Classical Observables of the Nambu String from the Many-time Approach.

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Abstract

In this talk we start with a short review of the many-time approach and its application to the open Nambu string. This is a preliminary and necessary step towards the search of a complete canonical set of observables, *i.e.* a set of canonical variables in strong evolution with the constraints of the string. This allows in turn to perform a canonical transformation to a new canonical basis in which gauge and physical degrees of freedom are completely separated into two canonical subsets.

1. Introduction.

This talk is based on work done in collaboration with G. Longhi and L. Lusanna[1]. We shall present a complete set of observables (*i.e* independent Cauchy data) for the open Nambu string; moreover, we shall give a canonical transformation in the phase-space of the string to a set of new canonical variables in which physical and gauge degrees of freedom are well separated. This is the canonical basis for a set of Dirac brackets[2] in an arbitrary gauge connected to the orthormal one.

It is worth stressing that this is not merely equivalent to the determination of the Dirac brackets. The algebraic algorithm for the computation of the Dirac brackets doesn't allow by itself the determination of the elementary canonical variables of the Dirac Symplectic structure. On the contrary, the complete set of these new canonical variables is such that, in terms of them, the Dirac brackets are simply the Poisson bracets restricted to the physical degrees of freedom.

The usefulness of such a canonical transformation stays in the fact that it allows to perform a canonical quantization of the theory in an arbitrary gauge.

The problem of finding a complete set of observables is of very general interest, since it is not yet completely solved for gauge theory nor general relativity.

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A preliminary step in the search of this complete set of observables is the discussion of the theory in an arbitrary gauge (*i.e.* without limiting ourselves to the orthonormal gauge, as it is usually done). This may be done through the many-time approach, which will be shortly reviewed.

2. The many-time Approach.

The many-time approach was developed in reference[3] in order to study systems of n nonrelativistic or relativistic particles with action-at-a-distance interactions described by n first-class constraints in strong involution

$$\{\chi_a, \chi_b\} = 0 \qquad a, b = 1, ..., n, \tag{2.1}$$

and for the general problem of a constrained system in[4].

Following the Dirac's approach [2], if $x^{\mu a}$, $P_{\mu a}$, $a = 1, ...n$, are the canonical variables, with $\{x^{\mu a}, P^{\nu}_{b}\} = -\eta^{\mu\nu}\delta^a_b$, $\eta^{\mu\nu} \equiv (1, -1, -1, -1)$, the Hamilton equations with respect to the Dirac Hamiltonian are

$$\begin{cases} \dfrac{d}{d\tau} x^{\mu a}(\tau) = \{x^{\mu a}, H_D\} \approx \sum_{b=1}^{n} \lambda^b(\tau) \{x^{\mu a}, \chi_b\}, \\ \dfrac{d}{d\tau} P_{\mu a}(\tau) = \{P_{\mu a}, H_D\} \approx \sum_{b=1}^{n} \lambda^b(\tau) \{P_{\mu a}, \chi_b\}. \end{cases} \tag{2.2}$$

These equations can be solved only after the first step in fixing a gauge, *i.e.* after assigning a set of multipliers $\lambda^a(\tau)$.

Another way to approach the problem of solving the system (2.2) is to introduce n "times" τ^a formally defined through the equations

$$d\tau^a = \lambda^a(\tau) d\tau, \tag{2.3}$$

and by redefining $x^{\mu a}(\tau) = \tilde{x}^{\mu a}(\tau^1, ..., \tau^n)$, $P_{\mu a}(\tau) = \tilde{P}_{\mu a}(\tau^1, ..., \tau^n)$, equations (2.2) are replaced by the n-times Hamilton equations

$$\begin{cases} \dfrac{\partial x^{\mu a}}{\partial \tau^b} = \{x^{\mu a}, \chi_b\}, \\ \dfrac{\partial P_{\mu a}}{\partial \tau^b} = \{P_{\mu a}, \chi_b\}, \end{cases} \tag{2.4}$$

whose integrability conditions are just equations (2.1). Each "time" τ^a has its own Hamiltonian. This formal derivation of the equations of motion can be justified in a more rigorous way, obtaining them as characteristic equations of the constraint's equations as in [4]. Since the system (2.4) is autonomous, apart from an initial constant each parameter τ^a is defined by the system (2.4) itself, and can eventually be eliminated in terms of some physical coordinate.

The physical coordinates $q^{\mu a}(\tau^a)$ in the configuration space are recovered [3] through the Droz-Vincent conditions[5]

$$\begin{cases} \{q^{\mu a}, \chi_b\} = 0, & \text{when a} \neq b, \\ q^{\mu a} = x^{\mu a}, & \text{when } \tau^{(a)} = \tau^{(a)}(\tau). \end{cases} \tag{2.5}$$

The no-interaction theorem[6] stems from the requirement $q^{\mu a} = x^{\mu a}$ when the times do not satisfy $\tau^{(a)} = \tau^{(a)}(\tau)$.

