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Introduction to the Bethe Ansatz Solvable Models

Lectures given in 1998-1999 by

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Lecture notes edited by

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Preface

In memory of Anatoli "Tolia" G. Izergin

It is still unbelievable for me to realize that Prof. Anatoli G. Izergin untimely died last December.

I met Tolia for the first time in Leningrad on October 1990. We immediately became friends.

A collaboration started by which I learned a lot on quantum integrable systems. We worked together for many years with exchange of visits. He visited many times our Department giving also lectures for our Ph.D. courses. My graduated students, young researchers, colleagues and myself enjoyed a lot attending his beautiful lectures. Last year, I invited him to give again a short course on integrable systems and Tolia gave 30 hours of lectures on this subject.

I would like to thank Filippo Colomo and Andrei Pronko who collected notes of these lectures and carefully edited them. I am sure that this will be very useful for all those wishing to start studying this exciting subject.

I hope that these notes can also represent a small monument to the memory of Tolia Izergin.

Florence, November 2000

Valerio Tognetti

Contents

1 The classical nonlinear Schrödinger equation.

The classical nonlinear Schrödinger (NS) equation is the following system of equations for the nonrelativistic Bose fields $\psi(x,t)$, $\psi^+(x,t)$, (in this Section the cross denotes complex conjugation of classical fields) in two space-time dimensions

$$\begin{aligned} i\partial_t\psi &= -\partial_x^2\psi + 2c\psi^+\psi\psi, \\ -i\partial_t\psi^+ &= -\partial_x^2\psi^+ + 2c\psi^+\psi^+\psi \end{aligned} \tag{1.1}$$

(the "ordering" of ψ, ψ^+ is not essential, of course, in the classical case). It is a Hamiltonian system. The Hamiltonian H is

$$H = \int dx(\partial_x\psi^+\partial_x\psi + c\psi^+\psi^+\psi\psi) \ , \tag{1.2}$$

the momentum P and the charge Q are given as

$$\begin{aligned} P &= \frac{i}{2}\int dx(\partial_x\psi^+\psi - \psi^+\partial_x\psi) = i\int dx(\partial_x\psi^+)\psi = -i\int dx\psi^+\partial_x\psi \ , \\ Q &= \int dx\psi^+\psi \end{aligned} \tag{1.3}$$

(the time argument t is fixed to be the same for all fields). At this level, boundary conditions for ψ, ψ^+ are not fixed, but it is assumed that integration by parts, as in the expressions for P, can be done.

Equations of motion (1.1) can be obtained by computing the Poisson brackets with the Hamiltonian. For functionals A, B of fields $\psi(x,t)$, $\psi^+(x,t)$ (at fixed time t) the Poisson brackets are defined as

$$\{A,B\} = -i\int dz\Big(\frac{\delta A}{\delta\psi(z)}\cdot\frac{\delta B}{\delta\psi^+(z)} - \frac{\delta A}{\delta\psi^+(z)}\cdot\frac{\delta B}{\delta\psi(z)}\Big) \ ; \tag{1.4}$$

the definition of the functional derivative is as usual: one represents the first variation of the functional in the form

$$\delta A = \int dx\Big(\frac{\delta A}{\delta\psi(x)}\delta\psi(x) + \frac{\delta A}{\delta\psi^+(x)}\delta\psi^+(x)\Big) \ ,$$

the coefficients of $\delta\psi$, $\delta\psi^+$ being the corresponding functional derivatives.

The canonical Poisson brackets are

$$\{\psi(x,t),\psi^+(y,t)\} = -i\delta(x-y), \qquad \{\psi^\pm(x,t),\psi^\pm(y,t)\} = 0, \qquad \psi^- \equiv \psi. \tag{1.5}$$

Equations

$$\partial_t\psi^\pm = \{\psi^\pm\, H\} \tag{1.6}$$

give just the equations of motion (1.1). It is easy to calculate also that

$$\{\psi^\pm\, P\} = -\partial_x\psi^\pm \ , \qquad \{\psi^\pm\, Q\} = \pm i\psi^\pm \ ,$$

and to show that Q, P are "commuting" integrals of motion,

$$\{H, P\} = \{H, Q\} = 0, \qquad \{P, Q\} = 0. \tag{1.7}$$

We will see later that there exist, in fact, infinitely many commuting integrals of motion for the nonlinear Schrödinger equation.

2 The quantization. The quantum NS model as the one-dimensional nonrelativistic Bose gas.

In the quantum case, the fields $\psi(x,t)$, $\psi^+(x,t)$ are operators, and the cross stands for the hermitean conjugation. The Poisson brackets (1.5), as usual in canonical quantization, are changed for the commutators ($\psi^- \equiv \psi$)

$$\Big[\psi(x,t), \psi^+(y,t)\Big] = \delta(x-y)\ , \qquad \Big[\psi^\pm(x,t), \psi^\pm(y,t)\Big] = 0 \tag{2.1}$$

(remind that under quantization the Poisson bracket between the coordinate q and the momentum p is changed for the commutator of the corresponding operators according to the rule $\{q,p\} = 1 \to [q,p] = i\hbar$; we put $\hbar = 1$). The Hamiltonian H, the momentum P, and the operator Q of the number of particles are given by the same expressions (1.2)–(1.3) as in the classical case (however, the normal ordering is now essential!). The equations of motion for operators ψ, ψ^+ are

$$-i\partial_t \psi^\pm = [H, \psi^\pm]\ , \tag{2.2}$$

which gives just the NS equation (1.1) for quantum fields. Also, we have the relations

$$i\partial_x \psi^\pm = [P, \psi^\pm]\ ,$$

and

$$[Q, \psi^\pm] = \pm\psi^\pm\ .$$

Operators P and Q commute with the Hamiltonian and between themselves,

$$[H, P] = [H, Q] = [P, Q] = 0\ . \tag{2.3}$$

We shall see later that also in the quantum case there exist infinitely many integrals of motion (conservation laws).

The space where operators ψ, ψ^+ act is the Fock space. The Fock vacuum $|0\rangle$ is defined as usual by the requirement

$$\begin{gathered} \psi(x)|0\rangle = 0, \qquad \forall x\ ; \\ \langle 0| \equiv (|0\rangle)^+\ , \qquad \langle 0|\psi^+(x) = 0\ , \qquad \langle 0|0\rangle \equiv 1\ . \end{gathered} \tag{2.4}$$

It is an eigenstate of H, P, Q with zero eigenvalues,

$$H|0\rangle = 0 \ , \quad P|0\rangle = 0 \ , \quad Q|0\rangle = 0 \ . \tag{2.5}$$

The other states in the Fock space are linear combinations of the states obtained by acting with operators ψ^+ on the Fock vacuum:

$$|\Psi_{_N}\rangle = \int d^N z \, \chi_{_N}(\{z\}) \, \psi^+(z_1)\psi^+(z_2)\dots\psi^+(z_{_N})|0\rangle \ , \qquad N = 0, 1, 2, \dots \tag{2.6}$$

Here, $\chi_{_N}(\{z\}) \equiv \chi_{_N}(z_1, \dots, z_{_N})$ is a wave function; $d^N z \equiv dz_1 dz_2 \dots dz_{_N}$. Due to the commutativity of operators $\psi^+(z_j)$ (2.1), we can restrict ourselves to the wave functions which are symmetrical under permutations of z_j,

$$\chi_{_N}(z_{S_1}, \dots, z_{S_{_N}}) = \chi_{_N}(z_1, \dots, z_{_N}), \quad \chi_{_N}(\{Sz\}) = \chi_{_N}(\{z\}), \tag{2.7}$$

where S is a permutation, $S : (1, 2, \dots, N) \to (S_1, S_2, \dots, S_{_N})$. In other words, one has Bose statistics.

Using commutation relations (2.1), it is not difficult to calculate the action of operators Q, P and H on the state (2.6), expressing it in terms of operators acting on the wave function. It is evident that

$$Q|\Psi_{_N}\rangle = N|\Psi_{_N}\rangle \ , \tag{2.8}$$

i.e., $|\Psi_{_N}\rangle$ is an "N-particle" state. Moreover, integration by parts gives:

$$\begin{aligned} P|\Psi_{_N}\rangle &= \left(i \int dx \partial_x \psi^+ \cdot \psi\right) |\Psi_{_N}\rangle \\ &= \int d^N z \, (\mathcal{P}_{_N} \chi_{_N}) \, \psi^+(z_1) \dots \psi^+(z_{_N})|0\rangle, \end{aligned}$$

where

$$\mathcal{P}_{_N} = -i \sum_{j=1}^{N} \partial_j, \qquad \partial_j \equiv \frac{\partial}{\partial z_j}, \tag{2.9}$$

and

$$H|\Psi_{_N}\rangle = \int d^N z \, (\mathcal{H}_{_N} \chi_{_N}) \, \psi^+(z_1) \dots \psi^+(z_{_N})|0\rangle,$$

with

$$\mathcal{H}_{_N} = -\sum_{j=1}^{N} \partial_j^2 + 2c \sum_{1 \le j < k \le N} \delta(z_j - z_k). \tag{2.10}$$

It follows from the expressions for $\mathcal{P}_{_N}$ and $\mathcal{H}_{_N}$ that in the sector containing N particles, our model describes a "gas" of N nonrelativistic bosons on a line (the mass of the boson is equal to $1/2$) interacting via the pair potentials, $2c\delta(z_j - z_k)$, of "zero radia". The fields ψ, ψ^+ are second quantized fields, describing the gas with any number of particles ($N = 0, 1, 2, \dots$).

Due to the commutativity of operators H, P, and Q, one can find their simultaneous eigenstates:

$$H|\Psi_N\rangle = E_N|\Psi_N\rangle \ , \qquad P|\Psi_N\rangle = P_N|\Psi_N\rangle \ , \qquad Q|\Psi_N\rangle = N|\Psi_N\rangle. \tag{2.11}$$

Here, E_N, P_N, and $Q_N \equiv N$ are the eigenvalues. Solving this problem is equivalent to solving the corresponding quantum mechanical problem for N particles:

$$\mathcal{H}_N\chi_N = E_N\chi_N \ , \qquad \mathcal{P}_N\chi_N = P_N\chi_N \ . \tag{2.12}$$

It is evident from (2.10) that if the coupling constant c is positive, $c > 0$, the potential is repulsive; the case $c < 0$ corresponds to the attraction between the particles.

3 A δ-potential as a boundary condition.

Due to Bose symmetry (2.7), it is sufficient to construct an eigenfunction χ_N (2.12) only in the "fundamental domain" F of coordinates $z_1, \ldots, z_N$,

$$F: \ z_1 < z_2 < \ldots < z_N, \tag{3.1}$$

continuing then the solution to all the values of z_j's, from symmetry. It follows from the form of the Hamiltonian $\mathcal{H}_N$ that an eigenfunction χ_N in equation $\mathcal{H}_N\chi_N = E_N\chi_N$ should be a continuous function of the coordinates z_j, and that there should be a discontinuity of the first derivatives to "compensate" the δ-function terms entering the potential.

In the fundamental domain F, the wave function satisfies the free Schrödinger equation,

$$\mathcal{H}_N^{(0)}\chi_N = E_N\chi_N, \quad \mathcal{H}_N^{(0)} = -\sum_{j=1}^{N}\partial_j^2. \tag{3.2}$$

The potential can be taken into account by imposing boundary conditions on the border of F, namely,

$$(\partial_{j+1} - \partial_j - c)\chi_N\Big|_{z_{j+1}=z_j+0} = 0, \quad j = 1, 2, ..., N-1 \tag{3.3}$$

(other z_k's are fixed in F). Here, the notation $z_{j+1} = z_j + 0$ means $z_{j+1} = z_j + \epsilon$, ($\epsilon > 0$, $\epsilon \to 0$).

Let us derive this boundary condition. Introduce variables z and Z,

$$z \equiv z_{j+1} - z_j, \qquad Z = \frac{1}{2}(z_{j+1} + z_j), \qquad \partial_z = \frac{1}{2}(\partial_{j+1} - \partial_j), \qquad \partial_Z = \partial_{j+1} + \partial_j \ ,$$

$$dz_j dz_{j+1} = dzdZ, \qquad \partial_j^2 + \partial_{j+1}^2 = \frac{1}{2}\partial_Z^2 + 2\partial_z^2.$$

Integrate now Schrödinger equation (2.12) over the small vicinity of point $z_{j+1} = z_j$, i.e., over the region $-\epsilon < z < \epsilon$ (all the other z's are in F):

$$\int\limits_{|z|<\epsilon} (\mathcal{H}_N \chi_N) dz = E_N \int\limits_{|z|<\epsilon} \chi_N dz.$$

In the limit $\epsilon \to 0$, one has (note that $\chi_N(\epsilon) = \chi_N(-\epsilon)$ due to the Bose symmetry)

$$-2\partial_z \chi_N(z)\Big|_{-\epsilon}^{+\epsilon} + 2c\chi_N(0) = 0,$$

which is just the boundary condition (3.3).

4 Bethe eigenfunctions. The coordinate Bethe Ansatz.

Constructing eigenstates of the Hamiltonian H (2.11) is equivalent to finding eigenfunctions (2.12) of the quantum mechanical Hamiltonians $\mathcal{H}_N$ (2.10). In turn, this problem is equivalent to finding solutions of the free Schrödinger equation (3.2) in the fundamental domain F (3.1) satisfying boundary conditions (3.3). To define the operator $\mathcal{H}_N$ properly, one should also impose the corresponding boundary conditions at the boundary of the domain of the coordinates $z_1, \ldots, z_N$ where the operator $\mathcal{H}_N$ acts. E.g., one can consider the problem in $\mathbf{R}^N$, $(-\infty < z_j < +\infty,\ j = 1, 2, \ldots, N)$ demanding that wave functions should not increase as $z_j \to \infty$, or impose the periodic boundary conditions. This is made in the following Section. Here we restrict ourselves to constructing formal solutions of the equation $\mathcal{H}_N \chi_N = E_N \chi_N$, following M.Gaudin.

Let us consider function $\chi_N^{(0)}$ (the "Slater determinant")

$$\chi_N^{(0)}(\{\lambda\}|\{z\}) \equiv \det(\exp[iz_j\lambda_k]) \tag{4.1}$$

In the right-hand side there is the determinant of a $N \times N$-dimensional matrix, the (j,k)-th matrix element being equal to $\exp[iz_j\lambda_k]$. Parameters λ_j $(j = 1, 2, \ldots, N)$ (complex numbers) can be regarded as "momenta" parametrizing the function $\chi_N^{(0)}$. Notations $\{\lambda\} \equiv \lambda_1, \ldots \lambda_N$ and $\{z\} \equiv z_1, \ldots, z_N$ are used.

This function satisfies the free Schrödinger equation in F (and also in the whole $\mathbf{R}^N$ space):

$$\mathcal{H}_N^{(0)} \chi_N^{(0)} = E_N \chi_N^{(0)}, \qquad E_N = \sum_{j=1}^N \lambda_j^2 \tag{4.2}$$

But it is antisymmetric under permutations of the z's, and the boundary conditions (3.3) are not satisfied (the function $\chi_N^{(0)}$ and all its derivatives are continuous functions of z).

Consider now the function $\chi_N^{(1)}$:

$$\chi_N^{(1)}(\{\lambda\}|\{z\}) = \left(\prod_{N\geq j>k\geq 1} (\partial_j - \partial_k + c) \right) \chi_N^{(0)}(\{\lambda\}|\{z\}). \tag{4.3}$$

This function also satisfies the free Schrödinger equation with the same eigenvalue,

$$\mathcal{H}_N^{(0)} \chi_N^{(1)} = E_N \chi_N^{(1)}, \tag{4.4}$$

and the boundary conditions (3.3) for it are fulfilled. To check this, let us apply to the function $\chi_N^{(1)}$ the differential operator entering the left hand side of (3.3). The result can be written as

$$(\partial_{j+1} - \partial_j - c)\chi_N^{(1)} = \left[(\partial_{j+1} - \partial_j)^2 - c^2 \right] \widetilde{\chi}_N \ , \tag{4.5}$$

with

$$\widetilde{\chi}_N = \left[\prod_{N\geq l>m\geq 1}{}' (\partial_l - \partial_m + c) \right] \chi_N^{(0)},$$

where the prime means that the operator $(\partial_{j+1} - \partial_j + c)$ is "extracted" from the product. Since integers $j+1$ and j are always both larger or both smaller than the other integers k ($k = 1, ..., N$, $k \neq j, j+1$), it is evident that the product $\prod'$ is symmetrical under the permutation $z_j \leftrightarrow z_{j+1}$. Due to the antisymmetricity of $\chi_N^{(0)}$, one concludes that function $\widetilde{\chi}_N$ is antisymmetric under $z_j \leftrightarrow z_{j+1}$ (as a function in the whole $\mathbf{R}^N$ space). It is also a continuous function in $\mathbf{R}^N$ (with all its derivatives). The right hand side in (4.5) is also an antisymmetric continuous function of z_j, z_{j+1}, and hence it is equal to zero at $z_{j+1} = z_j$. The boundary conditions (3.3) are thus fulfilled.

Thus, the solution χ_N of our Schrödinger equation coincides with $\chi_N^{(1)}$ in the fundamental domain, up to a "constant" factor which can depend on parameters λ_k and the coupling constant c but should be independent of the coordinates z_k,

$$\chi_N(\{\lambda\}|\{z\}) = \text{const}\, \chi_N^{(1)}(\{\lambda\}|\{z\}), \qquad \{z\} \in F. \tag{4.6}$$

To get the solution χ_N in the whole $\mathbf{R}^N$ space, one should "continue" it using Bose symmetry. Let us do this.

Using the explicit representation of the determinant, one can write for $\chi_N^{(0)}$:

$$\chi_N^{(0)} = \sum_P (-1)^{[P]} \exp\left[i \sum_{j=1}^N z_j \lambda_{P_j} \right],$$

where the sum is over all the permutations of N integers, $P : (1, 2, ..., N) \to (P_1, P_2, ..., P_N)$; $[P]$ is the parity of a permutation. Thus, in F one has

$$\chi_N = \text{Const} \sum_P (-1)^{[P]} \exp\left[i \sum_{j=1}^N z_j \lambda_{P_j} \right] \prod_{j>k} (\lambda_{P_j} - \lambda_{P_k} - ic), \qquad \{z\} \in F. \tag{4.7}$$

Using Bose symmetry to continue χ_N outside F, one gets the wave function in the whole $\mathbf{R}^N$ space:

$$\chi_N = \text{const} \sum_P (-1)^{[P]} \exp\left[i \sum_{j=1}^N z_j \lambda_{P_j} \right] \prod_{j>k} \left(\lambda_{P_j} - \lambda_{P_k} - ic\epsilon(z_j - z_k) \right), \quad \{z\} \in \mathbf{R}^N, \tag{4.8}$$

where $\epsilon(z)$ is the sign function, $\epsilon(z) = +1$, if $z > 0$ and $\epsilon(z) = -1$, if $z < 0$.

