b,c,d;u) = \delta_{ac}\sin(\lambda-u) + \delta_{bd}\sin(u)\frac{\psi(a)^{\frac{1}{2}}\psi(c)^{\frac{1}{2}}}{\psi(b)} \quad . \tag{2.30}$$

where

$$\psi(a) \equiv \sin(\lambda(a+1)) \quad , \qquad \lambda \equiv \frac{\pi}{5} \quad . \tag{2.31}$$

The restrictions on the values a,b,c,d, assigned to the plaquettes of a square lattice, are that they can take only the values 0 and 1, and that nearest neighbor sites can never be both 0. We also realize immediately that the model obtained from the analysis of Smirnov also belongs to this class. In order to link the spectral parameter u to the rapidity, one makes the ansatz

$$S^{ac}_{bd} = S_0(\beta)\,\omega(a,b,c,d;i\Lambda\beta) \tag{2.32}$$

that is one puts $u = i\Lambda\beta$, where the constant Λ has to be determined from the constraints of scattering theory. First of all one needs to satisfy crossing symmetry, that is the replacement $\beta \to i\pi - \beta$ must correspond to $u \to \frac{\pi}{5} - u$. Secondly, a pole is wanted at $\beta = \frac{2\pi i}{3}$ (since the bound state has the same mass m as the fundamental kink-excitations). But this pole must not appear in all amplitudes, because it would give rise also to a bound state K_{00}. Therefore there must be a zero at the rapidity $\beta = \frac{2\pi i}{3}$ in the Boltzmann weight of the amplitude S^{11}_{00} (again note that this conditions are also fulfilled by the S-matrix resulting from the Smirnov approach). This means one needs to fulfill the following system of equations:

$$\begin{aligned} \frac{\pi}{5} + \Lambda\pi &= 0 \,(\mathrm{mod}\,\pi) \\ \frac{\pi}{5} - \frac{2\pi\Lambda}{3} &= 0 \,(\mathrm{mod}\,\pi) \quad . \end{aligned} \tag{2.33}$$

Zamolodchikov chooses[2] the solution $\Lambda = -\frac{6}{5}$, which leads him to the S-matrix

$$\begin{aligned} S^{ac}_{bd} &= S_0(\beta)\left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]}\right)^{-\frac{\beta}{2\pi i}} \times \\ & \left\{\delta_{ac}\sinh(\frac{6\beta}{5})\left(\frac{[2a+1][2c+1]}{[2d+1][2b+1]}\right)^{\frac{1}{2}} + \sinh\left(\frac{6}{5}(\beta - i\pi)\right)\right\} \quad , \end{aligned} \tag{2.34}$$

with S_0 given by

$$S_0 = \frac{\sinh(\frac{6\beta}{5} - \frac{3i\pi}{5})}{\sinh(\frac{6\beta}{5} - \frac{2i\pi}{5})\sinh(\frac{6\beta}{5} + \frac{i\pi}{5})} \quad . \tag{2.35}$$

He requires strict crossing symmetry and includes therefore an oscillating factor, which (as in the case of the Smirnov S-matrix) corresponds to (2.30), differing only by a gauge transformation.

Now let us discuss some features of this S-matrix. The residues of the amplitudes at the simple pole at $\beta = \frac{2\pi i}{3}$ are given by

$$\begin{aligned} \tau_1 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\; S^{11}_{11}(\beta) = i\,\frac{5}{6}\,\frac{\left(\sin\left(\frac{\pi}{5}\right)\right)^3}{\left(\sin\left(\frac{2\pi}{5}\right)\right)^2} \;; \\ \tau_2 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\; S^{10}_{11}(\beta) = -i\,\frac{5}{6}\,\frac{\left(\sin\left(\frac{\pi}{5}\right)\right)^{\frac{13}{6}}}{\left(\sin\left(\frac{2\pi}{5}\right)\right)^{\frac{7}{6}}} \;; \end{aligned}$$