When we have a set of first-class constraints in weak involution

$$\{\chi_a, \chi_b\} = C^c_{ab}(x, P)\chi_c \approx 0, \tag{2.6}$$

equations (2.4), which are a direct consequence of definition (2.3), are no more integrable. To recover integrable equations we have to replace the constraints χ_a with equivalent constraints $\tilde{\chi}_a$ satisfying equations (2.1)[7]. In the finite-dimensional case their existence is ensured (at least locally) by theorems about function groups[8,9], which however have no general extension to the infinite-dimensional case. However, constraints in strong involution always exist when the original constraints can be solved in terms of a subset of the canonical variables, and this is equivalent to the use of the BRST method [8].

3. The Classical Bosonic String.

Let us consider the action for the open Nambu string ($\hbar = c = 1$)

$$S = -N \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \sqrt{-h(\sigma, \tau)}, \qquad L = -N\sqrt{-h}, \tag{3.1}$$

where $N = \frac{1}{2\pi\alpha'}$, and

$$-h = -\det \parallel h_{\alpha\beta} \parallel = (\dot{x} \cdot x')^2 - \dot{x}^2 x'^2 \geq 0. \tag{3.2}$$

The strip $0 < \sigma < \pi$ is mapped in the world-sheet spanned by the string in the Minkowski space, which is described by the coordinates $x^\mu(\sigma, \tau)$. $h_{\alpha\beta}(\sigma, \tau)$ is the induced metric.

Let us define the following two quantities

$$\begin{cases} P^\mu(\sigma, \tau) = -\dfrac{\partial L}{\partial \dot{x}_\mu(\sigma, \tau)} = N\sqrt{-h}h^{0\alpha}\partial_\alpha x^\mu = \dfrac{N}{\sqrt{-h}}((\dot{x} \cdot x')x'^\mu - x'^2 \dot{x}^\mu), \\ \Pi^\mu(\sigma, \tau) = -\dfrac{\partial L}{\partial {x'}_\mu(\sigma, \tau)} = N\sqrt{-h}h^{1\alpha}\partial_\alpha x^\mu = \dfrac{N}{\sqrt{-h}}((\dot{x} \cdot x')\dot{x}^\mu - \dot{x}^2 x'^\mu), \end{cases} \tag{3.3}$$

where P^μ is the canonical momentum, which satisfies the identities

$$\begin{cases} P^2(\sigma, \tau) + N^2 x'^2(\sigma, \tau) = 0, \\ (P(\sigma, \tau) \cdot x'(\sigma, \tau)) = 0. \end{cases} \tag{3.4}$$

The variational principle for the action (3.1), with the variations $\delta_0 x^\mu(\sigma,\tau)$ vanishing at $\tau=\tau_1,\tau_2$, is

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma L_\mu \delta_0 x^\mu - \int_{\tau_1}^{\tau_2} d\tau \Pi_\mu \delta_0 x^\mu|_0^\pi = 0, \tag{3.5}$$

and gives the following Euler-Lagrange equations and boundary conditions

$$L^\mu(\sigma,\tau) = \dot{P}^\mu + \Pi'^\mu = 0 \tag{3.6}$$

$$\Pi_\mu \delta_0 x^\mu|_0^\pi = 0. \tag{3.7}$$

These boundary conditions have been studied in detail in references [1–11], taking into account the fact that the variation $\delta_0 x^\mu$ are not all independent nor allowed, since a variation of the boundaries must not violate the condition $-h \geq 0$.

At this point the main problem in the classical theory is that the equation of motion is highly non-linear and very difficult to solve.

What is usally done at this stage is to impose the orthonormal gauge, that is a choice of the parametrization of the string world sheet which satisfies, beside

$$-h(\sigma,\tau) \geq 0, \tag{3.8}$$

the conditions

$$\dot{x}^2 + {x'}^2 = (\dot{x}\cdot x') = 0, \tag{3.9}$$

that is

$$h_{\alpha\beta} = \dot{x}^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{with} \quad \dot{x}^2 \geq 0. \tag{3.10}$$

The equation of motion now reduce to the well-known D'Alembert equation:

$$L^\mu = N(\ddot{x}^\mu - x''^\mu) = 0 \tag{3.11}$$

The usual o.g. boundary conditions are

$$x'^\mu(0,\tau) = x'^\mu(\pi,\tau) = 0, \tag{3.12}$$

which turn out to be more restrictive than the orthonormal gauge translation of the general boundary condition (3.7).

4. The Canonical Formalism.

In order to define a Poisson structure in the phase-space we will use the following extension of the coordinates outside the interval $(0,\pi)$, mainly in order to make contact with the usual extension used in the literature on string models,

$$\begin{cases} x^\mu(\sigma,\tau) = x^\mu(-\sigma,\tau) = x^\mu(\sigma+2n\pi,\tau), \\ P^\mu(\sigma,\tau) = P^\mu(-\sigma,\tau) = P^\mu(\sigma+2n\pi,\tau), \end{cases} \tag{4.1}$$

where n is an integer, with the points $\sigma = 2n\pi$, for any n (or $\sigma = (2n+1)\pi$, for any n), corresponding to the boundary values $x^\mu(0,\tau)$ and $P^\mu(0,\tau)$ (or $x^\mu(\pi,\tau)$ and $P^\mu(\pi,\tau)$) as limit values from the open interval $(0,\pi)$.