Let us prove equation (4.8). In the domain F, all the sign functions in (4.8) are equal to $+1$, and one reproduces (4.7). To show that (4.8) gives the correct expression for the wave function at any $\{z\}$, it is sufficient to check the symmetry,

$$\chi_N(\{\lambda\}|\{Qz\}) = \chi_N(\{\lambda\}|\{z\}), \tag{4.9}$$

where the set of the coordinates $\{Qz\} = z_{Q_1} \dots z_{Q_N}$ is obtained from the set $\{z\}$ by some permutation Q. Let us show that it is indeed the case. By definition, up to a constant factor,

$$\chi_N(\{\lambda\}|\{Qz\}) \cong \sum_P (-1)^{[P]} \exp\left[i \sum_{j=1}^N z_{Q_j} \lambda_{P_j} \right] \prod_{j>k} \left(\lambda_{P_j} - \lambda_{P_k} - ic\epsilon(z_{Q_j} - z_{Q_k}) \right)$$

Now make a shift of all the permutations P in the sum by the permutation Q, putting $P \equiv QP'$, and go from the summation over P to the summation over P'. Using the obvious properties,

$$(-1)^{[P]} = (-1)^{[P']+[Q]},$$

$$\sum_{j=1}^N z_{Q_j} \lambda_{P_j} = \sum_{j=1}^N z_j \lambda_{P'_j},$$

$$\prod_{j>k} \left(\lambda_{P_j} - \lambda_{P_k} - ic\epsilon(z_{Q_j} - z_{Q_k}) \right) = (-1)^{[Q]} \prod_{j>k} \left(\lambda_{P'_j} - \lambda_{P'_k} - ic\epsilon(z_j - z_k) \right),$$

one easily proves (4.9). Function χ_N (4.8) is a continuous function of z in $\mathbf{R}^N$; it has the discontinuities of the first derivatives corresponding to the boundary conditions (3.3) whenever two coordinates coincide.

So we established that for any set $\{\lambda\} = \lambda_1, \dots, \lambda_N$ $(N = 0, 1, 2, \dots)$ the function χ_N (4.8) is a "formal" solution of the Schrödinger equation; it is easily seen that this function is also an eigenfunction of the momentum operator $\mathcal{P}_N$:

$$\begin{aligned} &\mathcal{H}_N \chi_N = E_N \chi_N, \quad E_N = \sum_{j=1}^N \epsilon_0(\lambda_j), \quad \epsilon_0(\lambda_j) \equiv \lambda_j^2, \\ &\mathcal{P}_N \chi_N = P_N \chi_N, \quad P_N = \sum_{j=1}^N p_0(\lambda_j), \quad p_0(\lambda_j) \equiv \lambda_j. \end{aligned} \tag{4.10}$$

This function is called the Bethe eigenfunction. The corresponding eigenstate can be regarded as consisting of N particles with momenta $p_0(\lambda_j)$ and energies $\epsilon_0(\lambda_j)$.

Choosing the constant in (4.6), (4.8) to be equal to

$$\frac{(-1)^{\frac{N(N-1)}{2}}}{\sqrt{N!}} \left(\prod_{j>k} \left[(\lambda_j - \lambda_k)^2 + c^2 \right] \right)^{-1/2},$$

one can write the Bethe eigenfunction as

$$\begin{aligned} \chi_N(\{\lambda\}|\{z\}) = \frac{1}{\sqrt{N!}} \prod_{j>k} \epsilon(z_j - z_k) \cdot \\ \cdot \sum_P (-1)^{[P]} \exp\left[i \sum_{j=1}^N z_j p_0(\lambda_{P_j}) \right] \exp\left[\frac{i}{2} \sum_{j>k} \epsilon(z_j - z_k)\, \theta(\lambda_{P_j} - \lambda_{P_k}) \right], \end{aligned} \tag{4.11}$$

where the "scattering phase" $\theta(\lambda)$ is defined as

$$\theta(\lambda) = i \log\left(\frac{ic + \lambda}{ic - \lambda} \right) \tag{4.12}$$

and the branch of the logarithm is chosen so that $\theta(\lambda)$ is a continuous antisymmetric function if λ is real ($\theta(\lambda) = 2\arctan(\lambda/c)$, $\operatorname{Im}\lambda = 0$); remind also that $p_0(\lambda) = \lambda$ in the model under consideration.

The form (4.11) of the wave function is quite typical of models solvable by means of the Bethe Ansatz. In different models of this kind, the "bare momentum" functions $p_0(\lambda)$ are different. The form of the "scattering phase" $\theta(\lambda)$ is also different for different models; in fact, it will be seen later that the function $\theta(\lambda)$ is defined by the R-matrix of the model, being the same for the models with the same R-matrix.

It is not difficult to show that function χ_N (4.11) is antisymmetric under permutations of λ,

$$\chi_N(\{Q\lambda\}, \{z\}) = (-1)^{[Q]} \chi_N(\{\lambda\}, \{z\}).$$

It is also a continuous function of λ. So it is equal to zero if two of the "momenta" coincide. This is the origin of the "Pauli principle" for interacting bosons in one space dimension: all the momenta should be different, $\lambda_j \neq \lambda_k$ ($j \neq k$).

The eigenstate $|\Psi_N(\{\lambda\})\rangle$ of the quantum Hamiltonian H (1.2) corresponding to the wave function constructed is

$$|\Psi_N(\{\lambda\})\rangle = \int d^N z \chi_N(|\{\lambda\}|\{z\}) \psi^+(z_1)\psi^+(z_2)\dots\psi^+(z_N)|0\rangle, \tag{4.13}$$

and thus

$$H|\Psi_N(\{\lambda\})\rangle = E_N(\{\lambda\})|\Psi_N(\{\lambda\})\rangle,$$

$$P|\Psi_N(\{\lambda\})\rangle = P_N(\{\lambda\})|\Psi_N(\{\lambda\})\rangle,$$

$$Q|\Psi_N(\{\lambda\})\rangle = N|\Psi_N(\{\lambda\})\rangle.$$

It is parametrized by the momenta $\lambda_1, \dots, \lambda_N$ of the N particles forming the state.

5 Particles on the whole axis. The spectrum.

If particles are allowed to move along the whole x-axis, $-\infty < z_j < +\infty$, then the natural boundary condition is that the wave function should remain finite for $-\infty < z_j < +\infty$. This requirement defines the possible values of momenta λ_j, i.e., the spectrum of the Hamiltonian $E_N = \sum_{j=1}^{N} \lambda_j^2$.

In the case of one particle ($N = 1$) the wave function is just a plane wave,

$$\chi_1(\lambda|z) = \exp\{i\lambda z\}. \tag{5.1}$$

If $\mathrm{Im}\,\lambda > 0$ ($\mathrm{Im}\,\lambda < 0$), then it increases as $z \to -\infty$ ($z \to +\infty$). Thus λ should be real, $-\infty < \lambda < +\infty$, and one has the continuous spectra of the momentum and of the Hamiltonian,

$$-\infty < P_1 = \lambda < +\infty, \quad 0 \le E_1 = \lambda^2 < +\infty.$$

The spectrum at $E_1 > 0$ is two-fold degenerate (λ and $-\lambda$ correspond to the same energy).

Consider now the wave function of two particles ($N = 2$). Omitting an ininfluent constant factor, one rewrites (4.8) as

$$\begin{aligned}
&\chi_2(\lambda_1\lambda_2|z_1z_2) = \\
&= e^{i(\lambda_1 z_1 + \lambda_2 z_2)}(\lambda_2 - \lambda_1 - ic\epsilon(z_2 - z_1)) \; - \; e^{i(\lambda_2 z_1 + \lambda_1 z_2)}(\lambda_1 - \lambda_2 - ic\epsilon(z_2 - z_1)) \\
&= e^{i(\lambda_1+\lambda_2)Z}\left[e^{i\frac{\lambda_2-\lambda_1}{2}z}(\lambda_2 - \lambda_1 - ic\epsilon(z)) + e^{-i\frac{\lambda_2-\lambda_1}{2}z}(\lambda_2 - \lambda_1 + ic\epsilon(z))\right] .
\end{aligned} \tag{5.2}$$

Here $Z \equiv \frac{1}{2}(z_1+z_2)$ and $z = (z_2-z_1)$ are the centre of mass and relative coordinates, respectively.

If λ_1 and λ_2 are both real, $\mathrm{Im}\,\lambda_1 = \mathrm{Im}\,\lambda_2 = 0$, the wave function is bounded, and there is a continuous spectrum ("plane waves" in the asymptotics),

$$-\infty < P_2 = \lambda_1 + \lambda_2 < +\infty, \quad 0 \le E_2 = \lambda_1^2 + \lambda_2^2 < +\infty, \quad \mathrm{Im}\lambda_1 = \mathrm{Im}\lambda_2 = 0.$$

Could there be bound states (complex λ's)? The answer is "yes" <u>if $c < 0$</u>, and "no" <u>if $c > 0$</u>.

Notice that the total momentum should be real, i.e., $\mathrm{Im}(\lambda_1 + \lambda_2) = 0$, otherwise χ_2 increases in the variable Z. So let

$$\lambda_1 \equiv \lambda + i\kappa, \quad \lambda_2 \equiv \lambda - i\kappa, \quad \mathrm{Im}\,\lambda = 0, \quad \mathrm{Re}\,\kappa > 0$$

(but κ is not assumed to be real at the moment!). The requirement $\mathrm{Re}\,\kappa > 0$ is just a matter of convenience, being related only to a possible relabeling of λ's. The wave function becomes

$$\chi_2 = e^{2i\lambda Z}\left[e^{\kappa z}\Big(-2i\kappa - ic\epsilon(z)\Big) + e^{-\kappa z}\Big(-2i\kappa + ic\epsilon(z)\Big)\right].$$

If $z > 0$ ($z < 0$), then $e^{\kappa z}$ ($e^{-\kappa z}$) is increasing as $z \to +\infty$ ($z \to -\infty$). To kill the increasing terms, one should require that the coefficient of the increasing exponential equals zero in the corresponding region, $z > 0$ or $z < 0$. This means that κ must be real, and $\kappa = -c/2$, which is possible only in the repulsive case $c < 0$ (remind that $\kappa > 0$ by the assumption). The corresponding wave function is

$$\chi_2 = 2i|c|e^{2i\lambda Z}e^{-\frac{|c|}{2}|z|} \qquad (c < 0), \tag{5.3}$$

and describes the bound state of two particles (the "2-string") with $\lambda_{1,2} = \lambda \pm i|c|/2$, $\mathrm{Im}\,\lambda = 0$.

The momentum and the energy of the state are

$$P_2 = \lambda_1 + \lambda_2 = 2\lambda, \qquad E_2 = \lambda_1^2 + \lambda_2^2 = \frac{P_2^2}{2} - \frac{|c|^2}{2} \qquad (c < 0).$$

The first term in the expression for the energy corresponds to the centre of mass energy, and the second term, $|c|^2/2$, gives the binding energy.

In the case of general N as well one should distinguish between the attractive and repulsive potentials, the consideration being quite similar to the case $N = 2$.

If $c > 0$ (repulsion), the corresponding λ's entering χ_N, should be all real, $\mathrm{Im}\lambda_j = 0$, and one has

$$P_N = \sum_{j=1}^{N} \lambda_j, \qquad E_N = \sum_{j=1}^{N} \lambda_j^2 \qquad \text{with } -\infty < \lambda_j < +\infty\ .$$

Thus, for positive values of the coupling constant, the spectrum consists of the elementary particles only.

If $c < 0$ (attraction), besides real λ's (corresponding to the unbounded elementary particles), there exist bound states of n ($n \le N$) particles (the "n-strings"). The (complex) momenta of individual particles in the n-string are

$$\lambda_l^{(n)} = \lambda + \tfrac{i|c|}{2}(n - 2l - 1), \quad l = 0, 1, \ldots, n-1, \quad \mathrm{Im}\,\lambda = 0, \quad -\infty < \lambda < +\infty. \tag{5.4}$$

The momentum and the energy of the string are readily calculated to be

$$\begin{aligned} P_n &= \sum_{l=0}^{n-1} \lambda_l^{(n)} = n\lambda, \\ E_n &= \sum_{l=0}^{n-1} \left(\lambda_l^{(n)}\right)^2 = \frac{(P_n^2)}{n} - \frac{|c|^2}{12}(n-1)n(n+1). \end{aligned} \tag{5.5}$$

Again, the first term in the expression for the energy is the c.m. energy, and the second term is the binding energy of the n-string, this energy being proportional to n^3 for large n. Thus for a negative value of the coupling constant c, the spectrum at a given N is formed by the elementary particles with real momenta λ_j ($j = 1, \ldots, N_1$), and by their bound states (strings). If N_n is the number of n-strings, then $N = N_1 + 2N_2 + \ldots = \sum_n nN_n$.

6 Particles on the whole axis. The S-matrix.

a) *The two-particle S-matrix*

Consider again the two-particle wave function (5.2) for the "elementary particles" ($\lambda_{1,2}$ are real). Let us choose the momenta so that $\lambda_1 > \lambda_2$. Consider the domain F : $z_1 < z_2$. In this domain, the two terms in the second line of (5.2) correspond to the two pictures in Figs. 6.1a,b (the profiles represent the "wave packets" concentrated in the vicinities of the corresponding positions). It is evident that in the remote past, $t \to -\infty$, only in the situation corresponding to Fig. 6.1a the particles do not interact. So the first term describes in F the "incoming" wave. Similarly, Fig. 6.1b (the second term) describes in F the "outgoing" wave (particles do not interact in the remote future, $t \to +\infty$). The two-particle S-matrix is given as the ratio,

$$S(\lambda_1, \lambda_2) = \frac{\text{amplitude of outgoing wave}}{\text{amplitude of ingoing wave}} = \frac{\lambda_1 - \lambda_2 - ic}{\lambda_1 - \lambda_2 + ic} = -\exp\{i\theta(\lambda_1 - \lambda_2)\}. \tag{6.1}$$

Thus, the function $\theta(\lambda)$ defined in (4.12) is, in essential, the scattering phase. The same result is obtained, of course, for the domain $z_1 > z_2$ ($\lambda_1 > \lambda_2$), due to the Bose symmetry (see Figs. 6.2a,b).

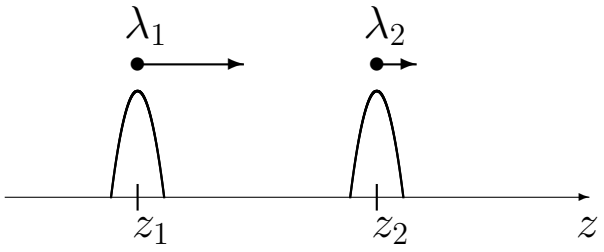

Fig. 6.1a: Incoming wave, $z_1 < z_2$.

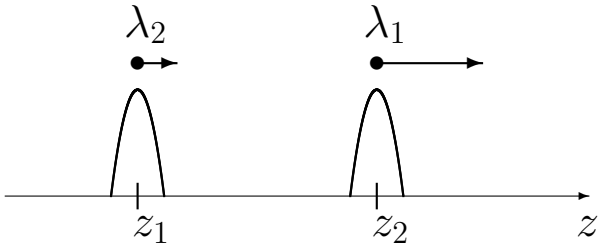

Fig. 6.1b: Outgoing wave, $z_1 < z_2$.

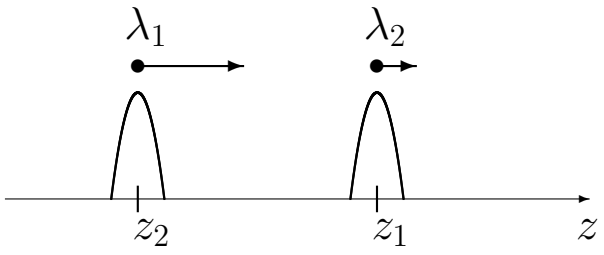

Fig. 6.2a: Incoming wave, $z_2 < z_1$.

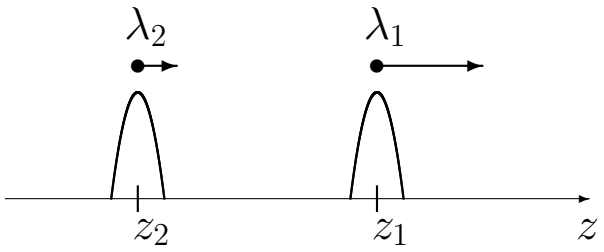

Fig. 6.2b: Outgoing wave, $z_2 < z_1$.

b) *Multiparticle scattering. The factorization of the S-matrix.*

Consider now the wave function χ_N (4.8), (4.11) describing the scattering of N particles (all λ_j are real). In the fundamental domain $F : z_1 < z_2 < \ldots < z_N$, it is given by expression (4.7) (all the sign functions are equal to $+1$). Choose the momenta $\lambda_1 > \lambda_2 > \ldots > \lambda_N$. Then in F only one term in χ_N (see Fig. 6.3a) describes the incoming wave (no interaction as $t \to -\infty$). The outgoing wave is as well described by one single term in F, see Fig. 6.3b.

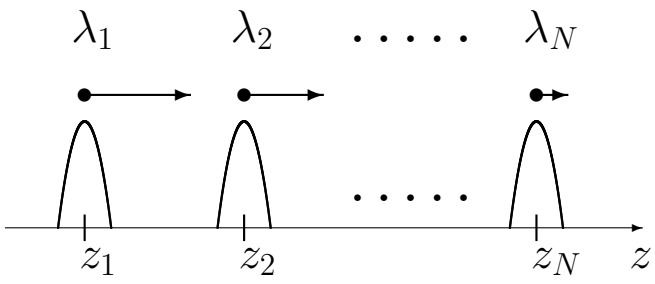

Fig. 6.3a: Ingoing wave, $\{z\} \in F$.

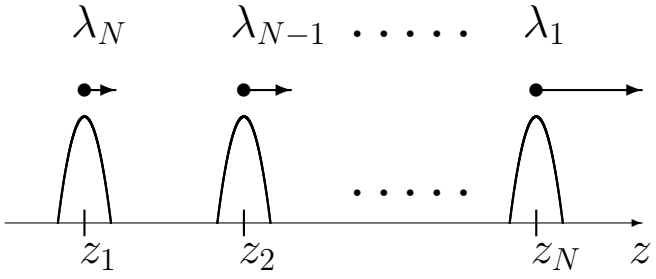

Fig. 6.3b: Outgoing wave, $\{z\} \in F$.

The S matrix is given by the ratio of the corresponding amplitudes which is calculated to be

$$S(\lambda_1, \lambda_2, \ldots, \lambda_N) = \prod_{1 \le j < k \le N} S(\lambda_j, \lambda_k), \tag{6.2}$$

where $S(\lambda_j, \lambda_k)$ are the two-particles S-matrices (6.1). The number of factors in the r.h.s. is $N(N-1)/2$, which is just the number of pair interactions between the N particles.

7 Periodic boundary conditions. Bethe equations.

One of the most interesting problems is to consider a boson gas of finite density in infinite volume. Usually, it is convenient to start by putting the gas into a "box" of length L, where quantities like the ground energy are well defined, and then increase the number N of the particles, $N \to \infty$ as well as the length of the box, $L \to \infty$, keeping the density $D = N/L$ fixed. It is very convenient to impose, for finite L, the periodic boundary conditions which permits to maintain translation invariance and to prescribe definite quasimomenta to the particles entering the eigenstates of the Hamiltonian.

Consider N particles in a "box" of length L ($0 \le z_j \le L$, $j = 1, 2, ..., N$) and impose periodic boundary conditions on the Bethe wave functions (4.8) or (4.11), demanding that they are periodic in each z_j with all other z_k ($k \ne j$) fixed

$$\chi_N(\{\lambda\}|z_1, \ldots, z_j + L, \ldots, z_N) = \chi_N(\{\lambda\}|z_1, \ldots, z_j, \ldots, z_N) \ . \tag{7.1}$$

In the domain $0 \le z_j \le L$, the wave function is given by equations (4.8) or (4.11). The requirement (7.1) is equivalent to the requirement that the values of function χ_N at $z_j = 0$ and $z_j = L$, as well as the values of its derivatives in z_j, are equal.

Equation (7.1) gives the system of equations for the permitted values of the momenta $\lambda_1, \ldots, \lambda_N$ parametrizing the Bethe wave function, these equations being called the Bethe equations.