[2]for a classification of scattering models arising from the HSLM geometry, see [21]

$$\begin{aligned}
\tau_3 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\ S_{10}^{11}(\beta) = i\,\frac{5}{6}\,\frac{\left(\sin\left(\frac{\pi}{5}\right)\right)^{\frac{4}{3}}}{\left(\sin\left(\frac{2\pi}{5}\right)\right)^{\frac{1}{3}}}\ ; \\
\tau_4 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\ S_{00}^{11}(\beta) = 0\ ; \\
\tau_5 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}}\ S_{11}^{00}(\beta) = i\,\frac{5}{6}\,\frac{\left(\sin\left(\frac{\pi}{5}\right)\right)^{\frac{4}{3}}}{\left(\sin\left(\frac{2\pi}{5}\right)\right)^{\frac{1}{3}}}\ .
\end{aligned} \tag{2.36}$$

Their numerical values are collected in Table 4.

In the asymptotic limit $\beta \to \infty$, all amplitudes but $S_{11}^{11}(\beta)$ have an oscillating behaviour

$$\begin{aligned}
S_{11}^{11}(\beta) &\sim e^{-2\pi i\delta}\ ; \\
S_{11}^{10}(\beta) &\sim e^{-\delta i\pi}\, e^{\delta\beta}\, e^{\frac{3\pi i}{5}}\ ; \\
S_{10}^{11}(\beta) &\sim e^{-\delta\beta}\, e^{-\frac{3\pi i}{5}}\ ; \\
S_{00}^{11}(\beta) &\sim e^{-2\delta\beta}\, e^{-\frac{i\pi}{5}}\ ; \\
S_{11}^{00}(\beta) &\sim e^{-2\delta i\pi}\, e^{2\delta\beta}\, e^{\frac{-4\pi i}{5}}\ .
\end{aligned} \tag{2.37}$$

where we defined

$$e^{2\pi i\delta} = \frac{\sin(\frac{\pi}{5})}{\sin(\frac{2\pi}{5})}\ .$$

This of course is due to the gauge transformation performed in order to insure strict crossing symmetry. But even not taking in account these oscillatory factors we were not able to link the asymptotic phase-shifts to an OPE of a conformal field of the model $\mathcal{M}_{4,5}$. Up to now the link of the asymptotic phase shifts to the CFT lies on a purely observational basis, *i.e.*, no rigorous theoretical explanation is available for this fact. Therefore we would like to stress, that the absence of such a link for Zamolodchikov's proposal does not imply the inconsistency of his conjecture.

3 Finite-Size Effects

The TCSA allows us to study the crossover from massless to massive behaviour in a theory with the space coordinate compactified on a circle of radius $R/2\pi$. As we discussed in the introduction, the method consists in truncating the infinite-dimensional Hilbert space of the CFT up to a level Λ in the Verma modules, diagonalizing the Hamiltonian

$$H_p(\lambda, R) = \frac{2\pi}{R}\,\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) + \mu\,\int_0^R \Phi_{r,s}(x)\,dx\ . \tag{3.1}$$

An efficient algorithm has been developed for performing such a computation [22]. In our case, the truncation Λ is fixed at level 5 in the Verma modules. The parameter μ in (3.1) is a dimensionful coupling constant, related to the mass scale of the perturbed theory

$$[\mu] = m^{2-2\Delta_{r,s}} \,. \tag{3.2}$$

In the following we fix the mass scale putting $\mu = 1$.

In a finite geometry the spectrum of the hamiltonian (3.1) is discrete. The energy eigenvalues take the scaling form

$$E_i(R) = \frac{1}{R} F_i(\rho) \,, \tag{3.3}$$

with the scaling variable $\rho = \frac{R}{\xi}$. The correlation length ξ is defined as the Compton wave length of the lightest particle in the thermodynamic limit, $\xi = \frac{1}{m}$. In the ultraviolet regime ($\rho \ll 1$), the spectrum is dominated by the conformal part of the Hamiltonian, and behaves as