Following reference[12] we introduce an even and odd delta function with period 2π:

$$\Delta_\pm(\sigma,\sigma') = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(e^{in(\sigma-\sigma')} \pm e^{-in(\sigma+\sigma')}\right) = \\ = \sum_{n=-\infty}^{\infty}\left(\delta(\sigma-\sigma'+2n\pi) \pm \delta(\sigma+\sigma'+2n\pi)\right) \longrightarrow \delta(\sigma-\sigma'), \quad \text{for } \sigma,\sigma' \in (0,\pi). \tag{4.2}$$

We introduce the following Poisson structure ($\eta^{\mu\nu} = (1;-1,-1,-1)$):

$$\{x^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = -\eta^{\mu\nu}\Delta_+(\sigma,\sigma') \quad \longrightarrow \quad -\eta^{\mu\nu}\delta(\sigma-\sigma'), \quad \text{for } \sigma,\sigma' \in (0,\pi). \tag{4.3}$$

The set of identity (3.4) are now constraints on the phase-space:

$$\begin{cases} P^2(\sigma) + N^2 x'^2(\sigma) \approx 0, \\ P(\sigma)\cdot x'(\sigma) \approx 0. \end{cases} \tag{4.4}$$

Let's define the following variables

$$A^\mu_\pm(\sigma,\tau) = A^\mu_\mp(-\sigma,\tau) = P^\mu(\sigma,\tau) \pm N x'^\mu(\sigma,\tau) = \frac{\partial}{\partial\sigma}B^\mu_\pm(\sigma,\tau), \tag{4.5}$$

$$B^\mu_\pm(\sigma,\tau) = -B^\mu_\mp(-\sigma,\tau) = \int_0^\sigma d\sigma' A^\mu_\pm(\sigma',\tau) \pm N x^\mu(0), \tag{4.6}$$

With this definition, the constraints (4.4) becomes

$$\chi_\pm(\sigma) = \chi_\mp(-\sigma) = A^2_\pm(\sigma) \approx 0, \tag{4.7}$$

with the following algebra:

$$\begin{cases} \{\chi_\pm(\sigma_1,\tau), \chi_\pm(\sigma_2,\tau)\} = \mp 2N\left(\chi_\pm(\sigma_1,\tau) + \chi_\pm(\sigma_2,\tau)\right)\cdot\left(\Delta'_+(\sigma_1,\sigma_2) + \Delta'_-(\sigma_1,\sigma_2)\right), \\ \{\chi_+(\sigma_1,\tau), \chi_-(\sigma_2,\tau)\} = -2N\left(\chi_+(\sigma_1,\tau) + \chi_-(\sigma_2,\tau)\right)\cdot\left(\Delta'_+(\sigma_1,\sigma_2) - \Delta'_-(\sigma_1,\sigma_2)\right). \end{cases} \tag{4.8}$$

Therefore the constraints are 1^{th}-class, but they are in weak involution; the algebra (4.16) is the universal Dirac algebra of reparametrization [2,13]. For the many-time approach a set of 1^{th}-class constraints in strong involution are needed.

One possible solution of the constraint makes use of lightcone variables; of course other solutions are possible. In terms of the following lightcone variables

$$\begin{cases} A^+_\pm(\sigma,\tau) = \frac{1}{\sqrt{2}}\left(A^0_\pm(\sigma,\tau) + A^3_\pm(\sigma,\tau)\right), \\ A^-_\pm(\sigma,\tau) = \frac{1}{\sqrt{2}}\left(A^0_\pm(\sigma,\tau) - A^3_\pm(\sigma,\tau)\right), \end{cases} \tag{4.9}$$

with the Lorentz indices now running over $\mu = +, 1, 2, -$, we may define the new constraints

$$\tilde{\chi}^{(+)}_{\pm}(\sigma,\tau) = \frac{\chi_{\pm}(\sigma,\tau)}{2A^{+}_{\pm}(\sigma,\tau)} = A^{-}_{\pm}(\sigma,\tau) - \frac{\vec{A}^2_{\pm}(\sigma,\tau)}{2A^{+}_{\pm}(\sigma,\tau)} \approx 0, \qquad \text{if} \qquad A^{+}_{\pm}(\sigma,\tau) \neq 0, \tag{4.10}$$

where $\vec{A}^2 = (A^1)^2 + (A^2)^2$.

If $A^{+}_{\pm}(\sigma)$ does vanish for some σ, we may do the same trick with the variable $A^{-}_{\pm}(\sigma)$. It is therefore important to outline that in order to solve the constraints we need in general at least two charts, and that a particular resolution of the constraints holds in general only locally. It may happen too that no one of the previous chart is good. We have the so-called exceptional solution which may be recovered by other metods [1].

Thus our constraints are only weakly Poincaré invariant and are only locally defined, in those regions of the constraints manifold $\chi_{\pm}(\sigma,\tau) \approx 0$ where the denominator don't vanish. But they are now in strong involution:

$$\{\tilde{\chi}_{\pm}(\sigma,\tau), \tilde{\chi}_{\pm}(\sigma',\tau)\} = 0 \tag{4.11}$$

5. Many-times Functional Equations of Motion.

Since we have found a set of strongly 1^{st}-class constraints, we are now able to build the many-time formalism for the Nambu string.