For $N = 1$, up to a factor independent of z, $\chi_1(\lambda|z) = \exp[i\lambda z]$, and the periodic boundary condition, $\exp[i\lambda L] = 1$, gives the standard quantization of the possible values of momentum for a free particle in a box:

$$\lambda_n = \frac{2\pi n}{L} \qquad (N = 1), \tag{7.2}$$

where $-\infty < n < +\infty$, n-integer. For $N = 2$, the wave function is

$$\begin{aligned}\chi_2(\lambda_1\lambda_2|z_1z_2) = e^{i(z_1\lambda_1+z_2\lambda_2)}\Big(\lambda_2 - \lambda_1 - ic\epsilon(z_2 - z_1)\Big) \\ -e^{i(z_1\lambda_2+z_2\lambda_1)}\Big(\lambda_1 - \lambda_2 - ic\epsilon(z_2 - z_1)\Big)\ ,\end{aligned} \tag{7.3}$$

and the periodic boundary conditions give the following equations for the permitted values of momenta λ_1, λ_2:

$$e^{iL\lambda_1} = \frac{\lambda_1 - \lambda_2 + ic}{\lambda_1 - \lambda_2 - ic}\ , \qquad e^{iL\lambda_2} = \frac{\lambda_2 - \lambda_1 + ic}{\lambda_2 - \lambda_1 - ic}\ . \tag{7.4}$$

For arbitrary N, one gets the system of N equations,

$$e^{iL\lambda_j} = \prod_{\substack{k=1\\k\neq j}}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} = -\prod_{k=1}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \qquad (j = 1, \dots, N). \tag{7.5}$$

These equations ensure the periodicity of the wave function; they are the Bethe equations for the nonrelativistic Bose gas.

In terms of the antisymmetric phase $\theta(\lambda)$ (4.12), they can be rewritten as

$$\exp\left[iLp_0(\lambda_j) + i\sum_{k=1}^{N}\theta(\lambda_j - \lambda_k)\right] = (-1)^{N+1} \qquad (j = 1, ..., N). \tag{7.6}$$

In this form, the Bethe equations are valid for many integrable models. The "bare momentum" $p_0(\lambda)$ is different in different models, the scattering phase θ being the same for the models with the same R-matrix.

Multiplying Bethe equations for $j = 1, \dots, N$, one gets

$$\exp\Big[iLP_{_N}\Big] = 1, \qquad P_{_N} = \sum_{j=1}^{N} p_0(\lambda_j), \tag{7.7}$$

i.e., the quantization of the total momentum is the same as for a free particle.

Let us sum up. The Bethe eigenstates of the second quantized Hamiltonian H (1.2) with periodic boundary conditions are of the form (see (2.6))

$$|\Psi_{_N}(\{\lambda\})\rangle = \int_0^L d^N z \chi_{_N}(\{\lambda\}|\{z\})\psi^+(z_1)\dots\psi^+(z_N)|0\rangle, \qquad N = 0, 1, 2, \dots\ , \tag{7.8}$$

where the Bethe eigenfunctions χ_N are given by (4.8) or (4.11) and momenta λ_j ($j = 1, 2, \ldots, N$) should satisfy Bethe equations (7.5), (7.6). All the λ_j's should be different. The energy, momentum and the number of particles are given as

$$H|\Psi_N(\{\lambda\})\rangle = E_N|\Psi_N(\{\lambda\})\rangle, \qquad E_N = \sum_{j=1}^{N} \epsilon_0(\lambda_j), \tag{7.9}$$

$$P|\Psi_N(\{\lambda\})\rangle = P_N|\Psi_N(\{\lambda\})\rangle, \qquad P_N = \sum_{j=1}^{N} p_0(\lambda_j), \tag{7.10}$$

$$Q|\Psi_N(\{\lambda\})\rangle = N|\Psi_N(\{\lambda\})\rangle \tag{7.11}$$

(remind that for the Bose gas, $p_0(\lambda) = \lambda$ and $\epsilon_0(\lambda) = \lambda^2$).

So the state $|\Psi_N(\{\lambda\})\rangle$ can be regarded as an N-particle state, each particle possessing momentum $p_0(\lambda_j) \equiv \lambda_j$ and energy $\epsilon_0(\lambda_j) = \lambda_j^2$. The total momentum and energy are sums of momenta and energies of the individual particles. This is similar to the case of free particles. But our theory is free only in the one-particle sector (see (7.2)). If there are several particles, the presence of other particles changes the permitted values of the momenta of particles, due to the Bethe equations; the particles are not free.

8 On the solutions of the Bethe equations for the one-dimensional Bose gas.

Let us discuss the solutions of Bethe equations (7.5) for the nonrelativistic Bose gas. Consider first the repulsive case (coupling constant $c > 0$). In this case it can be proved that the Bethe equations possess only real solutions, $\mathrm{Im}\lambda_j = 0$ ($j = 1, 2, \ldots, N$).

Let us demonstrate this in the case of $N = 2$. Multiplying the two equations in (7.4), one gets $\exp\{i(\lambda_1 + \lambda_2)L\} = 1$, so that $\lambda_1 + \lambda_2$ is real, and one can take

$$\lambda_{1,2} = \lambda_{1,2}^{(0)} \pm i\kappa\ , \qquad \mathrm{Im}\,\lambda_{1,2}^{(0)} = 0\ , \qquad \mathrm{Im}\,\kappa = 0. \tag{8.1}$$

From (7.4), one gets, e.g.,

$$e^{i\lambda_1^{(0)}L - \kappa L} = \frac{\lambda_1^{(0)} - \lambda_2^{(0)} + i(2\kappa + c)}{\lambda_1^{(0)} - \lambda_2^{(0)} + i(2\kappa - c)}. \tag{8.2}$$

Comparing the moduli of the left-hand side and the right-hand side, one concludes that for $c > 0$ this equality can be fulfilled only for $\kappa = 0$. Thus, both λ_1 and λ_2 should be real. The same idea can be used to prove that all the solutions should be real for any N in the case of $c > 0$.

Turn now to the case of the attractive potential ($c < 0$). In this case, besides the real solutions of the Bethe equations, there exist also solutions with complex

λ's. If the length L of the box is large, $L \to \infty$, the solutions with complex λ's form the so-called "strings"; they become just the string solutions considered earlier in Section 5. The individual momenta of the particles forming the string of "length" n (an "n-string") are given in the limit just by equation (5.4), the total momentum and energy of the string being given by equation (5.5). Let us emphasize once more that these formulae are valid in the limit $L \to \infty$ (within "exponential" accuracy).

Let us demonstrate the existence of these solutions by an example of a two-string. Suppose that $\lambda_{1,2}$ are complex solutions of the two-particle Bethe equations; they can be represented in the form (8.1). Suppose also that $\kappa > 0$ (if $\kappa < 0$, everything is considered similarly with the same result). As $L \to \infty$, the left-hand side of (8.2) becomes equal to zero; hence, it is necessary that

$$\lambda_1^{(0)} \to \lambda_2^{(0)} \to \lambda \qquad (\mathrm{Im}\,\lambda = 0), \qquad 2i\kappa + ic \to 0.$$

Thus, $\kappa = \frac{|c|}{2}$ in the limit $L \to \infty$, and one gets just equation (5.4) with $n = 2$. The real number λ in this case should satisfy equation $\exp[2i\lambda L] = 1$; i.e., λ could take any value in the limit $L = \infty$. The "corrections" to $\lambda_{1,2}$ at L finite but large can be seen to be on the order of $\exp[-|c|L/2]$.

9 Classical inverse scattering method. The Lax representation.

The classical inverse scattering method (CISM) is a method for solving many nonlinear partial differential equations. Only the aspects of this method necessary for quantization will be considered below.

Let us consider a classical nonlinear Hamiltonian evolution equation. As an example, one has in mind the nonlinear Schrödinger (NS) equation (1.1). The base for applying the inverse scattering method to the NS equation is its representation in the Lax form.

Consider the "auxiliary" matrix valued function $\Phi(x,t|\lambda)$ with matrix elements $\Phi_{jk}(x,t|\lambda)$ $(j,k = 1,\dots,N)$ and $N \times N$ matrices ("potentials") $U(x,t|\lambda)$ and $V(x,t|\lambda)$ depending on the space-time variables x, t, and on the complex spectral parameter $\lambda \in \mathbf{C}$. The system of linear differential equations for the function Φ is defined in the following way

$$\Big(\partial_x + V(x,t|\lambda)\Big)\Phi(x,t|\lambda) = 0, \tag{9.1}$$

$$\Big(\partial_t + U(x,t|\lambda)\Big)\Phi(x,t|\lambda) = 0. \tag{9.2}$$

We suppose that each column of matrix Φ forms a linear independent solution, so that $\det \Phi \neq 0$. This system of equations has solutions, if and only if the potentials

V and U satisfy, for any values of λ, the compatibility condition

$$\partial_t V - \partial_x U + [U, V] = 0, \quad \forall\lambda \tag{9.3}$$

(this follows from the requirement $\partial_x \partial_t \Phi = \partial_t \partial_x \Phi$, if one calculates the derivatives by means of (9.1), (9.2)).

Equation (9.3) is called the Lax representation (or the zero curvature representation) for a given nonlinear partial differential equation (or, for a system of equations) if one finds potentials V and U, for which the compatibility condition (9.3) is equivalent to the nonlinear equation itself. The fields satisfying the nonlinear equations enter the expressions for the matrix elements of the potentials giving the dependence on x, t of the matrices $V(x,t|\lambda)$ and $U(x,t|\lambda)$.

As an example, let us give the corresponding construction for the NS equation (1.1). Consider the following 2×2-matrices ($N = 2$) U and V depending on x, t via a complex scalar field $\psi(x,t)$, and its complex conjugated $\psi^*(x,t)$ (in Sections 9–17, the star will denote complex conjugation):

$$V(x,t|\lambda) = \begin{pmatrix} \dfrac{i\lambda}{2} & i\sqrt{c}\,\psi^*(x,t) \\ -i\sqrt{c}\,\psi(x,t) & -\dfrac{i\lambda}{2} \end{pmatrix}, \tag{9.4}$$

$$U(x,t|\lambda) = \begin{pmatrix} -\dfrac{i\lambda^2}{2} - ic\,\psi^*\psi & -i\lambda\sqrt{c}\,\psi^* + \sqrt{c}\partial_x\psi^* \\ i\lambda\sqrt{c}\,\psi + \sqrt{c}\,\partial_x\,\psi & \dfrac{i\lambda^2}{2} + ic\,\psi^*\psi \end{pmatrix}. \tag{9.5}$$

One easily computes

$$\partial_t V - \partial_x U + [U, V] = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0,$$

with $A_3 = A_2 = A_1 \equiv 0$,

$$A_0 = \begin{pmatrix} 0 & \sqrt{c}(i\partial_t\psi^* - \partial_x^2\psi^* + 2c\psi^*\psi^*\psi) \\ \sqrt{c}(-i\partial_t\psi - \partial_x^2\psi + 2c\psi^*\psi\psi) & 0 \end{pmatrix}.$$

Thus, the compatibility condition (9.3) is equivalent to the NS equation.

Operators $\partial_x + V$ and $\partial_t + U$ are usually called the "$L - M$ pair" (or the "$L - A$ pair") for a nonlinear partial differential equation. Representations of this kind are known now for a lot of interesting nonlinear equations (and systems of equations) in two space-time dimensions. These equations are usually called "completely integrable" equations (though the name may be not very strict in this context).

As an example, let us give here also the Lax representation for the famous sine-Gordon equation which is the nonlinear relativistic equation for a real scalar field $u(x,t)$ in $1+1$ space-time dimensions,

$$(\partial_t^2 - \partial_x^2)u(x,t) + \frac{m^2}{\beta}\sin\beta u(x,t) = 0. \tag{9.6}$$

The L–M pair for the sine-Gordon equation is defined by the following 2×2 matrices V and U:

$$V(x,t) = \frac{i\beta}{4}\partial_t u\sigma_3 + \frac{m}{2}\sinh\left(\frac{i\beta u}{2} - \lambda\right)\sigma_+ + \frac{m}{2}\sinh\left(\frac{i\beta u}{2} + \lambda\right)\sigma_-, \qquad (9.7)$$

$$U(x,t) = \frac{i\beta}{4}\partial_x u\sigma_3 + \frac{m}{2}\cosh\left(\frac{i\beta u}{2} - \lambda\right)\sigma_+ - \frac{m}{2}\cosh\left(\frac{i\beta u}{2} + \lambda\right)\sigma_-. \qquad (9.8)$$

The zero curvature condition gives

$$\partial_t V - \partial_x U + [U,V] = \frac{i\beta}{4}\left((\partial_t^2 - \partial_x^2)u(x,t) + \frac{m^2}{\beta}\sin\beta u(x,t)\right)\sigma_3 = 0,$$

which is fullfilled, if and only if the field $u(x,t)$ satisfies the sine-Gordon equation (9.6).

The Lax representation, if it exists, reduces the problem of solving a nonlinear partial differential equation to the problem of solving the linear ones. This is briefly explained in the next Section.

10 Why "the inverse scattering method"?

The name "inverse scattering method" is related to the inverse scattering problem of quantum mechanics, i.e., to the restoration (or reconstruction) of the potential from the knowledge of the "scattering data". In our case (one space dimension) this means the knowledge of the reflection and transition coefficients (and also, the information about the bound states is needed). This problem of restoration of the potential is solved, even in three dimensions. In the one-dimensional case, one can restore the potential from the scattering data by means of the Gelfand-Levitan-Marchenko equations. These are linear integral equations. Here, these equations will not be used or even written (for them, see, e.g., the book by Faddeev and Takhtadjan cited in the end of this Section). Only an idea of the solution of a nonlinear evolution equation by means of the classical inverse scattering method is described below. The method can be applied if one can represent the evolution equation in the Lax form (9.3).

Let us explain the method for the example of the nonlinear Schrödinger equation.

(i) The first step is to consider equation (9.1), $(\partial_x + V)\Phi = 0$, where time t is just a parameter, as an "auxiliary" (linear !) "Schrödinger equation" (better to say, "Dirac equation") at some time t:

$$(\partial_x + V(x,t|\lambda))\Phi(x,t|\lambda) = 0. \qquad (10.1)$$

The potential V is defined by the fields $\psi(x,t)$, $\psi^*(x,t)$ (given arbitrarily at time t), and the spectral parameter λ plays the role of an "energy" in equation (10.1).

If $\Phi(x,t|\lambda)$ is a solution of equation (10.1), then $\Phi(x,t|\lambda)A(t|\lambda)$ is also a solution of (10.1) for any matrix $A(t|\lambda)$ (not depending on x) and all the solutions can be obtained in this way (if $\det\Phi \neq 0$).

Impose boundary conditions in the coordinate x, demanding that ψ, $\psi^* \to 0$, as $x \to \pm\infty$. Then the asymptotics $\Phi^{\pm}$ of Φ at $x \to \pm\infty$,

$$\Phi^{\pm}(x,t_0|\lambda) \cong \Phi(x,t_0|\lambda)\Big|_{x\to\pm\infty}, \tag{10.2}$$

satisfies (see (9.4)) the "free equation" in x

$$\left(\partial_x + \frac{i\lambda}{2}\sigma_3\right)\Phi^{\pm}(x,t|\lambda) = 0, \qquad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10.3}$$

So at a given time t, $\Phi^+(x,t|\lambda) = \Phi^-(x,t|\lambda)S_\Phi(t|\lambda)$, where S_Φ is the S-matrix ("scattering data") for the solution Φ.

To fix the explicit form of S, one should state the scattering problem more precisely. Let us define the solution $\Phi_0(x,t|\lambda)$ of the problem (10.1), fixing its behavior at $x \to -\infty$ at time t as

$$\Phi_0^-(x,t|\lambda) \equiv \Phi_0^-(x|\lambda) = \exp\left\{-\frac{i\lambda x}{2}\sigma_3\right\} = \begin{pmatrix} \exp\left\{-\frac{i\lambda}{2}x\right\} & 0 \\ 0 & \exp\left\{\frac{i\lambda}{2}x\right\} \end{pmatrix}. \tag{10.4}$$

(this condition does not, in fact, depend on t). Condition (10.4) fixes the solution of equation (10.1) completely.

Let us denote $S(t|\lambda)$ the S-matrix for the solution Φ_0 of (10.1) with this behaviour, i.e.,

$$\Phi_0^+(x,t|\lambda) = \Phi_0^-(x,t|\lambda)S(t|\lambda)\,. \tag{10.5}$$

Assume that we have found the "scattering data" $S(t_0|\lambda)$ at time t_0 for the given $\psi(x,t_0)$, $\psi^+(x,t_0)$ solving the linear auxiliary Schrödinger equation (10.1) (i.e., solving the "direct problem"). This is much simpler, of course, than solving the original nonlinear Schrödinger equation. One should deal with a linear equation to find the scattering data.

(ii) It appears that the time evolution of the scattering data is very simple and can be easily found by using equation (9.2),

$$(\partial_t + U(x,t|\lambda))\Phi(x,t|\lambda) = 0, \qquad \Phi(x,t|\lambda)\Big|_{t=t_0} = \Phi(x,t_0|\lambda)\ . \tag{10.6}$$

Taking the asymptotics at $x \to \pm\infty$ ($U(x,t|\lambda) \cong -\frac{i\lambda^2}{2}\sigma_3$), one gets

$$\Phi(x,t|\lambda)\Big|_{x\to\pm\infty} \cong \Phi^{\pm}(x,t|\lambda) = \exp\left[\frac{i\lambda^2}{2}(t-t_0)\sigma_3\right]\Phi_0^{\pm}(x,t_0|\lambda)\ .$$

This is not the function $\Phi_0(x,t|\lambda)$ yet; by definition, it should be given at any t by the boundary condition in x, $\Phi_0^-(x,t|\lambda) \equiv \Phi_0^-(x|\lambda) = \exp[-\frac{i\lambda}{2}x\sigma_3]$, see (10.4),

(10.5). For the S-matrix at the t, one obtains (multiplying the solution $\Phi(x,t|\lambda)$ (10.6) by the corresponding x-independent matrix from the right)

$$S(t|\lambda) = \exp\left\{\frac{i\lambda^2}{2}(t-t_0)\sigma_3\right\} S(t_0|\lambda) \exp\left\{-\frac{i\lambda^2}{2}(t-t_0)\sigma_3\right\}.$$

If one writes explicitly

$$S(t|\lambda) = \begin{pmatrix} a(t|\lambda) & b(t|\lambda) \\ c(t|\lambda) & d(t|\lambda) \end{pmatrix},$$

(a, d are the "transition coefficients", and b, c — the "reflection coefficients"), then their evolution law is,

$$a(t|\lambda) = a(t_0|\lambda), \qquad d(t|\lambda) = d(t_0|\lambda),$$

$$b(t|\lambda) = \exp\Big[i\lambda^2(t-t_0)\Big] b(t_0|\lambda), \quad c(t|\lambda) = \exp\Big[-i\lambda^2(t-t_0)\Big] c(t_0|\lambda),$$

i.e., the evolution in time of the scattering data is very simple, the reflection and transition coefficients playing the role of the "angle-action" variables for the original nonlinear system (i.e., for the NS equation).

(iii) Now, the most important step should be done. The Gelfand-Levitan-Marchenko linear integral equations can be used to restore the potential $V(x,t|\lambda)$, i.e., the fields $\psi(x,t)$, $\psi^*(x,t)$ at time t. In this way, the problem of solving the Cauchy problem for the nonlinear partial differential equation is reduced to the solving of linear problems only.