$$E_i(R) \simeq \frac{2\pi}{R}\left(2\Delta_i - \frac{c}{12}\right) \,, \qquad mR \ll 1 \,, \tag{3.4}$$

whereas in the infrared region ($\rho \gg 1$) it is characterized by a set of stable particles. There the scaling functions F_i become

$$F_i \sim \frac{1}{2\pi}\left[\epsilon_0 (\frac{R}{\xi})^2 + \frac{M_i}{m}\frac{R}{\xi}\right] \quad , \tag{3.5}$$

so that

$$E_i(R) \simeq \epsilon_0 m^2 R + M_i \,, \qquad mR \gg 1 \ , \tag{3.6}$$

where M_i is the mass gap of the ith level. However, the above infrared asymptotic behaviour holds only in the ideal situation when the truncation parameter Λ goes to infinity. In practice, for finite Λ, the linear behaviour of eq. (3.6) is realized only within a finite region of the R axis. The large R behaviour is dictated by truncation effects. In order to find the physical regions, we make use of the parameter (introduced in ref. [7])

$$\rho_i(R) = \frac{d\,\log E_i(R)}{d\,\log R} \,. \tag{3.7}$$

The parameter ρ_i lies between the values $\rho_i = -1$ (in the ultraviolet region) and $\rho_i = 1$ (in the infrared one). In the limit of large R (the truncation-dominated regime), $\rho_i = 1-2\Delta$.

The "window" in R where the linear infrared behaviour holds depends upon the perturbing field and, for the case of operators with anomalous dimension $\Delta \geq \frac{1}{2}$, it can be completely shrunk away. This phenomenon is related to the divergencies which appear in a perturbative expansion of the Hamiltonian (3.1), which must be renormalized. Under these circumstances, it is more convenient to consider the differences of energies, which are not renormalized.

In the case of the subleading magnetic perturbation of TIM, the anomalous dimension of $\Phi_{2,1}$ is $\Delta = \frac{7}{16}$, which is near $\frac{1}{2}$. Looking at fig. 1, we see that the onset of the infrared region of the two lowest levels is around $R \sim 2$ and persists only for few units in R. It is well known in statistical mechanics, that the largest two eigenvalues of the transfer matrix of two-dimensional ferromagnetic systems have an exponential energy splitting for large R [23]. As was shown in [7] this is also the case in TCSA. That is, in the region of the on-set of the infra-red behaviour, the ground state levels approach each other exponentially [7]

$$E_1 - E_0 \sim e^{-mR} \,. \tag{3.8}$$

But this allows us to extract the mass "experimentally" by measuring the splitting of the first two lines. We find

$$m = 0.98 \pm 0.02 \,. \tag{3.9}$$

From fig. 5, we see that for the third level, that of one-particle state, the ultraviolet behaviour extends till $R \sim 0.5$. The crossover region is in the interval $0.5 \leq R \leq 2$. Beyond this interval, the infrared regime begins but the "window" of infrared behaviour is quite narrow, in the neighbourhood of $R \sim 3$. Considering the differences of energies with respect to those of the degenerate ground states (fig. 6) one can also read off the mass-gap and see that it is consistent with the value extracted from the exponential approach of the two lowest levels. In fig. 6, the third line defines the threshold, with a mass-gap $2m$.

In the ideal situation, when the truncation level $\Lambda \to \infty$, the crossover between the intermediate region ($mR \sim 1$) and the infrared one ($mR \gg 1$) is controlled by off-mass shell effects and has an exponential behaviour. The computation of these finite-size corrections has been put performed by Lüscher [6], and we refer the reader to this reference for a detailed discussion of the subject. Rigorously speaking, this analysis is valid for the case of only one vacuum in the theory, but the degeneracy of the ground state gives only subleading contribution (see below), so that we can use Lüsher's results, at least at leading order. The idea is to consider perturbative corrections to the propagator, which is known exactly for the infinite volume theory. The leading corrections come from topologically non-trivial diagrams, which wind around the cylinder exactly once. The analysis is independent of the details of the interaction, and the above statement is very general.