First of all we must introduce one "time" for each constraint, that is:

$$\tau_{\pm}(\sigma), \qquad \text{with } \tau_{\pm}(\sigma) = \tau_{\mp}(-\sigma) = \tau_{\pm}(\sigma + 2n\pi), \tag{5.1}$$

with the boundary condition

$$\delta\tau_{+}(\sigma_n) = \delta_{-}\tau(\sigma_n), \qquad \sigma_n = n\pi, \qquad n \in \mathbf{Z}, \tag{5.2}$$

reflecting the fact that the constraints are not independent on the boundaries (see equation (4.7)). This is to be compared to the Dirac approach, where we would have the Dirac Hamiltonian

$$H_D = \int_0^{\pi} d\sigma \big[\lambda_{+}(\sigma)\tilde{\chi}_{+}(\sigma) + \lambda_{-}(\sigma)\tilde{\chi}_{-}(\sigma)\big]. \tag{5.3}$$

As we are in an infinite-dimensional case, the many-time equations of motion become functional one. For a general dynamical variable $F(\sigma|\tau_{\pm}(\sigma)]$ we shall have:

$$\frac{\delta F(\sigma|\tau_{\pm}(\sigma)]}{\delta\tau_{\pm}(\bar{\sigma})} = \Big\{F(\sigma), \tilde{\chi}_{\pm}(\bar{\sigma})\Big\}. \tag{5.4}$$

In the particular case of the Nambu string (by restricting ourselves to the interval $\sigma, \sigma' \in (0,\pi)$, so that $\Delta_{\pm}(\sigma,\sigma') = \delta(\sigma \mp \sigma')$ we get:

$$\begin{cases} \dfrac{\delta A^{\mu}_{+}(\sigma|\tau_{\pm}]}{\delta\tau_{-}(\sigma')} = 0 \\ \dfrac{\delta A^{\mu}_{-}(\sigma|\tau_{\pm}]}{\delta\tau_{+}(\sigma')} = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} A^{\mu}_{+} = A^{\mu}_{+}(\sigma|\tau_{+}] \\ \\ A^{\mu}_{-} = A^{\mu}_{-}(\sigma|\tau_{-}]. \end{cases} \tag{5.5}$$

The equations for $A^+_+(\sigma|\tau_+]$ and $A^+_-(\sigma|\tau_-]$ are

$$\begin{cases} \dfrac{\delta A^+_+(\sigma|\tau_+]}{\delta\tau_+(\sigma')} = -2N\delta'(\sigma-\sigma') \\ \dfrac{\delta A^+_-(\sigma|\tau_-]}{\delta\tau_-(\sigma')} = 2N\delta'(\sigma-\sigma'), \end{cases} \tag{5.6}$$

and their solution are

$$\begin{cases} A^+_+(\sigma|\tau_+] = -2N\dfrac{d}{d\sigma}\left[\tau_+(\sigma) - c_+(\sigma)\right] \\ A^+_-(\sigma|\tau_-] = 2N\dfrac{d}{d\sigma}\left[\tau_-(\sigma) - c_-(\sigma)\right], \end{cases} \tag{5.7}$$

where $c_\pm(\sigma)$ are a double infinity of "integration constants" which do not depend on $\tau_\pm$. The remaining equations are

$$\frac{\delta A^a_\pm(\sigma)}{\delta\tau_\pm(\bar\sigma)} = \mp 2N\frac{A^a_\pm(\bar\sigma)}{A^+_\pm(\bar\sigma)}\delta'(\sigma-\sigma'), \qquad\qquad a=1,2,-. \tag{5.8}$$

If we remember the definition (4.6) for the quantities $B^\mu_\pm(\sigma)$, we immediately see that the solution of the + components of the many-time functional equation is such that

$$B^+_\pm(\sigma = \mp 2N\left[\tau_\pm(\sigma) - c_\pm(\sigma)\right] \equiv \mp 2NT_\pm(\sigma), \tag{5.9}$$

that is the $B^+_\pm(\sigma)$ are just the canonical equivalents of the many-times. Moreover, in the orthonormal gauge they reduce to:

$$B^+_\pm(\sigma) \quad \xrightarrow{O.N.} \quad \pm Nq^+ \pm \frac{P^+}{\pi}(\tau\pm\sigma) = \pm NQ^+(\tau\pm\sigma), \tag{5.10}$$

where $Q^\mu(u)$ is the coordinate of one boundary of the string, in terms of which the solutions are expressed in the orthonormal gauge[14]. As we shall see in the following, this point is crucial as a guide to the construction of the generalization of the Del Giudice-Di Vecchia-Fubini[15] to an arbitrary gauge.

Anyway the whole set of the many-time functional equations of motion may be solved [1]; let's only quote the result for assigned initial data (denoted by an overbar) at the initial times $\bar\tau_\pm(\sigma)$:

$$\begin{cases} x^\mu(\sigma|\tau_+\tau_-]|_{\tau_\pm=\bar\tau_\pm} = \bar x^\mu(\sigma) \\ P^\mu(\sigma|\tau_+\tau_-]|_{\tau_\pm=\bar\tau_\pm} = \bar P^\mu(\sigma). \end{cases} \tag{5.11}$$