To conclude, let us discuss in more detail the meaning of the S-matrix introduced above, stating the scattering problem in a different, more physical way. Let us define the solution $\Phi_s(x,t|\lambda)$ of the problem (10.1) fixing its behavior at $x \to \pm\infty$ at time t as

$$\Phi_s^-(x,t|\lambda) \equiv \begin{pmatrix} \alpha(t|\lambda) \exp\left\{-\frac{i\lambda}{2}x\right\} & \gamma(t|\lambda) \exp\left\{-\frac{i\lambda}{2}x\right\} \\ 0 & \exp\left\{\frac{i\lambda}{2}x\right\} \end{pmatrix}, \tag{10.7a}$$

and

$$\Phi_s^+(x,t|\lambda) \equiv \begin{pmatrix} \exp\left\{-\frac{i\lambda}{2}x\right\} & 0 \\ \beta(t|\lambda) \exp\left\{\frac{i\lambda}{2}x\right\} & \delta(t|\lambda) \exp\left\{\frac{i\lambda}{2}x\right\} \end{pmatrix}. \tag{10.7b}$$

Having in mind that two columns of matrix $\Phi_s(x,t|\lambda)$ correspond to the two linear independent solutions ("vectors") of equation (10.1), we see that the first solution corresponds to the scattering problem, where at $x \to -\infty$ one has incident wave of amplitude 1 going in the positive x direction (corresponding to the element (11) of matrix (10.7a)). The matrix element (21) of this matrix describes the reflected wave, and the matrix element (21) of matrix (10.7b) presents the transition wave, the quantities $\beta(t|\lambda)$ and $\alpha(t|\lambda)$ being the corresponding reflection and transition coefficients.

On the other hand, the second column of matrix $\Phi_s(x,t|\lambda)$ gives the solution where the incident wave (with amplitude equal to 1) is going along the negative direction of the x axis (matrix element (22) of matrix (10.7b)). The matrix element (12) of this matrix describes the reflected wave for this solution, ($\gamma(t|\lambda)$ is the corresponding reflection coefficient), and the matrix element (22) of the matrix (10.7a) gives the transition wave with the transition coefficient $\delta(t|\lambda)$. The conditions (10.7a) and (10.7b) fix the solution of equation (10.1) completely by fixing the amplitude of the incident waves to unity (the transition and reflection coefficients should be, of course, defined from equation (10.1)). It is not difficult to express the new coefficients in terms of the old ones,

$$\alpha(t|\lambda) = a^{-1}(t|\lambda),$$

$$\beta(t|\lambda) = -a^{-1}(t|\lambda)b(t|\lambda),$$

$$\gamma(t|\lambda) = a^{-1}(t|\lambda)c(t|\lambda),$$

$$\delta(t|\lambda) = d(t|\lambda) - a^{-1}(t|\lambda)b(t|\lambda)c(t|\lambda),$$

so that the new transition coefficients do not depend on time at all, and the dependence on time of the new reflection coefficients is simple.

Let us mention also that the matrix elements of the S-matrix are not, in fact, independent; they should satisfy some relations. In particular,

$$\alpha(t|\lambda) = \delta(t|\lambda) \ .$$

This follows from the fact that the determinant of a solution of the equation (10.1) does not depend on x due to the relation

$$\partial_x \det \Phi = -(\mathrm{tr}\, V) \det \Phi = 0$$

which is a consequence of (10.1). The symmetry property of the potential V,

$$V(x,t|\lambda) = \sigma_1 V^*(x,t|\lambda^*)\sigma_1 \ , \tag{10.8}$$

(the star denotes the complex conjugation of the matrix elements, not including their transposition) results in the relations

$$a(t|\lambda) = d^*(t|\lambda^*) \ , \qquad b(t|\lambda) = c^*(t|\lambda^*) \ .$$

It should be emphasized again that here only the idea of the method is given. For greater detail see, e.g., the excellent book by Faddeev and Takhtajan (L.D.Faddeev, L.A.Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Berlin, Springer-Verlag (1987)).

11 The transition matrix and the monodromy matrix.

Let us consider equation (9.1),

$$(\partial_x + V(x|\lambda))\Phi(x|\lambda) = 0 \tag{11.1}$$

(the time argument t is fixed and is not written explicitly in what follows). From now, on we assume that periodic boundary conditions are imposed in the interval $0 \le x \le L$ on the solutions of the nonlinear evolutionary equation (e.g., for the NS equation $\psi(x+L) \equiv \psi(x)$, $\psi^*(x+L) = \psi^*(x)$). This does not imply, of course, the periodicity of Φ.

Equation (11.1) is an ordinary differential equation ($\partial_x = d/dx$) for an $N \times N$ matrix Φ. The transition matrix $T(x, y|\lambda)$ (from point y to point x) is defined as a special solution of this equation, namely,

$$(\partial_x + V(x|\lambda))T(x, y|\lambda) = 0\ , \qquad T(y, y|\lambda) = I\ , \qquad x \ge y\ . \tag{11.2}$$

It can be represented in terms of any solution $\Phi(x|\lambda)$ of (11.1) ($\det\Phi \neq 0$) as

$$T(x, y|\lambda) = \Phi(x|\lambda)\Phi^{-1}(y|\lambda), \tag{11.3}$$

since the right hand side satisfies both the differential equation and boundary condition in (11.2). From (11.3), one sees that the transition matrix really gives the "transition" of any solution from point y to point x, $\Phi(x|\lambda) = T(x, y|\lambda)\Phi(y|\lambda)$.

Let us discuss the important properties of the transition matrix. Since matrix $\Phi^{-1}(x|\lambda)$ satisfies the equation $\partial_x\Phi^{-1}(x|\lambda) - \Phi^{-1}(x|\lambda)V(x|\lambda) = 0$ (remind that $\partial_x\Phi^{-1} = -\Phi^{-1}(\partial_x\Phi)\Phi^{-1}$, due to the identity $\partial_x(\Phi\Phi^{-1}) = 0$), the transition matrix satisfies the following equation in the second argument:

$$\partial_y T(x, y|\lambda) - T(x, y|\lambda)V(y|\lambda) = 0, \qquad T(x, x|\lambda) = I. \tag{11.4}$$

It can be used, together with equation (11.2), to establish the "group property" of the transition matrix,

$$T(x, z|\lambda)T(z, y|\lambda) = T(x, y|\lambda), \qquad y \le z \le x \tag{11.5}$$

(the derivatives in z of both sides are here equal to zero, and both sides are equal at $z = y$). Using the relation $\det T = \exp(\mathrm{tr}\ \ln T)$, one computes $\partial_x \det T$ and gets a "scalar" differential equation for the determinant of the transition matrix,

$$\partial_x \det T = -(\mathrm{tr}\, V) \det T$$

which gives

$$\det T(x, y|\lambda) = \exp\Big\{-\int_y^x \Big(\mathrm{tr}\, V(z)\Big)dz\Big\} \neq 0. \tag{11.6}$$

In particular, for the NS equation $\operatorname{tr} V = 0$ (see (9.4)), and $\det T(x, y|\lambda) \equiv 1$ in this case.

The monodromy matrix $T(\lambda)$ is defined as the transition matrix for the whole interval:

$$T(\lambda) \equiv T(x = L, y = 0|\lambda). \tag{11.7}$$

The trace $\tau(\lambda)$ of the monodromy matrix,

$$\tau(\lambda) \equiv \operatorname{tr} T(\lambda) = \sum_{i=1}^{N} T_{ii}(\lambda)\,, \tag{11.8}$$

is of particular importance. The Hamiltonian of the initial nonlinear equation can be usually expressed in terms of $\tau(\lambda)$. The corresponding formulae are called "trace identities". In next Section, this is demonstrated for the example of the NS equation.

12 The trace identities for the classical nonlinear Schrödinger equation.

For the NS equation, the monodromy matrix $T(\lambda)$ is a 2×2-matrix,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{12.1}$$

Matrix elements of $T(\lambda)$ are functionals of the fields $\psi(x)$, $\psi^*(x)$ $(0 \le x \le L)$; periodic boundary conditions are imposed on ψ, $\psi(x + L) = \psi(x)$. As shown below, the following asymptotic expansion for $\tau(\lambda) = \operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda)$ at $\lambda \to +i\infty$ holds true:

$$\ln\Big(\tau(\lambda) \exp\Big\{i\frac{\lambda}{2}L\Big\}\Big)\Big|_{\lambda\to+i\infty} = ic \sum_{n=1}^{\infty} \lambda^{-n} I_n + O(|\lambda|^{-\infty}), \tag{12.2}$$

where the first coefficients are charge Q, momentum P and Hamiltonian H (see (1.2), (1.3))

$$I_1 = Q = \int_0^L \psi^*(x)\psi(x)dx, \tag{12.3}$$

$$I_2 = P = -i \int_0^L \psi^*(x)\partial_x\psi(x)dx, \tag{12.4}$$

$$I_3 = H = \int_0^L (\partial_x\psi^*\partial_x\psi + c\psi^*\psi^*\psi\psi)dx. \tag{12.5}$$

Formulae of the kind (12.2), expressing the Hamiltonian and other integrals of motion in terms of the trace of the monodromy matrix of the auxiliary linear

problem, are called "trace identities". The derivation should be done by means of direct calculation.

Consider the transition matrix $T(x,y|\lambda)$ for the NS equation,

$$\Big(\partial_x + V(x|\lambda)\Big)T(x,y|\lambda) = 0 \ , \qquad T(y,y|\lambda) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{12.6}$$

Potential $V(x|\lambda)$ here can be written as

$$V(x|\lambda) = \frac{i\lambda}{2}\sigma_3 + \Omega(x) \ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the antidiagonal matrix $\Omega(x)$ is

$$\Omega(x) = \begin{pmatrix} 0 & i\sqrt{c}\,\psi^*(x) \\ -i\sqrt{c}\,\psi(x), & 0 \end{pmatrix}. \tag{12.7}$$

The periodic boundary conditions for the NS equation imply the periodicity of the potential, $V(x=0|\lambda) = V(x=L|\lambda)$. It is worth mentioning that $\det T = 1$ (see (11.6)), and, due to property (10.8), there exists an involution,

$$\sigma_1 T^*(x,y|\lambda^*)\sigma_1 = T(x,y|\lambda) \ . \tag{12.8}$$

Let us make a "gauge" transformation with a matrix $G(x|\lambda)$,

$$T(x,y|\lambda) = G(x|\lambda)D(x,y|\lambda)G^{-1}(y|\lambda) \ , \tag{12.9}$$

demanding that D is a diagonal matrix, and the matrix G has the form

$$G(x|\lambda) = I + A(x|\lambda) \ , \qquad A(x|\lambda) = \sum_{n=1}^{\infty} \lambda^{-n} G_n(x) \tag{12.10}$$

where $A(x|\lambda)$ and G_n are antidiagonal matrices (i.e., matrices with the diagonal elements equal to zero). For the diagonal matrix D, one gets the following equation

$$\Big(\partial_x + W(x|\lambda)\Big)D(x,y|\lambda) = 0 \ , \qquad D(y,y|\lambda) = I \ . \tag{12.11}$$

with the "local" potential W given as

$$\begin{aligned} W(x|\lambda) = \frac{i\lambda}{2}G^{-1}(x|\lambda)\sigma_3 G(x|\lambda) + \\ + G^{-1}(x|\lambda)\Omega(x)G(x|\lambda) + G^{-1}(x|\lambda)\partial_x G(x|\lambda). \end{aligned} \tag{12.12}$$

One defines G demanding that W is diagonal. Multiplying (12.12) by $G = I + A$, one has

$$W + AW = \frac{i\lambda}{2}\sigma_3 + \frac{i\lambda}{2}\sigma_3 A + \Omega + \Omega A + \partial_x A.$$

Since W is a diagonal matrix, and A is an antidiagonal matrix, one writes for the diagonal and the antidiagonal parts of this equation

$$W = \frac{i\lambda}{2}\sigma_3 + \Omega A, \tag{12.13}$$

$$AW = \frac{i\lambda}{2}\sigma_3 A + \Omega + \partial_x A. \tag{12.14}$$

By means of (12.13) (taking into account that σ_3 anticommutes with any antidiagonal matrix), equation (12.14) is put into the form:

$$A = \frac{i\sigma_3}{\lambda}\Omega + \frac{i\sigma_3}{\lambda}\partial_x A - \frac{i\sigma_3}{\lambda}A\Omega A, \tag{12.15}$$

so that the coefficients in the $1/\lambda$-expansion of A (12.10) are easily found by recursion, e.g.,

$$\begin{aligned} G_1 &= i\sigma_3\Omega, \\ G_2 &= i\sigma_3(\partial_x i\sigma_3\Omega) = -\partial_x\Omega, \\ G_3 &= i\sigma_3\partial_x G_2 - i\sigma_3 G_1\Omega G_1 = i\sigma_3(-\partial_x^2\Omega + \Omega^3), \\ G_4 &= \partial_x^3\Omega - 2\partial_x\Omega\cdot\Omega^2 - 2\Omega^2\partial_x\Omega - \Omega\cdot\partial_x\Omega\cdot\Omega, \ \ldots\ . \end{aligned} \tag{12.16}$$

For the potential $W(x|\lambda)$, one then gets from (12.13):

$$W_n(x|\lambda) = \frac{i\lambda}{2}\sigma_3 + \sum_{n=1}^{\infty}\lambda^{-n}W_n(x)\ , \qquad W_n(x) = \Omega(x)G_n(x)\ . \tag{12.17}$$

It is easily seen that matrices $G_n(x)$ are, in fact, antidiagonal matrices, and W_n are diagonal matrices.

Since W is a diagonal matrix (hence $W(x|\lambda)$ commutes with $W(y|\lambda)$), one has the solution of equation (12.11),

$$D(x,y|\lambda) = \exp\Big[-\int_y^x W(z|\lambda)dz\Big],$$

and due to the periodic boundary conditions,

$$\tau(\lambda) = \operatorname{tr} T(L,0|\lambda) = \operatorname{tr} D(L,0|\lambda). \tag{12.18}$$

Since $\det T(L,0|\lambda) = \det D(L,0|\lambda) = 1$, one can represent $D(L,0|\lambda) = \exp[\sigma_3 Z(\lambda)]$, with a scalar $Z(\lambda)$. Using (12.16), (12.17), it is easy to calculate explicitly that

$$Z(\lambda) = \frac{-i\lambda L}{2} + ic\sum_{n=1}^{\infty}\lambda^{-n}I_n,$$

where $I_1 = Q$, $I_2 = P$, $I_3 = H$,

$$I_4 = \int_0^L\Big(-i\psi^*\partial_x^3\psi + \frac{3ic}{2}\psi^*\psi^*\partial_x(\psi\psi)\Big)dx, \quad \text{etc.}$$

For $\tau(\lambda)$, one gets

$$\tau(\lambda) = \mathrm{tr}\exp[\sigma_3 Z(\lambda)] = 2\cosh Z(\lambda).$$

If $\lambda \to +i\infty$ (in the upper half plane), the ratio of the matrix elements of the diagonal matrix D is

$$D_{22}(L,0|\lambda)/D_{11}(L,0|\lambda) = \exp[i\lambda L] = O(|\lambda|^{-\infty}),$$

so that one obtains just the trace identities (12.2).

We know from Section 1 that the Poisson brackets between the functionals Q, P, and H are equal to zero (1.7). In fact, we shall see later that also $\{I_m, I_n\} = 0, \forall m, n$. The simplest way to establish this fact is to use the classical r-matrix (see Section 15). This means that, in fact, there are infinitely many "commuting" integrals of motion for the NS equation.

13 Tensor products; notations.

To go further, we need to introduce some notations.

Let A be an $N \times N$-dimensional matrix acting as a linear operator in an N-dimensional linear space V of vectors $x = \{x_i\}$, $i = 1, \ldots, N$:

$$(Ax)_{i_1} = \sum_{i_2=1}^{N} A_{i_1 i_2} x_{i_2} \equiv A_{i_1 i_2} x_{i_2} \ . \qquad x \in V \tag{13.1}$$

(the sum over the repeated indices is usually suppressed in the notations). Components of an N^2-dimensional vector z in the tensor product $V \otimes V$ of two linear spaces V will be labeled as z^i_k, indices i and k being related to the first and second component spaces in the tensor product. In particular, if $x, y \in V$, then

$$(x \otimes y)^i_k \equiv x_i y_k \ . \tag{13.2}$$

Linear operators in $V \otimes V$ are $N^2 \times N^2$-dimensional matrices:

$$\begin{aligned} (Rz)^{i_1}_{k_1} &= R^{i_1 i_2}_{k_1 k_2} z^{i_2}_{k_2} \ , \\ (RS)^{i_1 i_2}_{k_1 k_2} &= R^{i_1 i_3}_{k_1 k_3} S^{i_3 i_2}_{k_3 k_2} \ . \end{aligned} \tag{13.3}$$

Thus, the upper indices i are related to the action in the first space V in $V \otimes V$, and the lower indices k are related to the second space. The tensor product of the two $N \times N$ matrices A and B is defined as

$$(A \otimes B)^{i_1 i_2}_{k_1 k_2} \equiv A_{i_1 i_2} B_{k_1 k_2} \ . \tag{13.4}$$

The unit matrix E acting in the tensor product space is the tensor product of the unit matrices $I (I_{i_1 i_2} = \delta_{i_1 i_2})$ in the component spaces:

$$E = I \otimes I \ , \qquad E^{i_1 i_2}_{k_1 k_2} = \delta_{i_1 i_2} \delta_{k_1 k_2} \ . \tag{13.5}$$

Another important matrix is the permutation matrix Π defined by the relation

$$\Pi(A\otimes B)\Pi = (I\otimes A)(B\otimes I)\ , \qquad \Pi^{\,i_1 i_2}_{k_1 k_2} = \delta_{i_1 k_2}\delta_{k_1 i_2} \tag{13.6}$$

for any $N\times N$ matrices A, B. It is to be noted that if the matrix elements of matrices A and B commute (which will not always be the case !), then also

$$\Pi(A\otimes B)\Pi = (B\otimes A) \qquad (\text{if } A_{i_1 i_2}B_{k_1 k_2} = B_{k_1 k_2}A_{i_1 i_2}). \tag{13.7}$$

If the matrix elements of matrices B and C do commute, then the following relation holds true

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD) \qquad (\text{if } B_{k_1 k_2}C_{i_1 i_2} = C_{i_1 i_2}B_{k_1 k_2}). \tag{13.8}$$

Let us also introduce the $N^2\times N^2$ matrix of the Poisson brackets, $\{A\overset{\otimes}{,}B\}$, between the matrix elements of the $N\times N$ matrices A and B:

$$\Big\{A\overset{\otimes}{,}B\Big\}^{i_1 i_2}_{k_1 k_2} \equiv \Big\{A_{i_1 i_2}, B_{k_1 k_2}\Big\}. \tag{13.9}$$

Consider in more detail the case of $N=2$ (related, in particular, to the NS equation). In this case, A and B are 2×2 matrices; their tensor product is a 4×4 matrix, $S = A\otimes B$; $S^{i_1 i_2}_{k_1 k_2} = A_{i_1 i_2}B_{k_1 k_2}$. We will arrange the matrix elements of matrix S in the following way

$$S = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}. \tag{13.10}$$

In other words, S is a "block" 2×2-matrix, whose matrix elements are also 2×2 matrices,

$$S = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix} = (A\otimes B). \tag{13.11}$$

In the same way, one can represent any 4×4 matrix R by matrix elements $R^{i_1 i_2}_{k_1 k_2}$

$$R = \begin{pmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{pmatrix}, \tag{13.12}$$

where $R^{i_1 i_2}$ are 2×2 matrices with the matrix elements given as $(R^{i_1 i_2})_{k_1 k_2} = R^{i_1 i_2}_{k_1 k_2}$. In particular, the permutation matrix Π for $N=2$ is written as

$$\begin{aligned} \Pi &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+\sigma_3) & \sigma_- \\ \sigma_+ & \frac{1}{2}(1-\sigma_3) \end{pmatrix} \\ &= \frac{1}{2} + \frac{1}{2}\sigma_3\otimes\sigma_3 + \sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+ \\ &= \frac{1}{2}(I + (\vec{\sigma}\otimes\vec{\sigma}))\ , \end{aligned} \tag{13.13}$$

where $\sigma_{1,2,3}$ are the Pauli matrices ($\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$).