The diagrams contributing to the corrections are shown in fig. 7. Note that the only state which can propagate is K_{11} since K_{00} is forbidden from the HSLM geometry, and K_{01} and K_{10} from the boundary conditions. The diagrams can be computed from the data extracted from the S-matrix. The first correction involves the on-mass-shell three-particle vertex Γ, which is extracted from the residue at $\beta = \frac{2\pi i}{3}$ of the amplitudes $S_{11}^{11}(\beta)$ for the Smirnov and Zamolodchikov S-matrix respectively. The second correction comes from an integral over the momentum of the intermediate virtual particle, interacting via the S-matrix ($S_{11}^{11}(\beta)$. The final result becomes

$$\begin{aligned}\Delta E(R) \;\; &\equiv \;\; E_2(R) - E_0(R) = m + i\,\frac{\sqrt{3}m}{2}\Gamma^2 \exp\left(-\frac{\sqrt{3}mR}{2}\right) \\ &\quad -m\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}\,e^{-mR\cosh\beta}\cosh\beta\left(S\left(\beta+\frac{i\pi}{2}\right)-1\right) \;\; .\end{aligned} \tag{3.10}$$

In this analysis we have to be aware that the ground state of our potential is degenerate, therefore (3.10) is correct up to further subleading corrections of the form $\mathcal{O}(e^{-mR})$. The problem of comparing the S-matrix with the truncation data, using this approach, has been addressed to in [12]. There we adopted the following procedure: first we computed numerically the integral on the intermediate particles in both cases of RSOS and Zamolodchikov's S-matrix and we subtracted it from the numerical data obtained from the TCSA. After this subtraction, we made a fit of the data with a function of the form

$$G(R) = A + Be^{-\frac{\sqrt{3}}{2}mR} + Ce^{-mR} \;\; . \tag{3.11}$$

The first term should correspond to the mass term. The coefficient of the second one is the quantity we need in order to extract the residue of the S-matrix at $\beta = \frac{2\pi i}{3}$

$$\frac{2}{\sqrt{3}m}\,B = i\,\mathrm{Res}_{\beta=\frac{2\pi i}{3}}\;\; S_{11}^{11}(\beta) \;\; . \tag{3.12}$$

The third term is a subleading one, which takes into account: a) the asymptotic exponential approach of the lowest levels of our TCSA data to the (unknown) theoretical vacuum energy $E_0(R)$; b) the possible subleading corrections to (3.10), arising from tunneling processes. These tunneling processes might not be strictly proportional to e^{-mR}, but in the region in which we are measureing, a term of the order $R^\alpha e^{-mR}$ will behave as e^{-mR}, since the exponential decay will overwhelm the polynomial behaviour.

In the case of RSOS S-matrix, the best fit gives the following values

$$\begin{aligned} A &= 0.97 \pm 0.02\,; \\ B &= -0.29 \pm 0.02\,; \\ C &= -0.36 \pm 0.02\,. \end{aligned} \tag{3.13}$$

The corresponding curve is drawn in fig. 8, together with the data obtained from TCSA. The mass term agrees with our previous calculation (eq. (3.9)). The second term gives for the residue at $\beta = \frac{2\pi i}{3}$ the value 0.34 ± 0.02. This is consistent with that of the RSOS S-matrix. In our fit procedure, the value of the residue we extracted through (3.12) is stable with respect to small variation of the mass value. Increasing (decreasing) m, B increases (decreases) as well, in such a way that the residue takes the same value (within the numerical errors). This a pleasant situation because it allows an iterative procedure to find the best fit of the data: one can start with a trial value for m (let's say $m = 1$) and plug it into (3.11). From the A-term which comes out from the fit, one gets a new determination of the mass m that can be again inserted into (3.11) and so on. Continued iteration does not affect significantly the value we extract for the residue, but converges to an accurate measurement of the mass. The values in (3.13) were obtained in this way.