If we remember the equalities

$$\begin{cases} x^\mu(\sigma) = \dfrac{1}{2N}(B^\mu_+(\sigma) - B^\mu_-(\sigma)), \\ P^\mu(\sigma) = \dfrac{1}{2}(A^\mu_+(\sigma) + A^\mu_-(\sigma)), \end{cases}$$

we get the classical solution

$$
\begin{aligned}
x^\mu(\sigma|\tau_+,\tau_-]\Big|_{\tau_\pm=\bar\tau_\pm} =& \frac{1}{2N}\left\{\bar B_+^\mu[\bar B_+^{+-1}(-2NT_+(\sigma))] - \bar B_-^\mu[\bar B_-^{+-1}(2NT_-(\sigma))]\right\}\Big|_{\tau_\pm=\bar\tau_\pm} = \\
=& \bar x^\mu(\sigma), \\
P^\mu(\sigma|\tau_+,\tau_-]\Big|_{\tau_\pm=\bar\tau_\pm} =& \frac{1}{2}\frac{d}{d\sigma}\left\{\bar B_+^\mu[\bar B_+^{+-1}(-2NT_+(\sigma))] + \bar B_-^\mu[\bar B_-^{+-1}(2NT_-(\sigma))]\right\}\Big|_{\tau_\pm=\bar\tau_\pm} = \\
=& \bar P^\mu(\sigma).
\end{aligned}
\tag{5.12}
$$

If we impose the orthonormal gauge to the many-times, the usual orthonormal gauge solutions are recovered.

6. The Observables.

Let us now look for a local set of observables with respect to the first class constraints $\tilde\chi_\pm$, in the (σ,τ) region where there are defined.

First of all we shall introduce the generalization to an arbitrary gauge of the Del Giudice-Di Vecchia-Fubini (DDF) oscillators [15], which commute with the Virasoro generators L_n in the orthonormal gauge. They are the transverse part of the following objects:

$$
A_n^\mu = \frac{1}{\sqrt{4\pi N}}\int_{-\pi}^{\pi} d\sigma A_\pm^\mu(\sigma,\tau)\exp\left[\pm i\omega_n\frac{B_\pm^+(\sigma,\tau)}{2NP^+}\right], \qquad n=\pm1,\pm2,..., \tag{6.1}
$$

where $\omega_n = 2\pi N n$ and $P^+\neq 0$. In the orthonormal gauge they reduce to

$$
A_n^\mu \quad \xrightarrow{O.N.} \quad {}^{DDF}a_n^\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma Q'(\sigma)e^{in\sigma}. \tag{6.2}
$$

The A_n^μ's are gauge invariant quantities; they are also constant of motion, due to the vanishing of the canonical hamiltonian:

$$
\begin{cases}
\left\{\vec A_n(\tau),\tilde\chi_\pm(\sigma,\tau)\right\} = 0 \\
\left\{A_n^-(\tau),\tilde\chi_\pm(\sigma,\tau)\right\} = -\frac{i\omega_n}{P^+}\exp\left[\pm i\omega n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]\cdot\tilde\chi_\pm(\sigma,\tau)\approx 0.
\end{cases}
\tag{6.3}
$$

For the A_n^μ's we have the following properties:

$$
A_n^+ = 0 \qquad \text{for } n\neq 0, \tag{6.4}
$$

$$
A_0^\mu = \frac{P^\mu}{\sqrt{\pi N}}. \tag{6.4}
$$

Moreover, the A_n^-'s satisfy the Virasoro algebra.

The inversion formula of equations (6.1) is

$$\begin{aligned} A^{\mu}_{\pm}(\sigma,\tau) =& \sqrt{4\pi N}\frac{A^{+}_{\pm}(\sigma,\tau)}{2P^{+}}\sum_{n=-\infty}^{+\infty} A^{\mu}_{n}(\tau)\exp\left[\mp i\omega_n \frac{B^{+}_{\pm}(\sigma,\tau)}{2NP^{+}}\right] = \\ =& \sqrt{4\pi N}\frac{A^{+}_{\pm}(\sigma,\tau)}{2P^{+}}\sum_{n=-\infty}^{+\infty} A^{\mu}_{\pm n}(\tau)\exp\left[-i\omega_n \frac{B^{+}_{\pm}(\sigma,\tau)}{2NP^{+}}\right]. \end{aligned} \tag{6.5}$$

We thus recognize in our observables a generalization to an arbitrary gauge of the Fourier modes of the D'Alembert solutions.

For the transverse oscillators we have the following Poisson algebra:

$$\{A^a_m, A^b_n\} = -im\delta^{ab}\delta_{m,-n}. \tag{6.6}$$

If we want a canonical basis of observables for the transverse oscillatory modes, we may define $(n > 0)$:

$$\begin{cases} P^a_n \equiv \left(A^a_{-n} + A^a_n\right), \\ X^a_n \equiv \left(A^a_{-n} - A^a_n\right)\dfrac{1}{2in}. \end{cases} \tag{6.7}$$

We obtain the following canonical algebra

$$\{X^a_n, P^b_m\} = \delta^{ab}\delta_{m,n}, \qquad \text{with } n, m > 0. \tag{6.8}$$