Often, different notations are useful. Let us relate upper indices (1) and (2) to the corresponding spaces in the tensor product $V \otimes V$. Then $\overset{(1)}{A}$ denotes a matrix acting nontrivially only in the first space,

$$\overset{(1)}{A} = \Big(A \otimes I\Big) , \qquad \left(\overset{(1)}{A}\right)^{i_1 i_2}_{k_1 k_2} = A_{i_1 i_2}\delta_{k_1 k_2},$$

$\overset{(2)}{A}$ stands for the matrix

$$\overset{(2)}{A} = I \otimes A , \qquad \left(\overset{(2)}{A}\right)^{i_1 i_2}_{k_1 k_2} = \delta_{i_1 i_2} A_{k_1 k_2} .$$

Also

$$\overset{(1)(2)}{A\,B} = (A \otimes I)(I \otimes B) = (A \otimes B), \quad \text{etc.}$$

These notations are especially convenient to use if tensor products of more than two spaces are under consideration. Let us consider the product of three vector spaces, $V \otimes V \otimes V$, relating indices (1), (2), (3) to the components. Matrix elements of matrix M acting in this space are naturally labeled as

$$M^{\,i_1 i_2}_{\,k_1 k_2\ .}\,{}_{l_1 l_2}$$

Then $\overset{(12)}{R}$ denotes the matrix

$$\overset{(12)}{R} = R \otimes I , \qquad \left(\overset{(12)}{R}\right)^{\,i_1 i_2}_{\,k_1 k_2}\,{}_{l_1 l_2} = R^{i_1 i_2}_{k_1 k_2}\delta_{l_1 l_2} , \qquad \left(\overset{(13)}{R}\right)^{\,i_1 i_2}_{\,k_1 k_2}\,{}_{l_1 l_2} = R^{i_1 i_2}_{l_1 l_2}\delta_{k_1 k_2} , \qquad \text{etc.}$$

14 The classical r-matrix.

The notion of classical r-matrix is the main point for further quantization. Consider a transition matrix $T(x, y|\lambda)$ satisfying the equation

$$(\partial_x + V(x|\lambda))T(x, y|\lambda) = 0 , \qquad T(y, y|\lambda) = I . \tag{14.1}$$

The transition matrix T and "potential" V are the $N \times N$ matrices whose matrix elements depend on x, y via the dynamical variables of the nonlinear problem ("fields"). E.g., for the NS equation ($N = 2$) the potential is

$$V(x|\lambda) = \begin{pmatrix} \frac{i\lambda}{2} & i\sqrt{c}\,\psi^*(x) \\ -i\sqrt{c}\,\psi(x) & -\frac{i\lambda}{2} \end{pmatrix}, \tag{14.2}$$

the transition matrix $T(x,y|\lambda)$ being a matrix-valued functional depending on values of fields $\psi(z)$, $\psi^*(z)$ in the interval $x \geq z \geq y$.

Usually, one calculates rather easily the Poisson brackets (PB) of the matrix elements of matrix V, $\{V_{i_1 i_2}(x|\lambda),\, V_{k_1 k_2}(y|\lambda)\}$, i.e., the matrix elements of the matrix $\{V(x|\lambda) \stackrel{\otimes}{,} V(y|\lambda)\}$ (see equation (13.9) for the definition). E.g., for the NS equation, using the canonical Poisson brackets (1.5), one calculates

$$\Big\{V(x|\lambda) \stackrel{\otimes}{,} V(y|\mu)\Big\} = \delta(x-y) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -ic & 0 \\ 0 & ic & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14.3}$$

The transition matrix T is a rather complicated functional (which is nonlocal in fields) of potentials, and it is very difficult to calculate this object explicitly. The PB between the matrix elements of T, however, can be easily calculated, if the r-matrix $r(\lambda,\mu)$ exists. It is an $N^2 \times N^2$ matrix acting in $V \otimes V$. Its matrix elements depend on spectral parameters λ, μ, but do not depend on the potentials, or on x, y explicitly. The existence of the r-matrix for a given model is not *a priori* evident. If it exists for a model, the model is integrable; in particular, its equations of motion can be represented in the Lax form (see Section 16).

Let us formulate and prove the following **Theorem.**

If the PB between the "potentials" can be represented in the form

$$\Big\{V(z_1|\lambda) \stackrel{\otimes}{,} V(z_2|\mu)\Big\} = \delta(z_1 - z_2)\Big[r(\lambda,\mu), V(z_1|\lambda) \otimes I + I \otimes V(z_1|\mu)\Big], \tag{14.4}$$

where

$$y \leq z_1, z_2 \leq x,$$

then the PB between the matrix elements of the transition matrix is given as

$$\Big\{T(x,y|\lambda) \stackrel{\otimes}{,} T(x,y|\mu)\Big\} = \Big[T(x,y|\lambda) \otimes T(x,y|\mu), r(\lambda,\mu)\Big]. \tag{14.5}$$

In the right-hand sides of (14.4) and (14.5), there are commutators of two $N^2 \times N^2$ dimensional matrices.

Proof.

(a) It has already been shown that besides equation (14.1) in x, the transition matrix $T(x,y|\lambda)$ satisfies also the differential equation (11.4) in y,

$$\partial_y T(x,y|\lambda) - T(x,y|\lambda)V(y|\lambda) = 0, \qquad T(x,x|\lambda) = I. \tag{14.6}$$

(b) The transition matrix depends on the dynamical variables of the nonlinear equation via the matrix elements of the potential V. Having in mind the definition

and the properties of the PB (see, e.g., Section 1), one concludes that

$$\Big\{T_{i_1i_2}(x,y|\lambda), T_{k_1k_2}(x,y|\mu)\Big\} = \\ = \int\limits_y^x dz_1 \int\limits_y^x dz_2 \, \frac{\delta T_{i_1i_2}(x,y|\lambda)}{\delta V_{p_1p_2}(z_1|\lambda)} \, \frac{\delta T_{k_1k_2}(x,y|\mu)}{\delta V_{q_1q_2}(z_2|\mu)} \Big\{V_{p_1p_2}(z_1|\lambda), V_{q_1q_2}(z_2|\mu)\Big\}. \tag{14.7}$$

(summing up over the repeated indices is implied).

(c) Let us calculate the functional derivatives. To do this, one should calculate the first variation δT due to changing the potential, $V \to V + \delta V$. It is easy to see that to first order in δV one has the following equation for the variation:

$$\Big(\partial_x + V(x|\lambda)\Big)\delta T(x,y|\lambda) = -\delta V(x|\lambda) T(x,y|\lambda) \\ \delta T(y,y|\lambda) = 0 \tag{14.8}$$

so that

$$\delta T(x,y|\lambda) = -\int\limits_y^x dz \, T(x,z|\lambda)\delta V(z|\lambda) T(z,y|\lambda) \tag{14.9}$$

(it is easily checked that (14.8) is fulfilled for δT given by (14.9)). From the last equation, one writes for the functional derivative:

$$\frac{\delta T_{i_1i_2}(x,y|\lambda)}{\delta V_{p_1p_2}(z|\lambda)} = -\, T_{i_1p_1}(x,z|\lambda)\, T_{p_2i_2}(z,y|\lambda). \tag{14.10}$$

Putting (14.10) into (14.7) and rewriting the result in the matrix notations of Section 13, one has for the PB of the transition matrices

$$\Big\{T(x,y|\lambda) \stackrel{\otimes}{,} T(x,y|\mu)\Big\} = \int\limits_y^x dz_1 \int\limits_y^x dz_2 \Big(T(x,z_1|\lambda) \otimes T(x,z_2|\mu)\Big) \cdot \\ \cdot \Big\{V(z_1|\lambda) \stackrel{\otimes}{,} V(z_2|\mu)\Big\}\Big(T(z_1,y|\lambda) \otimes T(z_2,y|\mu)\Big).$$

(d) Now taking into account the known PB (14.4) for the potentials, one easily performs the integration over z_2, due to the δ-function. Then using equations (14.1) and (14.6), it is not difficult to exclude the potentials ($V(z)T(z,y) = -\partial_z T(z,y)$, $T(x,z)V(z) = \partial_z T$) and to represent the integrand as a derivative:

$$\Big\{T(x,y|\lambda) \stackrel{\otimes}{,} T(x,y|\mu)\Big\} \\ = -\int\limits_y^x dz \, \frac{d}{dz}\Big((T(x,z|\lambda) \otimes T(x,z|\mu)\Big) r(\lambda,\mu)\Big(T(z,y|\lambda) \otimes T(z,y|\mu)\Big).$$

Taking into account that $T(x,x) = T(y,y) = I$, one gets (14.5). Thus, the theorem is proved.

Let us also give the formulation of the theorem proved above using different notations introduced in the end of Section 13. It is convenient to relate spectral parameter λ to the first component of the tensor product $V \otimes V$ of two vector spaces, and μ to the second component and not to write the spectral parameters explicitly, having in mind that, e.g.,

$$\overset{(12)}{r} \equiv \overset{(12)}{r}(\lambda,\mu), \quad \overset{(1)}{T}(x,y) \equiv \overset{(1)}{T}(x,y|\lambda), \quad \overset{(2)}{V}(x) \equiv \overset{(2)}{V}(x|\mu).$$

Then the theorem is formulated in the following way.

If

$$\left\{\overset{(1)}{V}(z_1), \overset{(2)}{V}(z_2)\right\} = \delta(z_1 - z_2)\left[\overset{(12)}{r}, \overset{(1)}{V}(z_1) + \overset{(2)}{V}(z_1)\right], \quad y \le z_1, z_2 \le x,$$

then

$$\left\{\overset{(1)}{T}(x,y), \overset{(2)}{T}(x,y)\right\} = \left[\overset{(1)}{T}(x,y)\overset{(2)}{T}(x,y), \overset{(12)}{r}\right]. \tag{14.11}$$

Let some $N^2 \times N^2$ matrix $r(\lambda,\mu)$ be given. Can this matrix be the r-matrix for some classical integrable system? The answer is, generally speaking, 'no'. From the Jacobi identities for the PBs, it can be derived that the classical r-matrix should satisfy the following functional equation which is called "the classical Yang-Baxter equation":

$$\left[\overset{(12)}{r}, \overset{(13)}{r}\right] + \left[\overset{(12)}{r}, \overset{(23)}{r}\right] + \left[\overset{(13)}{r}, \overset{(23)}{r}\right] = 0. \tag{14.12}$$

It involves three spectral parameters λ, μ, ν related to the corresponding components of the tensor product $V \otimes V \otimes V$ of the three vector spaces:

$$\overset{(12)}{r} \equiv \overset{(12)}{r}(\lambda,\mu) = r(\lambda,\mu) \otimes I, \qquad \left(\overset{(12)}{r}\right)^{\substack{i_1 i_2 \\ k_1 k_2 \\ l_1 l_2}} \equiv r^{i_1 i_2}_{k_1 k_2}(\lambda,\mu)\delta_{l_1 l_2},$$

$$\overset{(23)}{r} \equiv \overset{(23)}{r}(\mu,\nu) \equiv I \otimes r(\mu,\nu), \qquad \left(\overset{(23)}{r}\right)^{\substack{i_1 i_2 \\ k_1 k_2 \\ l_1 l_2}} \equiv r^{k_1 k_2}_{l_1 l_2}(\mu,\nu)\delta_{i_1 i_2},$$

$$\overset{(13)}{r} \equiv \overset{(13)}{r}(\lambda,\nu), \qquad \left(\overset{(13)}{r}\right)^{\substack{i_1 i_2 \\ k_1 k_2 \\ l_1 l_2}} \equiv r^{i_1 i_2}_{l_1 l_2}(\lambda,\nu)\delta_{k_1 k_2}.$$

The classical Yang-Baxter equation is not derived in this Section. Later it will be obtained as a quasiclassical limit of the quantum Yang-Baxter equation (see Section 22).

Example.

For the NS equation ($N = 2$), the PBs between the potentials V are given by (14.3), and it is not difficult to find the r-matrix using the definition (14.4). It is

$$r(\lambda,\mu) = \frac{c}{\lambda - \mu}\Pi\ , \tag{14.13}$$

where Π is the permutation matrix (13.13). It is indeed easy to check that it satisfies the classical Yang-Baxter equation for any value of the parameters.

It can be shown that formula (14.13) gives the solution of the classical YB equation also in the case $N > 2$, where r is an $N^2 \times N^2$ matrix. This is easy to establish using the obvious identities

$$\left(\overset{(ab)}{\Pi}\right)^2 = \overset{(ab)}{E}, \qquad \overset{(ab)}{\Pi}\,\overset{(cb)}{\Pi}\,\overset{(ab)}{\Pi} = \overset{(ac)}{\Pi},$$

and

$$\frac{1}{(\lambda-\mu)}\cdot\frac{1}{(\lambda-\nu)} = \frac{1}{\mu-\nu}\left[\frac{1}{\lambda-\mu} - \frac{1}{\lambda-\nu}\right].$$

15 Trace identities and conservation laws.

Consider the monodromy matrix $T(\lambda) \equiv T(x = L; y = 0|\lambda)$. The Poisson brackets between its matrix elements are given by the r-matrix

$$\Big\{T(\lambda) \overset{\otimes}{,} T(\mu)\Big\} = \Big[T(\lambda)\otimes T(\mu), r(\lambda,\mu)\Big]. \tag{15.1}$$

Consider the trace tr of the $N \times N$ matrix $T(\lambda)$,

$$\tau(\lambda) \equiv \operatorname{tr} T(\lambda) = \sum_{j=1}^{N} T_{jj}(\lambda), \tag{15.2}$$

which is a scalar functional of the dynamical variables. Now let us take the trace of the both sides of equation (15.1). The trace "Tr" in the $N^2 \times N^2$ dimensional space of the tensor product of two $N \times N$ matrices is the product of the traces "tr" of these matrices,

$$\operatorname{Tr}(A\otimes B) = \sum_{i=1}^{N}\sum_{k=1}^{N} A_{ii}B_{kk} = \operatorname{tr} A\cdot\operatorname{tr} B.$$

Taking also into account that the trace of the commutator of any two finite-dimensional matrices equals zero, one gets remarkable result: the existence of the r-matrix ensures that the traces of the monodromy matrix at any points λ and μ have the Poisson brackets (PB) equal to zero:

$$\{\tau(\lambda), \tau(\mu)\} = 0, \quad \forall\lambda,\mu \tag{15.3}$$

This relation is of primary importance. It results in the existence of infinitely many conservation laws for the nonlinear dynamical system.

Turn for an example to the NS equation. In Section 12 we got the "trace identities", i.e., the following expansion for $\tau(\lambda)$:

$$\ln\left(\tau(\lambda)\right)\Big|_{\lambda\to+i\infty} = -\frac{i\lambda}{2}L + ic\sum_{n=1}^{\infty}\lambda^{-n}I_n, \qquad I_1 \equiv Q, \quad I_2 = P, \quad I_3 = H, \quad \ldots \tag{15.4}$$

(each I_n in this expansion is a local λ-independent functional of fields $\psi(x)$, $\psi^*(x)$; $0 \le x \le L$). The PBs between the matrix elements of the monodromy matrix are given by formula (15.1), with the r-matrix $r(\lambda,\mu)$ (14.13). So equation (15.3) is fulfilled for $\tau(\lambda)$ in the NS model. Putting expansion (15.4) into (15.3), one concludes that the PBs between I_n's are all equal to zero. In particular, the PB of any I_n with the Hamiltonian equals zero:

$$\{I_n, I_m\} = 0, \quad \{I_n, H\} = 0. \tag{15.5}$$

Thus, we have proved the existence of infinitely many conservation laws in the NS model just from the fact that the r-matrix exists for it.

This is a general fact: if one can express the Hamiltonian of the nonlinear system in terms of $\tau(\lambda)$ (the corresponding formulae are called "trace identities"), and if there exists the r-matrix for this system, then the functional $\tau(\lambda)$ generates many conservation laws for the nonlinear dynamical system under consideration. Strictly speaking, this fact by itself does not mean yet the complete integrability of the nonlinear system; for the complete integrability it is necessary to have a "complete" set of integrals of motion. One can prove that in the case of the NS equation a complete set of conservation laws is thus obtained, and therefore the model is completely integrable.

It is to be said that getting trace identities in many cases requires a lot of work, and there is no general principle allowing to express a given Hamiltonian in terms of $\tau(\lambda)$ (except for some kind of models).

16 On the r-matrix and the M-operator.

The starting point for solving a nonlinear dynamical equation by the inverse scattering method is to represent it in the Lax form (9.1), (9.2). Till now, we used mostly equation (9.1), $(\partial_x + V)\Phi = 0$, containing the space derivative, and practically did not use equation (9.2), $(\partial_t + U)\Phi = 0$. The remarkable fact is that the r-matrix (if there are also trace identities expressing explicitly the Hamiltonian in terms of $\tau(\lambda)$) "replaces" this second equation. Intuitively, it's quite obvious, since the time derivative can be replaced by the PB with the Hamiltonian given by the r-matrix. Formally, one can represent the PB of $\tau(\mu)$ with $V(x|\lambda)$ as

$$\Big\{\tau(\mu), V(x|\lambda)\Big\} = \partial_x\tilde{U}(x|\lambda,\mu) + \Big[V(x|\lambda), \tilde{U}(x|\lambda,\mu)\Big] \tag{16.1}$$

($\tau(\mu)$ is a scalar, V and $\tilde{U}$ are $N \times N$ matrices), where, in the notations introduced in the end of Section 13,

$$\tilde{U}(x|\lambda,\mu) = \mathrm{tr}_1(\overset{(1)}{T}(L,x|\mu)\overset{(12)}{r}(\mu,\lambda)\overset{(1)}{T}(x,0,|\mu)). \tag{16.2}$$

Here "tr_1" means the trace in the first matrix space (of dimension $N \times N$) of the $N^2 \times N^2$ matrix; for any $N^2 \times N^2$ matrix A,

$$\mathrm{tr}_1\,(A_{k_1k_2}) \equiv \sum_i A^{ii}_{k_1k_2},$$

so that $\mathrm{tr}_1\,A$ is an $N \times N$ matrix, $\mathrm{tr}_1\,A$: $V \to V$. The proof of the equation (16.1) is a good exercise for the reader.

Equation (16.1) is quite similar to the zero curvature condition (9.3), it generates, in fact, many different integrable equations.

As already discussed in Section 15, the Hamiltonian of an integrable system is usually expressed in a natural way in terms of the logarithm of the trace of the monodromy matrix. For any function $F(\tau(\mu))$, due to the definition of the Poisson brackets (see, e.g., equation (1.4) for the case of the NS equation), its PB with any functional A of dynamical variables is given as

$$\Big\{F(\tau(\mu)), A\Big\} = F'_\tau(\tau(\mu))\Big\{\tau(\mu), A\Big\},$$

where F'_τ denotes the derivative of F with respect to τ (recall that $\{\tau(\mu_1), \tau(\mu_2)\} = 0$).