With Zamolodchikov's S-matrix the best fit of the data (with the same iterative procedure as before) gives the result

$$\begin{aligned} A &= 0.96 \pm 0.02\,; \\ B &= -1.10 \pm 0.02\,; \\ C &= 1.14 \pm 0.02\,. \end{aligned} \tag{3.14}$$

The residue extracted from these data (1.29 ± 0.01) is not consistent with that one of the amplitude $S_{11}^{11}(\beta)$. The situation does not improve even if we *fix* the coefficient of $e^{-\frac{\sqrt{3}m}{2}}$ to be that one predicted by Table 2, namely $B = -0.158$ and leave as free parameters for a best fit A and C. In this case, our best determination of A and C were $A = 0.965$ and $C = -0.046$. The curve is plotted in fig. 8 together with the data obtained from TCSA.

4 Conclusions

We reviewed recent work on the S matrix theory for kink excitations and the use of finite size effects. We discussed these techniques in the framework of the tricritical Ising model. There exist two proposals for the scattering matrix of this model, one deriving from a general framework describing $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations of CFT, introduced by Smirnov [8], and one by Zamolodchikov [19]. The two resulting models turn out to be very similar, both exhibiting the same spectrum and both being determined by the geometry of the hard square lattice gas. Nevertheless the analytic structure in the whole complex plane is different.

As an additional feature the proposal of Smirnov allows one to relate the asymptotic phase shifts to the operator product expansion of the field $\Phi_{1,3}$ of the underlying field theory. We also discussed that this is a general property of the RSOS-reduced S-matrix constructed from the Izergin-Korepin R-matrix. This feature should be analyzed in more detail, and could give hints on the ultraviolet limit of the kink-fields.

In order to decide which proposal fits better to the physical picture, we compared the results from the TCSA with those coming from considering the theoretical predictions on finite size effects for a quantum field theory put onto a cylinder. These calculations have been carried out by Lüscher [6] and we applied them to the TIM, including unknown subleading corrections coming from tunneling effects between the two degenerate vacua. Our analysis shows that the S matrix of Smirnov fits the data, where it is impossible to achieve a reasonable result for the proposal of Zamolodchikov.

Though we have settled the question of which of the two S-matrices describes the $\phi_{2,1}$-perturbation of $\mathcal{M}_{4,5}$ the topic is far from being exhausted. The main open question is which kind of system does the S matrix of Zamolodchikov describe. To address this last point, it seems very important to develop the TBA for systems with HSLM geometry. This could give a much deeper insight in the structure of the previously discussed scattering theories.

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Figure Captions

Figure 1 . First energy levels for the subleading magnetic perturbation of TIM with periodic boundary conditions on the strip.

Figure 2 . Landau-Ginzburg potential for the subleading magnetic perturbation of TIM.

Figure 3 . Pole structure of $S_0(\beta)$: $*$ are the location of the poles and **o** the position of the zeroes.

Figure 4 . Intermediate states in the s-channel and t-channel of the scattering amplitudes.

Figure 5 . Energy level which corresponds to one-particle state in the infrared region.

Figure 6 . The lowest energy differences with respect to the ground state.

Figure 7 . Finite volume off-mass-shell corrections to one-particle energy: (a) leading correction, (b) sub-leading correction.

Figure 8 . Energy difference of the one-particle state with respect to the double degenerate ground state, $\Delta E(R) \equiv E_2(R) - E_0(R)$, compared with off-mass-shell corrections. The dots are the numerical data obtained from TCSA, a is the curve for the RSOS S-matrix, with A, B and C given in (6.13), b is the curve for Zamolodchikov's S-matrix with the B term equal to the theoretical value and A and C coming from a best fit.