7. The Canonical Transformation.

Let us define relative and center of mass variables for the string. The center-of-mass coordinates of the string are

$$\begin{cases} X^{\mu}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\sigma x^{\mu}(\sigma,\tau), \\ P^{\mu} = \dfrac{1}{2}\displaystyle\int_{-\pi}^{\pi} d\sigma P^{\mu}(\sigma,\tau), \end{cases} \tag{7.1}$$

where P^{μ} is the conserved generator of the space-time translations.
Let us introduce the following relative coordinates

$$\begin{cases} y^{\mu}(\sigma,\tau) = -x'^{\mu}(\sigma,\tau) = -y^{\mu}(-\sigma,\tau), \\ \mathcal{P}^{\mu}(\sigma,\tau) = \int_0^{\sigma} d\sigma' P^{\mu}(\sigma',\tau) - \dfrac{\sigma}{\pi}P^{\mu} = -\mathcal{P}^{\mu}(-\sigma,\tau) = \mathcal{P}^{\mu}(\sigma+2n\pi,\tau), \end{cases} \tag{7.2}$$

with the following properties

$$\begin{cases} \int_{-\pi}^{\pi} d\sigma y^{\mu}(\sigma,\tau) = \int_{-\pi}^{\pi} d\sigma \mathcal{P}^{\mu}(\sigma,\tau) = 0, \\ \mathcal{P}^{\mu}(0) = \mathcal{P}^{\mu}(\pm\pi) = 0 \rightarrow \int_{-\pi}^{\pi} d\sigma \mathcal{P}'^{\mu}(\sigma,\tau) = 0. \end{cases} \tag{7.3}$$

It may be checked that the coordinates (7.1) and (7.2) constitute a basis of canonical variables

$$\begin{cases} \{X^\mu, P^\nu\} = -\eta^{\mu\nu}, \\ \\ \{y^\mu(\sigma,\tau), \mathcal{P}^\nu(\sigma',\tau)\} = -\eta^{\mu\nu}\Delta_-(\sigma,\sigma'), \end{cases} \tag{7.4}$$

with all the other Poisson brackets vanishing.

The inverse formulae are

$$\begin{cases} x^\mu(\sigma,\tau) = X^\mu(\tau) + \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 y^\mu(\sigma_2,\tau) - \int_0^{\sigma} d\sigma_2 y^\mu(\sigma_2,\tau), \\ P^\mu(\sigma,\tau) = \dfrac{1}{\pi}P^\mu + \mathcal{P}'^\mu(\sigma,\tau). \end{cases} \tag{7.5}$$

Let us now look for a canonical transformation such that gauge and physical degrees of freedom are completely separated.

As for the physical sector, since we already got the transverse oscillators observables $\vec{A}_n(\tau)$, we now only need to find 6 observables for the center-of-mass of the string. Three of them are P^+, $\vec{P}$. Their conjugate observable variables, Z^-, $\vec{Z}$, are:

$$\begin{cases} Z^-(\tau) = X^-(\tau) - \dfrac{1}{2P^+}\int_{-\pi}^{\pi} d\sigma \left\{ \dfrac{x^+(\sigma,\tau)}{2}\left(\dfrac{\vec{A}_-^2(\sigma,\tau)}{2A_-^+(\sigma,\tau)} + \dfrac{\vec{A}_+^2(\sigma,\tau)}{2A_+^+(\sigma,\tau)}\right) + \right. \\ \qquad\qquad \left. - \dfrac{\mathcal{P}^+(\sigma,\tau)}{2N}\left(\dfrac{\vec{A}_-^2(\sigma,\tau)}{2A_-^+(\sigma,\tau)} - \dfrac{\vec{A}_+^2(\sigma,\tau)}{2A_+^+(\sigma,\tau)}\right)\right\}, \\ \vec{Z}(\tau) = \vec{X}^+(\tau) - \dfrac{1}{2P^+}\int_{-\pi}^{\pi} d\sigma \left[x^+(\sigma,\tau)\vec{P}(\sigma,\tau) - \vec{y}(\sigma,\tau)\mathcal{P}^+(\sigma,\tau)\right]. \end{cases} \tag{7.6}$$

The so defined quantities are effectively observables, since it can be checked that:

$$\{Z^-(\tau), \tilde{\chi}_\pm(\sigma,\tau)\} = \left\{\vec{Z}(\tau), \tilde{\chi}_\pm(\sigma,\tau)\right\} = 0. \tag{7.7}$$

Besides, the non vanishing Poisson brackets among these 6 observables are

$$\begin{cases} \{Z^-, P^+\} = -1, \\ \{Z^a, P^b\} = -\delta^{ab}, \qquad\qquad a,b = 1,2. \end{cases} \tag{7.8}$$

We can now define a canonical transformation from the variables $x^\mu(\sigma,\tau)$, $P^\mu(\sigma,\tau)$ to a new canonical base fitting to the multitemporal approach of the previous Section. This new base should be an appropriate point of departure toward the construction of Dirac brackets [12] associated to gauge-fixing constraints like those of the orthonormal gauge. This kind of canonical transformation for a system with first class constraints[16] generates new canonical variables divided into two sets. In one set, half of the canonical variables are functions of the first class constraints, hence vanishing on the manifold defined by the constraints in the phase-space, while the other half constitute a possible choice of the gauge degrees of freedom of the theory. In the second set we have those observables which have vanishing Poisson brackets with the chosen gauge degrees of freedom. The gauge sector of the new variables is composed by

$$\begin{cases} Y^-(\sigma,\tau) = \dfrac{1}{2N}\left(\tilde{\chi}_-(\sigma,\tau) - \tilde{\chi}_+(\sigma,\tau)\right), \\ \mathcal{P}^+(\sigma,\tau), \end{cases} \tag{7.9}$$

$$\begin{cases} x^+(\sigma,\tau), \\ \Pi^-(\sigma,\tau) = \dfrac{1}{2}\left(\tilde{\chi}_-(\sigma,\tau) + \tilde{\chi}_+(\sigma,\tau)\right). \end{cases} \tag{7.10}$$