So one readily rewrites the relation (16.1) in the form

$$\Big\{\ln\tau(\mu), V(x|\lambda)\Big\} = \partial_x U(x|\lambda,\mu) + \Big[V(x|\lambda), U(x|\lambda,\mu)\Big], \tag{16.4}$$

where the $N \times N$ matrix $U(x|\lambda,\mu)$ ($U(x|\lambda,\mu) : V \to V$) is defined as the product of the matrix $\tilde{U}(x|\lambda,\mu)$ by the scalar $\tau^{-1}(\mu)$,

$$U(x|\lambda,\mu) = \tau^{-1}(\mu)\tilde{U}(x|\lambda,\mu). \tag{16.5}$$

Let us demonstrate explicitly how the operator $U(x|\lambda,\mu)$ generates the operator $U(x|\lambda)$ (9.5) entering the Lax representation for the NS equation. The r-matrix for the NS equation is given by equation (14.13) so that one obtains the matrix $U(x|\lambda,\mu)$, up to a scalar factor, as the product of two transition matrices, $T(x,0|\mu)$ and $T(L,x|\mu)$ (which are 2×2 matrices for the NS equation):

$$U(x|\lambda,\mu) = \frac{c}{\mu-\lambda}\tau^{-1}(\mu)\left(T(x,0|\mu)T(L,x|\mu)\right). \tag{16.6}$$

For the NS equation one has (see (15.4)) the following trace identities:

$$\ln\tau(\mu)\,|_{\mu\to+i\infty} = -\frac{i\mu L}{2} + ic\sum_{n=1}^{\infty}\mu^{-n}I_n, \qquad\qquad I_3 \equiv H. \tag{16.7}$$

So to calculate the quantity $\{H, V\} = \partial_t V$ from equation (16.4), one needs the coefficient U_3 at μ^{-3} in the expansion,

$$U(x|\lambda, \mu) = ic \sum_{n=1}^{\infty} \mu^{-n} U_n(x|\lambda), \tag{16.8}$$

at $\mu \to +i\infty$. It is not difficult to get expansion (16.8) by means of the technique of "diagonalizing" the gauge transformation described in detail in Section 12. Using equation (12.9) and taking into account the fact that $G(0|\mu) = G(L|\mu)$ due to the periodic boundary conditions, as well as the commutativity of the diagonal matrices D, and the property

$$D(x, 0|\mu) D(L, x|\mu) = D(L, 0|\mu),$$

one obtains the following representation

$$U(x|\lambda, \mu) = \frac{c}{\mu - \lambda} \tau^{-1}(\mu) G(x|\mu) D(L, 0|\mu) G^{-1}(x|\mu).$$

At $\mu \to +i\infty$, the ratio of the matrix elements, $D_{22}(L, 0|\mu)/D_{11}(L, 0|\mu) = O(\mu^{-\infty})$, is small, and one gets the following asymptotical equality,

$$U(x|\lambda, \mu) = \frac{c}{\mu} \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n G(x|\mu) \frac{1}{2}(I + \sigma_3) G^{-1}(x|\mu).$$

The unit matrix I in the right-hand side can be omitted, since the corresponding contribution does not depend on x, and I commutes with any 2×2 matrix. It does not change the validity of the "Lax representation" (16.4) (or, of (9.3)). Thus, equivalently, one can use the following representation for matrix $U(x|\lambda, \mu)$

$$U(x|\lambda, \mu) = \frac{c}{2\mu} \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n G(x|\mu) \sigma_3 G^{-1}(x|\mu). \tag{16.9}$$

The matrix $G(x|\mu)$ has the asymptotical expansion (12.10), the expansion for the matrix $G^{-1}(x|\mu)$ is

$$G^{-1}(x|\mu) = I + \sum_{n=1}^{\infty} \mu^{-n} G_n^{(-1)}(x),$$

and one easily calculates the coefficients,

$$G_1^{(-1)} = -G_1, \quad G_2^{(-1)} = -G_2 + G_1^2, \quad \dots \ .$$

Now, it is quite easy to calculate the coefficient U_3 at the expansion (16.8) with the result

$$U_3(x|\lambda) = U(x|\lambda),$$

where $U(x|\lambda)$ is given just by equation (9.5) entering the Lax representation (9.3) for the NS equation.

Functional $U(x|\lambda, \mu)$ generates, in fact, the "hierarchy" of the "higher" NS equations. If one takes I_n ($n \neq 3$) as a new Hamiltonian, then one obtains the Lax representation of a new integrable equation (with the same operator $V(x|\lambda)$, but with different operator $U(x|\lambda)$) which is given by the corresponding coefficient in expansion (16.8)).

17 The Lax representation for lattice systems.

Consider a dynamical system given on a one-dimensional space lattice with sites m, $m \in \mathbf{Z}$ (the time variable remains continuous). The auxiliary $N \times N$ matrix Φ of the Lax representation is now $\Phi(m, t|\lambda)$ (instead of $\Phi(x, t|\lambda)$ in (9.1), (9.2)), and the representation is written in the form

$$\Phi(m+1, t|\lambda) = L(m, t|\lambda)\Phi(m, t|\lambda), \tag{17.1}$$

$$(\partial_t + U(m, t|\lambda))\Phi(m, t|\lambda) = 0. \tag{17.2}$$

The $N \times N$ matrix $L(m, t|\lambda)$ defines the shift of function Φ by one step of the lattice. It is usually called "the L-operator". The matrix elements of matrices $L(m, t|\lambda)$ and $U(m, t|\lambda)$ depend on distance m and time t via the dynamical variables of the nonlinear dynamical system.

The compatibility condition for the system (17.1), (17.2) reads (compare with (9.3) in the continuous case):

$$\partial_t L(m, t|\lambda) = L(m, t|\lambda)U(m, t|\lambda) - U(m+1, t|\lambda)L(m, t|\lambda). \tag{17.3}$$

Remark. One can also consider the case where the time t is also discrete, $t \in \mathbf{Z}$. In this case, the Lax representation contains an M-operator, analogous to the L-operator,

$$\Phi(m+1, t|\lambda) = L(m, t|\lambda)\Phi(m, t|\lambda),$$

$$\Phi(m, t+1|\lambda) = M(m, t|\lambda)\Phi(m, t|\lambda),$$

and the compatibility condition reads

$$M(m+1, t|\lambda)L(m, t|\lambda) - L(m, t+1|\lambda)M(m, t|\lambda) = 0.$$

We shall not consider this kind of systems further, though it appears to be rather useful for the description of quantum integrable systems.

Let us turn again to the systems with discrete space and continuous time variables. There are many models of this kind which are of fundamental importance in physics; an example is the Heisenberg spin chain, further considered in more detail. It is often very convenient (especially for the quantization) to put onto a lattice even a continuous system. Consider a classical continuous system (e.g., the NS equation) on the interval $0 \le x \le L$, introducing an infinitesimal lattice with M sites and spacing Δ: $M\Delta = L$, $\Delta \to 0$, $M \to \infty$.

Let $x_m = m\Delta$ be the coordinate of the m-th site. Function $\Phi(x, t|\lambda)$ is approximated by $\Phi(m, t|\lambda) \equiv \Phi(x_m, t|\lambda)$, and it follows from (9.1) that

$$\Phi(m+1, t|\lambda) = L(m, t|\lambda)\Phi(m, t|\lambda), \tag{17.4}$$

where (I denotes the unit 2×2 matrix)

$$L(m, t|\lambda) = I - V(x_m, t|\lambda)\Delta + 0(\Delta^2), \quad \Delta \to 0 \tag{17.5}$$

is an infinitesimal L-operator. The exact expression for the L-operator is, of course,

$$L(m, t|\lambda) = T(x_{m+1}, x_m|\lambda),$$

where T is the transition matrix from the point x_m to the point x_{m+1} for the continuous NS equation. In the limit $\Delta \to 0$, $M \to \infty$ (and $\Delta M = L$ fixed), all the formulae of the continuous case are restored.

The transition matrix in the lattice case is defined as the product of the corresponding L-operators

$$T(n, m|\lambda) = L(n|\lambda)L(n-1|\lambda)\dots L(m+1|\lambda)L(m|\lambda) \equiv \prod_{j=m}^{\overleftarrow{n}} L(j|\lambda). \tag{17.6}$$

It describes the transition from the m-th site to the $(n+1)$-th site of the lattice:

$$\Phi(n+1|\lambda) = T(n, m|\lambda)\Phi(m|\lambda). \tag{17.7}$$

In particular, the L-operator itself is an elementary transition matrix which gives the transition for one step of the lattice, $L(n|\lambda) = T(n, n|\lambda)$; see equation (17.1).

The lattice monodromy matrix $T(\lambda)$ gives the transition over all the sites of the lattice,

$$T(\lambda) = \prod_{m=1}^{\overleftarrow{M}} L(m|\lambda). \tag{17.8}$$

Usually, the Hamiltonian of the lattice nonlinear dynamical system is expressed in terms of the trace $\tau(\lambda)$ of the monodromy matrix,

$$\tau(\lambda) = \operatorname{tr} T(\lambda), \tag{17.9}$$

by means of the trace identities.

The main theorem regarding the r-matrix is formulated for lattice systems in the following way.

If the PB's of the matrix elements of L-operators at different sites of the lattice equal zero, and at the same site they are given by the r-matrix, i.e.,

$$\Big\{ L(k|\lambda) \overset{\otimes}{,} L(l|\mu) \Big\} = \delta_{kl} \Big[L(k|\lambda) \otimes L(k|\mu), r(\lambda, \mu) \Big], \quad m \le k, l \le n, \tag{17.10}$$

then the PBs of matrix elements of the transition matrix are also given by the r-matrix as

$$\Big\{ T(n, m|\lambda) \overset{\otimes}{,} T(n, m|\mu) \Big\} = \Big[T(n, m|\lambda) \otimes T(n, m|\mu), r(\lambda, \mu) \Big]. \tag{17.11}$$

The proof is left as an exercise for the reader.

As an example, let us consider in more detail the NS model, putting it onto a lattice. For the NS model, the r-matrix is a 4×4 matrix which is proportional to the permutation matrix,

$$\Big[T(\lambda) \mathop{\otimes}_{,} T(\mu)\Big] = \Big[T(\lambda) \otimes T(\mu), r(\lambda,\mu)\Big],$$

$$r(\lambda,\mu) = \frac{c}{\lambda-\mu}\Pi, \qquad \Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{17.12}$$

The infinitesimal L-operator is given by formula (17.5) with $L(m|\lambda)$ being a 2×2-matrix,

$$L(n|\lambda) = I - V(x_n|\lambda)\cdot\Delta + O(\Delta^2) = \begin{pmatrix} 1 - \dfrac{i\lambda\Delta}{2} & -i\sqrt{c}\,\psi_n^*\Delta \\ i\sqrt{c}\,\psi_n\Delta & 1 + \dfrac{i\lambda\Delta}{2} \end{pmatrix} + O(\Delta^2). \tag{17.13}$$

The local boson fields ψ_n, ψ_n^* on the lattice are defined as

$$\psi_n^{\pm} \equiv \frac{1}{\Delta}\int\limits_{x_{n-1}}^{x_n} \psi^{\pm}(x)dx \approx \psi^{\pm}(x_n) \qquad (\psi^+ \equiv \psi^*,\ \psi^- \equiv \psi),$$

with Poisson brackets

$$\{\psi_n, \psi_m\} = \{\psi_m^*, \psi_n^*\} = 0, \qquad \{\psi_n, \psi_m^*\} = \frac{1}{\Delta}\delta_{mn}. \tag{17.14}$$

It is easy to calculate explicitly (or, to understand after recalling the formulae for the continuous case) that

$$\begin{aligned} &\Big\{L(n|\lambda) \mathop{\otimes}_{,} L(m|\mu)\Big\} = 0, \qquad m \neq n, \\ &\Big\{L(n|\lambda) \mathop{\otimes}_{,} L(n|\mu)\Big\} = \Big[L(n|\lambda) \otimes L(m|\mu), r(\lambda,\mu)\Big] + O(\Delta^2), \end{aligned} \tag{17.15}$$

with the r-matrix (17.12).

Let us put a question: can one write the "exact" lattice L-operator (linear in λ) for this r-matrix which goes to the infinitesimal L-operator (17.13)? The answer is 'yes'. The L-operator

$$L(n|\lambda) = \begin{pmatrix} 1 - \dfrac{i\lambda\Delta}{2} + \dfrac{c\Delta^2}{2}\psi_n^*\psi_n & -i\sqrt{c}\,\Delta\psi_n^*\rho_n^+ \\ i\sqrt{c}\,\Delta\rho_n^-\psi_n & 1 + \dfrac{i\lambda\Delta}{2} + \dfrac{c\Delta^2}{2}\psi_n^*\psi_n \end{pmatrix} \tag{17.16}$$

satisfies relation (17.10) exactly, if $\rho_n^{\pm}$ are the functions of the product $\psi_n^*\psi_n$ satisfying the relation

$$\rho_n^+\rho_n^- = 1 + \frac{c\Delta^2}{4}\psi_n^*\psi_n \tag{17.17}$$

In particular, one can choose $\rho_n^+ = \rho_n^- = \sqrt{1 + \frac{c\Delta^2}{4}\psi_n^*\psi_n}$ (which is well defined if $c > 0$).

18 The quantum inverse scattering method. The quantum monodromy matrix and the transfer matrix.

Our aim now is to quantize the scheme of the inverse scattering method. It is convenient to consider models on a periodical one-dimensional space lattice with M sites ($m = 1, ..., M$). For continuous models, one can use the infinitesimal lattice, as explained in the previous Section. Let us consider some quantum nonlinear dynamical system for which the equations of motion can be represented in the Lax form (the time argument t will usually not be written explicitly):

$$\Phi(m+1|\lambda) = L(m|\lambda)\Phi(m|\lambda), \tag{18.1}$$

$$\Big(\partial_t + U(m|\lambda)\Big)\Phi(m|\lambda) = 0. \tag{18.2}$$

In the quantum case the dynamical variables of the model are noncommuting operators. In the case of the NS model, e.g., one has the operators ψ^+ and ψ with the commutation relations (2.1) instead of the classical fields with Poisson brackets (1.5). The Hamiltonian H of the quantum model and the matrix elements of $N \times N$ matrices $L(m|\lambda)$, $U(m|\lambda)$ and $\Phi(m|\lambda)$ are now quantum operators. So one has to consider matrices L, U, and Φ with noncommuting matrix elements, these matrix elements being quantum operators acting in the linear space where the Hamiltonian H acts.

The transition matrix $T(n, m|\lambda)$ from the m-th site to the $(n+1)$-th site of the lattice,

$$T(n, m|\lambda) \equiv L(n|\lambda)L(n-1|\lambda)\dots L(m+1|\lambda)L(m|\lambda), \quad T(n, n|\lambda) \equiv L(n|\lambda), \tag{18.3}$$

and the monodromy matrix $T(\lambda)$ corresponding to the transition over the whole lattice,

$$T(\lambda) \equiv T(M, 1|\lambda)\,, \tag{18.4}$$

are defined similarly to the classical case. The trace $\tau(\lambda)$ of the monodromy matrix in the matrix space, i.e., the sum of its diagonal elements,

$$\tau(\lambda) = \mathrm{tr}\, T(\lambda) = \sum_{i=1}^{N} T_{ii}(\lambda)\ , \tag{18.5}$$

is called "the transfer matrix" in the quantum case (though it is a scalar in the matrix space!). Usually, the Hamiltonian of the model is expressed in terms of $\tau(\lambda)$, similar to the classical case.

The matrix elements T_{ik} of the monodromy matrix are "quantum operators" acting in the same "quantum space" $\mathcal{H}$ where the Hamiltonian H acts:

$$T_{ik}(\lambda):\ \mathcal{H} \longrightarrow \mathcal{H}\ .$$

Thus, the monodromy matrix $T(\lambda)$ acts in the tensor product of the N-dimensional vector space V (where it acts as an $N \times N$-dimensional matrix) and of the quantum space $\mathcal{H}$;

$$T(\lambda): \ V \otimes \mathcal{H} \longrightarrow V \otimes \mathcal{H} \ ,$$

in other words, $T(\lambda)$ is a matrix with noncommuting matrix elements. Its trace $\tau(\lambda)$—the transfer matrix—is a quantum operator (being a scalar in V):

$$\tau(\lambda): \ \mathcal{H} \to \mathcal{H}$$

The linear operators in $\mathcal{H}$ can be represented as matrices (either finite dimensional or infinite dimensional). To do this, one chooses some orthonormal basis, $|\varphi_\alpha\rangle$, in $\mathcal{H}$ and represents the quantum operator A by its matrix elements, $A^{\alpha\beta} = \langle\varphi_\alpha|A|\varphi_\beta\rangle$. If state $|\Psi\rangle \in \mathcal{H}$ is given as

$$|\Psi\rangle = \sum_\alpha a_\alpha |\varphi_\alpha\rangle \ ,$$

then the action of operator A on $|\Psi\rangle$ corresponds to the action of matrix $A^{\alpha\beta}$ on vector a_α,

$$A|\Psi\rangle \equiv \sum_\alpha b_\alpha |\varphi_\alpha\rangle, \quad b_\alpha \equiv \sum_\beta A^{\alpha\beta} a_\beta \ .$$

The matrix product of the corresponding matrices gives the matrix for the product of operators,

$$(AB)^{\alpha\beta} = \sum_\gamma A^{\alpha\gamma} B^{\gamma\beta}.$$

In this sense, one can say that $T(\lambda)$ is a matrix both in matrix space V (the corresponding matrix indices are i, k) and in quantum space $\mathcal{H}$ (the corresponding "quantum" indices are α, β), each matrix element of $T(\lambda)$ being a matrix in the quantum space, $T_{ik}^{\alpha\beta}(\lambda)$.

19 The quantum R-matrix.

To quantize a model, one usually changes the fundamental Poisson brackets for the commutators. E.g., for the NS model, one changes the PB (1.5) between the fields for the commutators (2.1) between the field operators. The remarkable property of classical integrable models is that the PB between the matrix elements of the monodromy matrix (which are usually very complicated functionals of the basic dynamical variables of the model) can be easily obtained quite explicitly, if an r-matrix exists. To quantize a model of this kind, one should have the quantum R-matrix which gives the commutation relations between the matrix elements of the quantum monodromy matrix. It should be emphasized that the matrix elements of the R-matrix are c-numbers, i.e., they commute with any quantum operator. Below, we give the main theorem, corresponding to the theorem on the classical r-matrix

given in Section 14 (see also (17.10), (17.11) for the lattice case). Consider the transition matrix $T(n,m|\lambda)$ (18.3) of which the monodromy matrix $T(\lambda)$ (18.4) is the most important particular case.

Theorem

If

(i) The matrix elements of the L-operators at different sites of the lattice commute as quantum operators, i.e.,

$$L_{pq}(k|\lambda)L_{rs}(l|\mu) = L_{rs}(l|\mu)L_{pq}(k|\lambda)$$
$$(l \neq k; \quad m \leq k, l \leq n; \quad p,q,r,s = 1,\dots,N); \tag{19.1}$$

(ii) The commutation relations between the matrix elements of all L-operators at the same site are given by the R-matrix as

$$\widetilde{R}(\lambda,\mu)(L(l|\lambda)\otimes L(l|\mu)) = (L(l|\mu)\otimes L(l|\lambda))\widetilde{R}(\lambda,\mu), \qquad (m \leq l \leq n), \tag{19.2}$$

then the commutation relations between the matrix elements of the transition matrix $T(n,m|\lambda)$ (18.3) are given by a similar formula:

$$\widetilde{R}(\lambda,\mu)\Big(T(n,m|\lambda)\otimes T(n,m|\mu)\Big) = \Big(T(n,m|\mu)\otimes T(n,m|\lambda)\Big)\widetilde{R}(\lambda,\mu). \tag{19.3}$$

In particular, for the monodromy matrix $T(\lambda)$ (18.4) one has

$$\widetilde{R}(\lambda,\mu)\Big(T(\lambda)\otimes T(\mu)\Big) = \Big(T(\mu)\otimes T(\lambda)\Big)\widetilde{R}(\lambda,\mu). \tag{19.4}$$

Let us make some comments. The property (19.1) is usually called "ultralocality" and means that the matrix elements of the L-operators do commute as quantum operators, i.e., that the matrix elements of the operator $L(l|\lambda)$ act nontrivially only in its own "local" quantum space $\mathcal{H}_l$ associated with the l-th site of the lattice. The matrix elements of the transition matrix $T(n,m|\lambda)$ act in the tensor product of the corresponding local quantum spaces, i.e., in $\mathcal{H}_{m,n} = \otimes_{l=m}^{n}\mathcal{H}_l$. The matrix elements of the monodromy matrix $T(\lambda)$ are quantum operators acting in quantum space $\mathcal{H}$ which is the tensor product of all the local spaces, $\mathcal{H} = \mathcal{H}_{1,M}$. The transfer matrix $\tau(\lambda) = \operatorname{tr} T(\lambda)$ and the Hamiltonian H of the quantum nonlinear system act also in the same space $\mathcal{H}$.