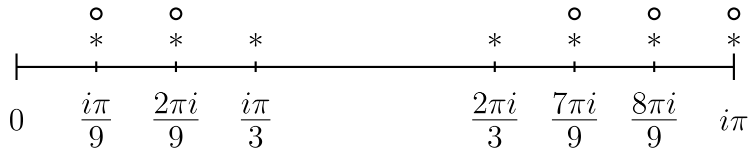

Figure 3

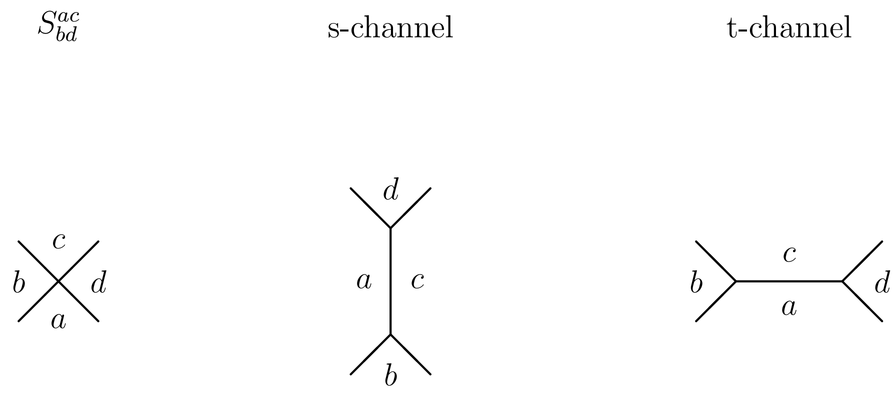

Figure 4

$\frac{3}{2}$	$\frac{7}{16}$	0
$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$
$\frac{1}{10}$	$\frac{3}{80}$	$\frac{3}{5}$
0	$\frac{7}{16}$	$\frac{3}{2}$

Table 1 - Kac-table of the minimal model $\mathcal{M}_{4,5}$

Field	Identification	Z_2-symmetry
identity	$1 \equiv \Phi_{0,0}$	even
leading energy	$\epsilon \equiv \Phi_{\frac{1}{10},\frac{1}{10}} =: \phi^2 :$	even
subleading energy	$\epsilon' \equiv \Phi_{\frac{6}{10},\frac{6}{10}} =: \phi^4 :$	even
irrelevant field	$\epsilon'' \equiv \Phi_{\frac{3}{2},\frac{3}{2}} =: \phi^6 :$	even
leading magnetization	$\sigma \equiv \Phi_{\frac{3}{80},\frac{3}{80}} =: \phi :$	odd
subleading magnetization	$\sigma' \equiv \Phi_{\frac{7}{16},\frac{7}{16}} =: \phi^3 :$	odd

Table 2 - Landau-Ginzburg identification of the primary fields of the model $\mathcal{M}_{4,5}$

	Res at $\frac{2\pi i}{3}$	Res at $\frac{i\pi}{3}$
$S_{00}^{11}(\beta)$	0	-0.957340 i
$S_{01}^{11}(\beta)$	0.957340 i	0.591669 i
$S_{11}^{11}(\beta)$	0.365671 i	-0.365671 i
$S_{11}^{01}(\beta)$	0.465141 i	0.752614 i
$S_{11}^{00}(\beta)$	0.591669 i	0

Table 3 - The residues on the poles $\frac{2\pi i}{3}$ and $\frac{i\pi}{3}$ of the RSOS S-matrix (2.15).

	Res at $\frac{2\pi i}{3}$	Res at $\frac{i\pi}{3}$
$S_{11}^{11}(\beta)$	0.187095 i	-0.187095 i
$S_{11}^{10}(\beta)$	-0.279395 i	-0.417229 i
$S_{10}^{11}(\beta)$	0.417229 i	0.279395 i
$S_{00}^{11}(\beta)$	0	-0.417229 i
$S_{11}^{00}(\beta)$	0.417229 i	0

Table 4 - The residues on the poles $\frac{2\pi i}{3}$ and $\frac{i\pi}{3}$ of the Zamolodchikov S-matrix (2.36).