If we separate $\Pi^-(\sigma,\tau)$ in is center-of-mass and relative part:

$$\Pi^-(\sigma,\tau) = \frac{\Xi^-_{tot}(\tau)}{\pi} + \Xi'^-_{rel}(\sigma,\tau), \tag{7.11}$$

where

$$\begin{cases} \Xi_{tot} = \dfrac{1}{2}\int_{-\pi}^{\pi} d\sigma \Pi^-(\sigma), \\ \Xi^-_{rel} = \int_0^{\sigma} d\sigma' \Pi^-(\sigma') - \dfrac{\sigma}{\pi}\Xi^-_{tot}, \end{cases} \tag{7.12}$$

we may replace $X^+(\sigma,\tau)$, $\Pi^-(\sigma,\tau)$ with the new gauge variables

$$\begin{cases} X^+(\tau), \\ \Xi^-_{tot}(\tau), \end{cases} \qquad \begin{cases} y^+(\sigma,\tau), \\ \Xi^-_{rel}(\sigma,\tau). \end{cases} \tag{7.13}$$

The constraints $\tilde{\chi}_\pm \approx 0$ are equivalent to $Y^-(\sigma,\tau) \approx 0$, $\Pi^-(\sigma,\tau) \approx 0$ (or $\Xi^-_{tot}(\tau) \approx 0$, $\Xi^-_{rel}(\sigma,\tau) \approx 0$).

The sector of the observables is composed by the $\vec{\alpha}_n$, P^+, Z^-, $\vec{P}$, $\vec{Z}$. It is only a matter of calculation to verify that the non-vanishing Poisson brackets are

$$\begin{array}{ll} \text{gauge sector} & \begin{cases} \{Y^-(\sigma,\tau), \mathcal{P}^+(\sigma',\tau)\} = -\Delta_-(\sigma,\sigma'), \\ \{x^+(\sigma,\tau), \Pi^-(\sigma',\tau)\} = -\Delta_+(\sigma,\sigma'), \\ \left\{X^+(\tau), \Xi^-_{tot}(\tau)\right\} = -1, \\ \left\{y^+(\sigma,\tau), \Xi^-_{rel}(\sigma',\tau)\right\} = -\Delta_-(\sigma,\sigma'), \end{cases} \\ \\ \text{physical sector} & \begin{cases} \left\{\alpha^a_n, \alpha^b_{-m}\right\} = -i\delta^{ab}\delta_{nm}, & a,b = 1,2, \\ \left\{Z^-, P^+\right\} = -1, \\ \left\{Z^a, P^b\right\} = \delta_{ab}, & a,b = 1,2. \end{cases} \end{array} \tag{7.14}$$

We notice that equations (5.9) connect the "times" $\tau_\pm(\sigma)$ to the observable P^+ and to the gauge variables $x^+(\sigma,\tau)$, $\mathcal{P}^+(\sigma,\tau)$, via $B^+_\pm(\sigma,\tau)$. As it is apparent from equation (7.14), we have separated into two diffferent sectors the gauge and physical degrees of freedom. This is in fact equivalent to the construction of Dirac brackets in an arbitrary gauge connected to the orthonormal one. This allows for instance to perform the canonical quantization procedure for the string, without gauge fixing.

With some calculation, it is possible to find the inverse canonical transformation, which is defined by the following expressions:

$$\begin{aligned}
&x^+(\sigma,\tau);\\
&P^-(\sigma,\tau) = \Pi^-(\sigma,\tau) + \frac{P^+(\sigma,\tau)}{2P^{+2}}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right) + \\
&\qquad + \frac{\pi N}{P^+}\sum_{n=1}^{\infty}\left\{\left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right] + \frac{A_-^+(\sigma,\tau)}{2P^+}\cdot\right.\right.\\
&\qquad\left.\cdot\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\tilde{U}_n^- + \left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[+i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right] + \right.\\
&\qquad\left.\left. + \frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\tilde{U}_{-n}^-\right\};\\
&x^-(\sigma,\tau) = Z^- + \frac{x^+(\sigma,\tau)}{2P^{+2}}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right) + \\
&\qquad + \frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma_1\int_0^{\sigma_1}d\sigma_2 Y^-(\sigma_2,\tau) - \int_0^{\sigma}d\sigma_2 Y^-(\sigma_2,\tau) + \\
&\qquad + \frac{i}{2P^+}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right] + \exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\tilde{U}_n^-(\tau) + \right.\\
&\qquad\left. - \left(\exp\left[-i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right] + \exp\left[i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]\right)\tilde{U}_{-n}^-(\tau)\right];\\
&P^+(\sigma,\tau) = \frac{P^+}{\pi} + \mathcal{P}'^+(\sigma,\tau); \qquad (7.15)\\
&\vec{x}(\sigma,\tau) = \vec{Z}(\tau) + \frac{\vec{P}}{P^+}x^+(\sigma,\tau) + \\
&\qquad + \frac{i}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{\omega_n}}\left[\left(\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right] + \exp[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_n + \\
&\qquad\left. - \left(\exp\left[-i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right] + \exp\left[i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_{-n}\right];\\
&\vec{P}(\sigma,\tau) = \frac{\vec{P}}{P^+}P^+(\sigma,\tau) + \sum_{n=1}^{\infty}\sqrt{\frac{\omega_n}{2}}\left[\left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right] + \right.\right.\\
&\qquad\left. + \frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_n +
\end{aligned}$$