The tensor product entering equations (19.2)–(19.4) is the tensor product of two $N \times N$ matrices acting each in its own vector space V (the dimension of V is equal to N). So, e.g., matrix $(T(\lambda)\otimes T(\mu))$ in (19.4) is an $N^2 \times N^2$ matrix (acting as a matrix in the space $V \otimes V$) whose matrix elements are noncommuting quantum operators acting in $\mathcal{H}$,

$$(T(\lambda)\otimes T(\mu)):\ V\otimes V\otimes\mathcal{H} \to V\otimes V\otimes\mathcal{H}\,.$$

To be completely precise, let us also write relation (19.4) in explicit form, relating indices i and k to the first and second matrix spaces ("matrix indices"), the Greek indices α being related to the quantum space ("quantum indices", see (18.6)):

$$\Big(\widetilde{R}(\lambda,\mu)\Big)^{i_1 i_2}_{k_1 k_2} T^{\alpha_1\alpha_2}_{i_2 i_3}(\lambda) T^{\alpha_2\alpha_3}_{k_2 k_3}(\mu) = T^{\alpha_1\alpha_2}_{i_1 i_2}(\mu) T^{\alpha_2\alpha_3}_{k_1 k_2}(\lambda) \Big(\widetilde{R}(\lambda,\mu)\Big)^{i_2 i_3}_{k_2 k_3} \ . \qquad (19.4a)$$

The R-matrix $\widetilde{R}(\lambda,\mu)$ is the $N^2 \times N^2$ matrix acting in $V \otimes V$, its matrix elements being not quantum operators, but just complex valued functions of spectral parameters λ and μ. These matrix elements commute between themselves and also with matrix elements of all other matrices (they are "c-numbers").

Proof of the theorem

Expressing the monodromy matrices as matrix product of L-operators by means of (18.3), one rewrites the left hand side of (19.3) as

$$\mathcal{L} \equiv \widetilde{R}(\lambda,\mu)\Big(L(n|\lambda)\dots L(l|\lambda)\dots L(m|\lambda)\Big) \otimes \Big(L(n|\mu)\dots L(l|\mu)\dots L(m|\mu)\Big).$$

The matrix elements of matrices $L(l|\lambda)$ and $L(k|\mu)$ commute if $l \neq k$ (19.1), and using the property (13.8) one can write

$$\mathcal{L} = \widetilde{R}(\lambda,\mu)\Big(L(n|\lambda) \otimes L(n|\mu)\Big) \dots \Big(L(l|\lambda) \otimes L(l|\mu)\Big) \dots \Big(L(m|\lambda) \otimes L(n|\mu)\Big).$$

Now using the commutation rules (19.2) between the L-operator at the same site, one can commute the R-matrix to the right, forming then again the transition matrices from the corresponding L-operators:

$$\begin{aligned}
\mathcal{L} &= \Big(L(n|\mu) \otimes L(n|\lambda)\Big)\widetilde{R}(\lambda,\mu) \dots \Big(L(l|\lambda) \otimes L(l|\mu)\Big) \dots \Big(L(m|\lambda) \otimes L(m|\mu)\Big) \\
&= \Big(L(n|\mu) \otimes L(n|\lambda)\Big) \dots \Big(L(l|\mu) \otimes L(l|\lambda)\Big) \dots \Big(L(m|\mu) \otimes L(m|\lambda)\Big)\widetilde{R}(\lambda,\mu) \\
&= \Big(L(n|\mu)\dots L(l|\mu)\dots L(m|\mu)\Big) \otimes \Big(L(n|\lambda)\dots L(l|\lambda)\dots L(m|\lambda)\Big)\widetilde{R}(\lambda,\mu) \\
&= \Big(T(n,m|\mu) \otimes T(n,m|\lambda)\Big)\widetilde{R}(\lambda,\mu) \ ,
\end{aligned}$$

which is just the right-hand side of (19.3). The theorem is thus proved.

Let us also give the formulation of the theorem using different notations introduced at the end of Section 13. Relating upper indices (1) and (2) to the vector spaces where the corresponding matrices act nontrivially, one writes

$$\overset{(1)}{T}(\lambda) \equiv T(\lambda) \otimes I, \quad \overset{(2)}{T}(\lambda) \equiv I \otimes T(\lambda),$$

so that, e.g., equation (19.4) is rewritten as

$$\overset{(1\underline{2})}{\widetilde{R}}(\lambda,\mu)\overset{(1)}{T}(\lambda)\overset{(2)}{T}(\mu) = \overset{(1)}{T}(\mu)\overset{(2)}{T}(\lambda)\overset{(1\underline{2})}{\widetilde{R}}(\lambda,\mu) \qquad (19.5)$$

(the superscript (12) over $\widetilde{R}$ means that it acts nontrivially in both vector spaces). Equation (19.5) shows that the R-matrix $\widetilde{R}(\lambda,\mu)$ not only changes the order of noncommuting matrix elements in the tensor product of matrices $T(\lambda)$ and $T(\mu)$ but also changes the vector spaces where these matrices act. Let us multiply both sides of equation (19.5) from the left by the permutation matrix Π (13.6) ($\Pi^2 = E = I \otimes I$). Introducing the R-matrix "without tilde",

$$R(\lambda,\mu) \equiv \Pi\widetilde{R}(\lambda,\mu), \quad \left(\widetilde{R}(\lambda,\mu)\right)^{i_1 i_2}_{k_1 k_2} = \left(R(\lambda,\mu)\right)^{k_1 i_2}_{i_1 k_2}, \tag{19.6}$$

one gets from (19.5)

$$\overset{(12)}{R}(\lambda,\mu)\overset{(1)}{T}(\lambda)\overset{(2)}{T}(\mu) = \overset{(2)}{T}(\mu)\overset{(1)}{T}(\lambda)\overset{(12)}{R}(\lambda,\mu), \tag{19.7}$$

so that matrix R only changes the order of matrices $T(\lambda)$ and $T(\mu)$ but each matrix remains in its own matrix space. The convenience of these notations is that one can relate the spectral parameter λ to the first space, and the spectral parameter μ to the second space, not writing them explicitly in the notations, so that

$$\overset{(12)}{R} \equiv \overset{(12)}{R}(\lambda,\mu), \quad \overset{(1)}{T} \equiv \overset{(1)}{T}(\lambda), \quad \overset{(2)}{T} \equiv \overset{(2)}{T}(\mu), \quad \overset{(1)}{L}(l) \equiv \overset{(1)}{L}(l|\lambda), \quad \overset{(2)}{L}(l) \equiv \overset{(2)}{L}(l|\mu). \tag{19.8}$$

The formulation of the theorem in these notations is:

If there is the ultralocality (19.1) and the commutation relations between the matrix elements of operators $L(l|\lambda)$ at the same site are given as

$$\overset{(12)}{R}\overset{(1)}{L}(l)\overset{(2)}{L}(l) = \overset{(2)}{L}(l)\overset{(1)}{L}(l)\overset{(12)}{R}, \quad m \le l \le n, \tag{19.9}$$

then also the commutation relations between the matrix elements of the transition matrix are of the same form,

$$\overset{(12)}{R}\overset{(1)}{T}(n,m)\overset{(2)}{T}(n,m) = \overset{(2)}{T}(n,m)\overset{(1)}{T}(n,m)\overset{(12)}{R}. \tag{19.10}$$

For the monodromy matrix $T(\lambda) \equiv T(M,1|\lambda)$ one also has

$$\overset{(12)}{R}\overset{(1)}{T}\overset{(2)}{T} = \overset{(2)}{T}\overset{(1)}{T}\overset{(12)}{R}, \tag{19.11}$$

or, for the matrix elements,

$$\overset{(12)}{R}{}^{i_1 i_2}_{k_1 k_2}\overset{(1)}{T}{}^{\alpha_1\alpha_2}_{i_2 i_3}\overset{(2)}{T}{}^{\alpha_2\alpha_3}_{k_2 k_3} = \overset{(2)}{T}{}^{\alpha_1\alpha_2}_{k_1 k_2}\overset{(1)}{T}{}^{\alpha_2\alpha_3}_{i_1 i_2}\overset{(12)}{R}{}^{i_2 i_3}_{k_2 k_3}. \tag{19.11a}$$

Of course, "bilinear" relations (19.2)–(19.4) and (19.9)–(19.11) are only different ways of writing the same commutation relations between the matrix elements of

matrices L and T. To use the R-matrix in the form $\widetilde{R}(\lambda,\mu)$ or $R(\lambda,\mu)$ is only a matter of convenience. Both of them are used in literature.

Let us consider now the transfer matrix, $\tau(\lambda) = \operatorname{tr} T(\lambda) = \sum\limits_{i=1}^{N} T_{ii}(\lambda)$. It is a scalar in the matrix space, being a quantum operator. At points λ and μ we have different operators, $\tau(\lambda) \neq \tau(\mu)$ $(\lambda \neq \mu)$. However, the consequence of our theorem is the commutativity of these operators at any values of λ and μ,

$$\tau(\lambda)\tau(\mu) = \tau(\mu)\tau(\lambda) \tag{19.12}$$

To prove this, one uses the obvious identity,

$$\operatorname{Tr}(A \otimes B) = \operatorname{tr} A \cdot \operatorname{tr} B\,,$$

or

$$\operatorname{Tr} \overset{(1)}{A}\ \overset{(2)}{B} = \operatorname{tr} A \cdot \operatorname{tr} B\,.$$

Here, "Tr" denotes the trace in the space of $N^2 \times N^2$ dimensional matrices acting in $V \otimes V$, and tr denotes the trace in the space of $N \times N$ matrices acting in V. Rewriting commutation relations (19.11) in the form

$$\overset{(12)}{R}\ \overset{(1)}{T}\ \overset{(2)}{T}\ \left(\overset{(12)}{R}\right)^{-1} = \overset{(2)}{T}\ \overset{(1)}{T} \tag{19.13}$$

and taking the trace Tr of both sides, one gets just equation (19.12).

The commutativity (19.12) of the transfer matrices is of primary importance. As in the classical case (see Section 15), it ensures the existence of many commuting quantum operators. If the Hamiltonian of the nonlinear quantum model is expressed in terms of the tranfer matrix $\tau(\lambda)$ by means of the "trace identities", it is among these operators. Then there exist many integrals of motions in the quantum model. The examples will be given further. Usually, the Hamiltonian is expressed in terms of the logarithmic derivatives of the transfer matrix $\tau(\lambda)$,

$$H = \sum_k \sum_\alpha c_{k\alpha} \frac{d^k}{d\lambda^k} (\ln \tau(\lambda))\Big|_{\lambda=\nu_\alpha}\,. \tag{19.14}$$

The points ν_α are suitably chosen to render the Hamiltonian local. Therefore it follows directly from equation (19.12) that $[H, \tau(\lambda)] = 0$.

It should be noted that also in the quantum case the existence of the Lax representation for the dynamical nonlinear system (the Hamiltonian of which can be related by means of the "trace identities" to the transfer matrix) is a result of the existence of an L-operator and of an R-matrix. Computing the commutator of the logarithmic derivative of the transfer matrix with the L-operators, one gets the quantum analog of equation (16.1),

$$i\left[\frac{d}{d\mu}\left(\ln \tau(\mu)\right), L(n|\lambda)\right] = U(n+1|\lambda,\mu)L(n|\lambda) - L(n|\lambda)U(n|\lambda,\mu)\,, \tag{19.15}$$

where the $N \times N$ matrix U is the generating function for the quantum M-operators and is given as

$$U(n|\lambda,\mu) = i\frac{d}{d\mu}\ln\tau(\mu)\cdot I - iq^{-1}(n|\lambda,\mu)\frac{d}{d\mu}q(n|\lambda,\mu)\,. \tag{19.16}$$

Here, I is the unit $N \times N$ matrix, and the $N \times N$ matrix q is obtained by taking the trace in the first vector space of the $N^2 \times N^2$ matrix acting in $V \otimes V$:

$$\overset{(2)}{q}(n|\lambda,\mu) = \mathrm{tr}_1\Big(\overset{(1)}{T}(M,n|\mu)\overset{(12)}{R}(\mu,\lambda)\overset{(1)}{T}(n-1,1|\mu)\Big)\,. \tag{19.17}$$

To get the M-operator for the system with the Hamiltonian given by trace identities (19.14), one should "generate" the $N \times N$ matrix $U(m|\lambda)$ entering (18.2) from $U(n|\lambda,\mu)$ similarly to the classical case considered in detail in Section 16. The compatibility condition for the Lax representation (18.1), (18.2) is of the form (17.3):

$$\partial_t L(m|\lambda) = L(m|\lambda)U(m|\lambda) - U(m+1|\lambda)L(m|\lambda). \tag{19.18}$$

The time derivative is proportional to the commutator in the quantum space with the Hamiltonian

$$-i\partial_t L(m|\lambda) = \Big[H, L(m|\lambda)\Big]\,. \tag{19.19}$$

Now it is sufficient to use (19.14) and (19.16) for constructing the $N \times N$ matrix $U(m|\lambda)$ from $U(m|\lambda,\mu)$.

20 The Yang-Baxter equation.

Let us put the following question: can some given $N^2 \times N^2$ matrix $R(\lambda,\mu)$ with c-number matrix elements serve as the R-matrix for some integrable model? The answer is, in general, 'no'. To be the R-matrix, matrix $R(\lambda,\mu)$ should satisfy the condition called "the Yang-Baxter equation".

To derive the Yang-Baxter equation, let us consider the tensor product of three monodromy matrices $\overset{(1)}{T}(\lambda)\overset{(2)}{T}(\mu)\overset{(3)}{T}(\nu) \equiv \overset{(1)}{T}\overset{(2)}{T}\overset{(3)}{T}$ and transform them into the order $\overset{(3)}{T}(\nu)\overset{(2)}{T}(\mu)\overset{(1)}{T}(\lambda) \equiv \overset{(3)}{T}\overset{(1)}{T}\overset{(2)}{T}$ using the commutation relations (19.11),

$$\overset{(i)}{T}\overset{(k)}{T} = \overset{(ik)}{\left(R\right)}^{-1}\overset{(k)}{T}\overset{(i)}{T}\overset{(ik)}{R}\quad, \tag{20.1}$$

where $\overset{(ik)}{R} \equiv \overset{(ik)}{R}(\lambda_i,\lambda_k)$; $\overset{(i)}{T} = \overset{(i)}{T}(\lambda_i)$; $i,k = 1,2,3$; $\lambda_1 \equiv \lambda$, $\lambda_2 = \mu$, $\lambda_3 = \nu$. It can be done in the two following ways, (a) and (b),

(a)
$$\overset{(1)}{T}\overset{(2)}{T}\overset{(3)}{T} = \left(\overset{(12)}{R}\right)^{-1} \overset{(2)}{T}\overset{(1)}{T}\overset{(12)}{R}\overset{(3)}{T}$$

$$= \left(\overset{(12)}{R}\right)^{-1} \overset{(2)}{T}\overset{(1)}{T}\overset{(3)}{T}\overset{(12)}{R}$$

$$= \left(\overset{(12)}{R}\right)^{-1} \left(\overset{(13)}{R}\right)^{-1} \overset{(2)}{T}\overset{(3)}{T}\overset{(1)}{T}\overset{(13)}{R}\overset{(12)}{R}$$

$$= \left(\overset{(12)}{R}\right)^{-1} \left(\overset{(13)}{R}\right)^{-1} \left(\overset{(23)}{R}\right)^{-1} \overset{(3)}{T}\overset{(2)}{T}\overset{(1)}{T}\overset{(23)}{R}\overset{(13)}{R}\overset{(12)}{R} \ ;$$

(b)
$$\overset{(1)}{T}\overset{(2)}{T}\overset{(3)}{T} = \left(\overset{(23)}{R}\right)^{-1} \overset{(1)}{T}\overset{(3)}{T}\overset{(2)}{T}\overset{(23)}{R}$$

$$= \left(\overset{(23)}{R}\right)^{-1} \left(\overset{(13)}{R}\right)^{-1} \overset{(3)}{T}\overset{(1)}{T}\overset{(2)}{T}\overset{(13)}{R}\overset{(23)}{R}$$

$$= \left(\overset{(23)}{R}\right)^{-1} \left(\overset{(13)}{R}\right)^{-1} \left(\overset{(12)}{R}\right)^{-1} \overset{(3)}{T}\overset{(2)}{T}\overset{(1)}{T}\overset{(12)}{R}\overset{(13)}{R}\overset{(23)}{R}$$

(one has to take into account that $\overset{(12)}{R}\overset{(3)}{T} = \overset{(3)}{T}\overset{(12)}{R}$, $\overset{(13)}{R}\overset{(2)}{T} = \overset{(2)}{T}\overset{(13)}{R}$, etc.). The sufficient condition for the results of (a) and (b) being equal is the famous Yang-Baxter equation, which the R-matrix should satisfy, identically, in λ, μ, and ν:

$$\overset{(12)}{R}(\lambda,\mu)\overset{(13)}{R}(\lambda,\nu)\overset{(23)}{R}(\mu,\nu) = \overset{(23)}{R}(\mu,\nu)\overset{(13)}{R}(\lambda,\nu)\overset{(12)}{R}(\lambda,\mu), \tag{20.2}$$

or, in brief notations,

$$\overset{(12)}{R}\overset{(13)}{R}\overset{(23)}{R} = \overset{(23)}{R}\overset{(13)}{R}\overset{(12)}{R} \tag{20.2a}$$

(arguments λ, μ, and ν are related to the first, second and third vector spaces in the tensor product $V \otimes V \otimes V \equiv \overset{(1)}{V} \otimes \overset{(2)}{V} \otimes \overset{(3)}{V}$, respectively). The equivalent relation for the R-matrix $\widetilde{R}(\lambda,\mu) = \Pi R(\lambda,\mu)$,

$$\widetilde{R}(\lambda,\mu)\Big(T(\lambda)\otimes T(\mu)\Big) = \Big(T(\mu)\otimes T(\lambda)\Big)\widetilde{R}(\lambda,\mu), \tag{20.3}$$

is written as

$$\begin{aligned}&\Big(I\otimes\widetilde{R}(\lambda,\mu)\Big)\Big(\widetilde{R}(\lambda,\nu)\otimes I\Big)\Big(I\otimes\widetilde{R}(\mu,\nu)\Big) = \\ &= \Big(\widetilde{R}(\mu,\nu)\otimes I\Big)\Big(I\otimes\widetilde{R}(\lambda,\nu)\Big)\Big(\widetilde{R}(\lambda,\mu)\otimes I\Big)\ ,\end{aligned} \tag{20.4}$$

or, in other notations,

$$\overset{(23)}{\widetilde{R}}\,(\lambda,\mu)\overset{(12)}{\widetilde{R}}\,(\lambda,\nu)\overset{(23)}{\widetilde{R}}\,(\mu,\nu) = \overset{(12)}{\widetilde{R}}\,(mu,\nu)\overset{(23)}{\widetilde{R}}\,(\lambda,\nu)\overset{(12)}{\widetilde{R}}\,(\lambda,\mu) \tag{20.4a}$$

(notice that for $\widetilde{R}$, in constrast to R, arguments λ, μ, ν are not related to the fixed matrix spaces !).

As already mentioned, to use the R-matrix in the form $R(\lambda,\mu)$ or $\widetilde{R}(\lambda,\mu)$ is a matter of convenience. They are related simply as $\widetilde{R} = \Pi R, R = \Pi\widetilde{R}$. Both forms have their own advantages. In what follows, the tilde over $\widetilde{R}$ will sometimes be omitted. Which is the normalization used for the R-matrix is quite clear from the form of basic relations ((20.1), (20.2) or (20.3), (20.4)).

Any solution of the Yang-Baxter equations is of great interest. E.g., we shall see further that any R-matrix immediately generates an integrable spin model on the one-dimensional lattice.