R	$\Delta E(R)$	$I_1(R)$	$I_2(R)$
2.5	0.95913	-0.0625322	-0.0174576
2.55	0.958681	-0.058955	-0.0164808
2.6	0.9583	-0.0555943	-0.0155615
2.65	0.957983	-0.0524359	-0.0146961
2.7	0.957722	-0.0494667	-0.0138811
2.75	0.957511	-0.0466744	-0.0131135
2.8	0.957345	-0.0440477	-0.0123902
2.85	0.957221	-0.0415761	-0.0117085
2.9	0.957132	-0.0392498	-0.011066
2.95	0.957076	-0.0370596	-0.0104601
3.	0.957048	-0.034997	-0.00988871
3.05	0.957046	-0.0330542	-0.00934973
3.1	0.957068	-0.0312238	-0.00884121
3.15	0.95711	-0.0294988	-0.00836133
3.2	0.95717	-0.0278729	-0.00790841
3.25	0.957247	-0.02634	-0.00748085
3.3	0.957337	-0.0248945	-0.00707716
3.35	0.95744	-0.0235313	-0.00669595
3.4	0.957553	-0.0222453	-0.0063359
3.45	0.957676	-0.021032	-0.00599579
3.5	0.957807	-0.019887	-0.00567447
3.55	0.957945	-0.0188064	-0.00537086

Table 5 - Difference of energy $\Delta E \equiv E_2 - E_0$ compared with the finite volume off-mass-shell corrections. In the first column, the values of R. In the second column, the numerical data obtained from TCSA. In the third and fourth columns, the numerical values of the integral $\int_{-\infty}^{\infty} \frac{d\beta}{2\pi}\, e^{-mR\cosh\beta} \cosh\beta \left(S\left(\beta + \frac{i\pi}{2}\right) - 1\right)$ for the RSOS S-matrix and for the Zamolodchikov's S-matrix, respectively. The integral is computed for a value of m, obtained self-consistently from the best fit of the data.

R	$\Delta E(R)$	$I_1(R)$	$I_2(R)$
3.6	0.958089	-0.0177864	-0.00508394
3.65	0.958239	-0.0168233	-0.00481276
3.7	0.958393	-0.015914	-0.00455642
3.75	0.958549	-0.0150553	-0.00431408
3.8	0.958709	-0.0142442	-0.00408495
3.85	0.95887	-0.013478	-0.00386828
3.9	0.959033	-0.0127541	-0.00366337
3.95	0.959198	-0.0120701	-0.00346957
4.	0.959363	-0.0114238	-0.00328625
4.05	0.959528	-0.0108129	-0.00311283
4.1	0.959693	-0.0102355	-0.00294875
4.15	0.959857	-0.00968962	-0.0027935
4.2	0.960021	-0.00917357	-0.00264659
4.25	0.960183	-0.00868562	-0.00250756
4.3	0.960346	-0.00822421	-0.00237597
4.35	0.960505	-0.00778784	-0.00225142
4.4	0.960663	-0.00737512	-0.00213352
4.45	0.96082	-0.00698473	-0.00202191
4.5	0.960973	-0.00661542	-0.00191624
4.55	0.961128	-0.00626603	-0.00181618
4.6	0.961279	-0.00593545	-0.00172144
4.65	0.961426	-0.00562265	-0.00163172
4.7	0.96158	-0.00532664	-0.00154676
4.75	0.961718	-0.0050465	-0.00146629
4.8	0.961862	-0.00478135	-0.00139007
4.85	0.962	-0.00453039	-0.00131787
4.9	0.962143	-0.00429282	-0.00124948
4.95	0.962279	-0.00406792	-0.00118469

Table 5 - (Continued).

R	$\Delta E(R)$	$I_1(R)$	$I_2(R)$
5.	0.962408	-0.003855	-0.00112331
5.05	0.962541	-0.0036534	-0.00106515
5.1	0.962667	-0.00346252	-0.00101005
5.15	0.962795	-0.00328177	-0.000957836
5.2	0.962916	-0.00311059	-0.000908356
5.25	0.963039	-0.00294848	-0.000861466
5.3	0.963154	-0.00279494	-0.000817027
5.35	0.96328	-0.00264952	-0.00077491
5.4	0.963387	-0.0025117	-0.00073499

Table 5 - (Continued).