$$+\left(\frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]+\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_{-n}\Bigg],$$

where $\tilde{U}_n^-$ is given by

$$\tilde{U}_n^- = \frac{1}{2}\sum_{n=-\infty}^{\infty}\vec{A}_{n-m}\cdot\vec{A}_m, \tag{7.16}$$

and $A_\pm^+$, $B_\pm^+$ are expressed in terms of P^+, $\mathcal{P}^+(\sigma,\tau)$, $x^+(\sigma,\tau)$ in equations (4.5), (4.6).

It is also possible to write the results of equations (7.15) in a more compact form through the use of the generalized DDF oscillators; we obtain:

$$\begin{aligned}
x^\mu(\sigma,\tau) =& Z^\mu(\tau)+\frac{P^\mu}{P^+}x^+(\sigma,\tau)+\\
&+\frac{i}{2\sqrt{\pi N}}\sum_{n\neq 0}\frac{A_n^\mu(\tau)}{n}\left(\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)+\\
&+\left(\eta^{\mu 0}+\eta^{\mu 3}\right)\Bigg\{\frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma_1\int_0^{\sigma_1}d\sigma_2 Y^-(\sigma_2,\tau)-\int_0^{\sigma}d\sigma_2 Y^-(\sigma_2,\tau)+\\
&-\frac{i}{2P^+}\sum_{n\neq 0}\frac{\tilde{L}_n(\tau)}{n}\left(\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\Bigg\},
\end{aligned} \tag{7.17}$$

where we have defined $Z^+ \equiv X^+$, and

$$\begin{aligned}
P^\mu(\sigma,\tau) =& \frac{P^\mu}{P^+}P^+(\sigma,\tau)+\sqrt{\pi N}\sum_{n\neq 0}A_n^\mu(\tau)\left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\right.\\
&\left.+\frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)+\\
&+\left(\eta^{\mu 0}+\eta^{\mu 3}\right)\Bigg[\Pi^-(\sigma,\tau)-\frac{P^+(\sigma,\tau)}{P^+}\Xi_{tot}^-(\tau)-\frac{\pi N}{P^+}\sum_{n\neq 0}\tilde{L}_n(\tau)\cdot\\
&\cdot\left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\Bigg],
\end{aligned} \tag{7.18}$$

where P^- is given by

$$\begin{aligned}
P^- =& \frac{1}{2P^+}\left(\vec{P}^2+\sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right)+\Xi_{tot-}(\tau)=\\
=& \frac{1}{2P^+}\left(\vec{P}^2+2\pi N\sum_{n=1}^{\infty}\vec{A}_n\cdot\vec{A}_{-n}\right)+\Xi_{tot-}(\tau).
\end{aligned} \tag{7.19}$$

8. Conclusions.

If we perform the canonical quantization in one chart, we recover the critical dimension $D = 26$ of the non-covariant approach. However in this way one looses the contribution of the non-overlapping part of the second chart, that is of the classical longitudinal modes of Patrascioiu. Moreover one also looses the contribution of the other exceptional charts, which contain only longitudinal modes [1]. This fact may explain why in the covariant quantization approach there are solutions of the no-ghost theorem with $D < 26$, but with the extra longitudinal modes of Brower (or the Liouville modesin the Polyakov path integral approach).

However the main motivation for this work lies in the strategy to solve the part of the dynamics connected to the 1^{st}-class constraints without fixing the gauge, and to find the observables (in a gauge theory) or the independent Cauchy data (in a generally covariant theory). This strategy is based on the search of symplectic bases in phase-space, adapted to the constraints, and which are the only relevant ones for a parametrization of the constraints presymolectic manyfold. Such bases generally exist only locally and this is connected to the possible abelanizations of the original 1^{st}-class constraints. In this way one should build an atlas for the constraints manyfold, and the quantization should be done in a way consistent with this atlas.

Many problems are still open: 1) find the Wigner covariant form of the constraints and the atlas; 2) study the classical action-angle variables and see whether there is one set such that the Poincaré Casimirs in D=4 are independent from the angle variables; 3) study the connection of our observables (independent Cauchy data for the string) with the Pohlmeyer-Rehren constants of motion[17]; 4) find a generalization of the concept of Green function in connection with the solutions (5.12).

This strategy may then be applied to other systems with 1^{st}-class constraints like Yang-Mills fields, Chern-Simons theory, 2+1 gravity and general relativity in the Ashtekar formulation[18]. New problems arise already at the classical level, like the relationship between global topological observables, and the one associated to the local adapted symplectic charts, describing the local independent physical degrees of freedom (when they exist).

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