A particular case of the R-matrices are "constant" R-matrices, not depending on spectral parameters at all. The corresponding constant solutions of the Yang-Baxter equation have a close relation to the representations of "quantum groups".

To conclude, let us discuss the following point. The Yang-Baxter equation ensures getting the same answer for the two different ways of transforming the triple product of monodromy matrices, $\overset{(1)}{T}\overset{(2)}{T}\overset{(3)}{T} \to \overset{(3)}{T}\overset{(2)}{T}\overset{(1)}{T}$. Do some new relations occur, if one transforms the product of a larger number of monodromy matrices, e.g., $\overset{(1)}{T}\overset{(2)}{T}\overset{(3)}{T}\overset{(4)}{T} \to \overset{(4)}{T}\overset{(3)}{T}\overset{(2)}{T}\overset{(1)}{T}$? The answer is "no": the Yang-Baxter equation is sufficient also for the compatibility of different possible ways of performing these transformations.

21 The examples of the R-matrices.

The trivial examples of the "constant" (not depending on spectral parameters) R-matrix are the unit matrix $E = I \otimes I$ (13.5) and the permutation matrix Π (13.6) which satisfy the Yang-Baxter equation both in the form (20.2) and (20.4) (it is evident from the fact that one of the matrices, R or $\widetilde{R} = \Pi R$, is equal to E in both cases). Of course, the Yang-Baxter equality for the permutation matrix,

$$\Pi^{(12)}\,\Pi^{(13)}\,\Pi^{(23)} \;=\; \Pi^{(23)}\,\Pi^{(13)}\,\Pi^{(12)}\,, \tag{21.1}$$

is easily verified directly. Since $\Pi^2 = E$, it is equivalent to the relation

$$\Pi^{(12)}\,\Pi^{(13)}\,\Pi^{(23)}\,\Pi^{(12)} \;=\; \Pi^{(23)}\,\Pi^{(13)}\,.$$

In the left-hand side here, the permutation matrices $\Pi^{(12)}$ just change the spaces 1 and 2 not touching space 3 which results just in the expression written down in the right-hand side.

Let us turn to R-matrices depending on spectral parameters. The simplest of them correspond to the case $\dim V = 2$, i.e., they are 4×4 matrices (acting in the space $V \otimes V$). We will arrange their matrix elements $R^{i_1 i_2}_{k_1 k_2}$ into a 4×4 array, as explained in Section 13, see (13.10)–(13.12),

$$R = \begin{pmatrix} R^{11}_{11} & R^{11}_{12} & R^{12}_{11} & R^{12}_{12} \\ R^{11}_{21} & R^{11}_{22} & R^{12}_{21} & R^{12}_{22} \\ R^{21}_{11} & R^{21}_{12} & R^{22}_{11} & R^{22}_{12} \\ R^{21}_{21} & R^{21}_{22} & R^{22}_{21} & R^{22}_{22} \end{pmatrix}. \tag{21.2}$$

The simplest R-matrices of this kind are of the form

$$\widetilde{R}(\lambda,\mu) = \begin{pmatrix} \mathrm{a} & 0 & 0 & 0 \\ 0 & \mathrm{b} & \mathrm{c} & 0 \\ 0 & \mathrm{c} & \mathrm{b} & 0 \\ 0 & 0 & 0 & \mathrm{a} \end{pmatrix}, \qquad R(\lambda,\mu) = \begin{pmatrix} \mathrm{a} & 0 & 0 & 0 \\ 0 & \mathrm{c} & \mathrm{b} & 0 \\ 0 & \mathrm{b} & \mathrm{c} & 0 \\ 0 & 0 & 0 & \mathrm{a} \end{pmatrix}, \tag{21.3}$$

where $\mathrm{a}, \mathrm{b}, \mathrm{c}$ are functions of spectral parameters λ and μ. Putting

$$\mathrm{a} \equiv f(\mu,\lambda) = \frac{\lambda - \mu + ic}{\lambda - \mu}, \qquad \mathrm{b} \equiv g(\mu,\lambda) = \frac{ic}{\lambda - \mu}, \qquad \mathrm{c} \equiv 1, \qquad c \in \mathbf{C} \tag{21.4}$$

one obtains a solution of the Yang-Baxter equation for any value of the complex parameter c. This R-matrix is called the XXX, or rational, R-matrix. It will be seen that it is the R-matrix for the Heisenberg XXX spin chain. The same R-matrix is also the R-matrix for the nonrelativistic one-dimensional Bose gas, parameter c should be identified with the coupling constant of the gas. Choosing

$$\mathrm{a} \equiv f(\mu,\lambda) = \frac{\sinh(\lambda - \mu + i\gamma)}{\sinh(\lambda - \mu)}, \qquad \mathrm{b} \equiv g(\mu,\lambda) = \frac{i \sin\gamma}{\sinh(\lambda - \mu)}, \qquad \mathrm{c} \equiv 1, \tag{21.5}$$

where, again, γ is a complex parameter, one gets another solution of the Yang-Baxter equation which is called the XXZ, or trigonometrical, R-matrix. It is the R-matrix for the Heisenberg XXZ spin chain and also for the sine-Gordon model. The rational R-matrix is the limiting case of the trigonometrical R-matrix ($\lambda \to \epsilon\lambda$, $\mu \to \epsilon\mu$, $\gamma \to \epsilon c$; $\epsilon \to 0$).

The most general 4×4 R-matrix with an "additive" dependence on the spectral parameter, $R(\lambda,\mu) = R(\lambda - \mu)$, is the Baxter XYZ R-matrix which was, in fact, used by him in solving the eight-vertex model and the XYZ Heisenberg chain. It is defined in terms of the Jacobi elliptic functions of modulus k,

$$\widetilde{R}(\lambda,\mu) = \begin{pmatrix} \mathrm{a} & 0 & 0 & \mathrm{d} \\ 0 & \mathrm{b} & \mathrm{c} & 0 \\ 0 & \mathrm{c} & \mathrm{b} & 0 \\ \mathrm{d} & 0 & 0 & \mathrm{a} \end{pmatrix}, \qquad R(\lambda,\mu) = \begin{pmatrix} \mathrm{a} & 0 & 0 & \mathrm{d} \\ 0 & \mathrm{c} & \mathrm{b} & 0 \\ 0 & \mathrm{b} & \mathrm{c} & 0 \\ \mathrm{d} & 0 & 0 & \mathrm{a} \end{pmatrix}, \tag{21.6}$$

where

$$\mathrm{a}(\lambda,\mu)=\mathrm{sn}(\lambda-\mu+2\eta,k),\qquad \mathrm{b}(\lambda,\mu)=\mathrm{sn}(2\eta,k),\qquad \mathrm{c}(\lambda,\mu)=\mathrm{sn}(\lambda-\mu,k),$$

$$\mathrm{d}(\lambda,\mu)=k\ \mathrm{sn}(2\eta,k)\,\mathrm{sn}(\lambda-\mu,k)\,\mathrm{sn}(\lambda-\mu+2\eta,k),\qquad \eta\in\mathbf{C}. \tag{21.7}$$

The definitions of these elliptical functions can be found in text books or in reference books on special functions. Elliptical functions are double-periodical functions of the first argument and satisfy the relations (in the first argument)

$$\mathrm{sn}(-u)=-\,\mathrm{sn}\,u,\qquad \mathrm{cn}(-u)=\mathrm{cn}\,u,\qquad \mathrm{dn}(-u)=\mathrm{dn}\,u,$$

$$\mathrm{sn}(2K-u)=\mathrm{sn}\,u,\qquad \mathrm{cn}(2K-u)=-\,\mathrm{cn}\,u,\qquad \mathrm{dn}(2K-u)=\mathrm{dn}\,u,$$

$$\mathrm{sn}(2iK'-u)=-\,\mathrm{sn}\,u,\qquad \mathrm{cn}(2iK'-u)=-\,\mathrm{cn}\,u,\qquad \mathrm{dn}(2iK'-u)=-\,\mathrm{dn}\,u.$$

Here,

$$K=\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},\quad K'=\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k'\sin^2\varphi}},$$

k being the modulus and $k'=\sqrt{1-k^2}$ the complementary modulus. The main properties:

$$\mathrm{sn}^2\,u+\mathrm{cn}^2\,u=1,\qquad k^2\,\mathrm{sn}^2\,u+\mathrm{dn}^2\,u=1,$$

and the addition theorems,

$$\mathrm{sn}(u+v)=\frac{\mathrm{sn}\ u\ \mathrm{cn}\ v\ \mathrm{dn}\ v+\mathrm{cn}\ u\ \mathrm{dn}\ u\ \mathrm{sn}\ v}{1-k^2\,\mathrm{sn}^2\,u\ \mathrm{sn}^2\,v},$$

$$\mathrm{cn}(u+v)=\frac{\mathrm{cn}\ u\ \mathrm{cn}\ v-\mathrm{sn}\ u\ \mathrm{dn}\ u\ \mathrm{sn}\ v\ \mathrm{dn}\ v}{1-k^2\,\mathrm{sn}^2\,u\ \mathrm{sn}^2\,v},$$

$$\mathrm{dn}(u+v)=\frac{\mathrm{dn}\ u\ \mathrm{dn}\ v-k^2\,\mathrm{sn}\ u\ \mathrm{cn}\ u\ \mathrm{sn}\ v\ \mathrm{cn}\ v}{1-k^2\,\mathrm{sn}^2\,u\ \mathrm{sn}^2\,v},$$

can be used to show that the XYZ R-matrix satisfies the Yang-Baxter equation. At $k\to 0$, they turn out to be just the usual trigonometrical functions, $\mathrm{sn}(u,k=0)=\sin u$, $\mathrm{cn}(u,k=0)=\cos u$, $\mathrm{dn}(u,k=0)=1$. Elliptical functions are "easy to differentiate",

$$\frac{d}{du}\,\mathrm{sn}\,u=\mathrm{cn}\,u\,\mathrm{dn}\,u,\qquad \frac{d}{du}\,\mathrm{cn}\,u=-\,\mathrm{sn}\,u\,\mathrm{dn}\,u,\qquad \frac{d}{du}\,\mathrm{dn}\,u=-k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u.$$

Let us put $\lambda=\mu$; then

$$\mathrm{a}=\mathrm{sn}(2\eta,k),\quad \mathrm{b}=\mathrm{sn}(2\eta,k),\quad \mathrm{c}=\mathrm{sn}(0,k)=0,\quad \mathrm{d}=0,$$

that is,

$$\widetilde{R}(\lambda,\lambda)=(\mathrm{sn}\,2\eta)\cdot E,\quad R(\lambda,\lambda)=(\mathrm{sn}\,2\eta)\cdot\Pi, \tag{21.8}$$

where Π is the permutation matrix.

Let us come back to the rational R-matrix. Formulae (21.3), (21.4) defining the rational R-matrix in the case of $N = 2$ can be put into the form in which they give a solution of the Yang-Baxter equation in the case of any N, namely,

$$\widetilde{R}(\lambda,\mu) = \Pi + \frac{ic}{\lambda-\mu}E, \quad R(\lambda,\mu) = E + \frac{ic}{\lambda-\mu}\Pi \tag{21.9}$$

(R and $\widetilde{R}$ here are $N^2 \times N^2$ dimensional matrices). To proof is left to the reader, as an exercise.

22 Some properties of the R-matrix.

Let us discuss some properties of the R-matrices (which are the solutions of the Yang-Baxter equation (20.2), (20.4)).

(a) Spectral parameters λ, μ, ν are usually complex numbers. One can make a change of the variable defining a "new" spectral parameter λ' as a function of the old one, $\lambda' = \lambda'(\lambda)$. The Yang-Baxter equation is invariant under this change of variables. In most cases, the spectral parameter can be chosen in such a way that the R-matrix depends on the difference of variables, i.e., only on one complex argument

$$R(\lambda,\mu) = R(\lambda-\mu)\ . \tag{22.1}$$

If it is possible, then the L-operator $L(l|\lambda-\nu_l)$ (ν_l is any fixed complex number) satisfies exactly the same "bilinear" relation (19.2) as operator $L(l|\lambda)$,

$$\widetilde{R}(\lambda,\mu)\Big(L(l|\lambda-\nu_l)\otimes L(l|\mu-\nu_l)\Big) = \Big(L(l|\mu-\nu_l)\otimes L(l|\lambda-\nu_l)\Big)\widetilde{R}(\lambda,\mu)\,. \tag{22.2}$$

The numbers ν_l can be chosen different at different l. It means, in particular, that the monodromy matrix

$$T(\lambda;\nu_1,\dots,\nu_M) \equiv L(M|\lambda-\nu_M)\dots L(l|\lambda-\nu_l)\dots L(1|\lambda-\nu_1) \tag{22.3}$$

satisfies the same bilinear relation (19.4) as the monodromy matrix $T(\lambda)$ (22.4).

(b) It is easily seen that the Yang-Baxter equation defines the R-matrix up to the arbitrary *scalar* factor $f(\lambda,\mu)$, i.e., if $R(\lambda,\mu)$ is a solution, then also $f(\lambda,\mu)R(\lambda,\mu)$ is a solution). Also, the bilinear relations ((19.2)–(19.4), (19.9)–(19.11)) remain unchanged. This property can be used to normalize the R-matrix in a convenient way. E.g., the rational R-matrix (21.11) can be also written as a polynomial,

$$R(\lambda,\mu) = (\lambda-\mu)E - ic\Pi, \qquad \widetilde{R}(\lambda,\mu) = (\lambda-\mu)\Pi - icE\ .$$

(c) Suppose that an R-matrix $R(\lambda,\mu)$ is a solution of the Yang-Baxter equation. Then the matrix $R'(\lambda,\mu)$ is also a solution,

$$\overset{(12)'}{R}(\lambda,\mu) \equiv \overset{(1)}{a}(\lambda)\overset{(2)}{a}(\mu)\overset{(12)}{R}(\lambda,\mu)\overset{(1)}{a}{}^{-1}(\lambda)\overset{(2)}{a}{}^{-1}(\mu)\ , \tag{22.4}$$

where $a(\lambda)$ is any $N \times N$ matrix whose matrix elements are the complex-valued (commuting) functions of the spectral parameter. Equivalently, for the matrix $\widetilde{R} \equiv \Pi R$, one has ($\widetilde{R}' \equiv \Pi R'$)

$$\widetilde{R}'(\lambda,\mu) = (a(\mu) \otimes a(\lambda))\widetilde{R}(\lambda,\mu)(a^{-1}(\lambda) \otimes a^{-1}(\mu)) \tag{22.4a}$$

which can be seen without doing much calculation. The Yang-Baxter equation (20.2) for the matrix R is the compatibility condition for the commutation relations of matrix elements of some monodromy matrix $T(\lambda)$,

$$\overset{(12)}{R}\ \overset{(1)}{T}\ \overset{(2)}{T} = \overset{(2)}{T}\ \overset{(1)}{T}\ \overset{(12)}{R}$$

Define a new monodromy matrix $T'(\lambda)$,

$$T'(\lambda) \equiv a(\lambda)T(\lambda)a^{-1}(\lambda)\ .$$

Then the commutation relations for it are obtained from the formula above as

$$\overset{(12)}{R}{}'\overset{(1)}{T}{}'\overset{(2)}{T}{}' = \overset{(2)}{T}{}'\overset{(1)}{T}{}'\overset{(12)}{R}{}'\ ,$$

with R' given just by equation (22.4). The compatibility of these, i.e., the Yang-Baxter equation for R', follows from the compatibility of the commutation relations for T. It is, however, instructive to check directly that the matrix $R'(\lambda,\mu)$ satisfies the Yang-Baxter equation, if the matrix $R(\lambda,\mu)$ does. In fact, matrices R and R' give the same set of commutation relations between the matrix elements of the monodromy matrix and can be regarded as the same R-matrix.

(d) If the R-matrix $\widetilde{R}(\lambda,\mu)$ "intertwines" the monodromy matrices $T(\lambda)$ and $T(\mu)$, i.e.,

$$\widetilde{R}(\lambda,\mu)\Big(T(\lambda) \otimes T(\mu)\Big) = \Big(T(\mu) \otimes T(\lambda)\Big)\widetilde{R}(\lambda,\mu)\ ,$$

then

$$\widetilde{R}(\mu,\lambda)\Big(T(\mu) \otimes T(\lambda)\Big) = \Big(T(\lambda) \otimes T(\mu)\Big)\widetilde{R}(\mu,\lambda)\ ,$$

and

$$\widetilde{R}^{-1}(\lambda,\mu)\Big(T(\mu) \otimes T(\lambda)\Big) = \Big(T(\lambda) \otimes T(\mu)\Big)\widetilde{R}^{-1}(\lambda,\mu)\ .$$

Thus, matrices $\widetilde{R}(\mu,\lambda)$ and $\widetilde{R}^{-1}(\lambda,\mu)$ should be proportional, i.e.,

$$\widetilde{R}^{-1}(\lambda,\mu) = \varphi(\lambda,\mu)\widetilde{R}(\mu,\lambda)\ , \tag{22.5}$$

up to the scalar factor $\varphi(\lambda,\mu)$. Usually, the normalization can be chosen in such a way that

$$\begin{aligned} &\widetilde{R}(\mu,\lambda) = \widetilde{R}^{-1}(\lambda,\mu)\,, && \widetilde{R}(\lambda,\lambda) = E\,, \\ &R(\mu,\lambda) = \Pi R^{-1}(\lambda,\mu)\Pi\,, && R(\lambda,\lambda) = \Pi\,. \end{aligned} \tag{22.6}$$

(e) In the quasiclassical limit, the quantum R-matrix of the quantum model should be related to the classical r-matrix of the classical model. It means that the Poisson brackets (14.11) of the classical monodromy matrices,

$$\left\{ \overset{(1)}{T}_{cl}(\lambda) \underset{,}{\otimes} \overset{(2)}{T}_{cl}(\mu) \right\} = \left[\overset{(1)}{T}_{cl}(\lambda) \overset{(2)}{T}_{cl}(\mu), \overset{(12)}{r}(\lambda,\mu) \right] , \tag{22.7}$$

should be restored from the commutation relations (19.7) of the quantum monodromy matrices,

$$\overset{(12)}{R}(\lambda,\mu) \overset{(1)}{T}(\lambda) \overset{(2)}{T}(\mu) = \overset{(2)}{T}(\mu) \overset{(1)}{T}(\lambda) \overset{(12)}{R}(\lambda,\mu) \tag{22.8}$$

(notice that it is more convenient to use the R-matrix without tilde here).

In the quantum case, the R-matrix is defined up to an arbitrary c-number scalar factor, $R(\lambda,\mu) \to f(\lambda,\mu)R(\lambda,\mu)$, where $f(\lambda,\mu)$ denotes any complex-valued function. In the classical case, one can add any matrix proportional to the unit matrix to the classical r-matrix, $r(\lambda,\mu) \to \varphi(\lambda,\mu)E + r(\lambda,\mu)$, which does not change the value of the commutator in the right hand-side of (22.6). Correspondingly, let us work out the quasiclassical limit ($\hbar$ is the Plank constant)

$$R(\lambda,\mu) = f(\lambda,\mu)\Big(E + \hbar\alpha(\lambda,\mu) + \hbar^2\beta(\lambda,\mu) + \dots \Big) \tag{22.9}$$

which gives for (22.8)

$$\overset{(1)}{T}(\lambda)\overset{(2)}{T}(\mu) - \overset{(2)}{T}(\mu)\overset{(1)}{T}(\lambda) = \hbar \left(-\alpha(\lambda,\mu)\overset{(1)}{T}(\lambda)\overset{(2)}{T}(\mu) + \overset{(2)}{T}(\mu)\overset{(1)}{T}(\lambda)\alpha(\lambda,\mu) \right) + 0(\hbar^2) \tag{22.10}$$

Remind that in quantum mechanics, the commutator of the operators of momentum p and coordinate q is equal to $(-i\hbar)$ times the value of the Poisson brackets of the classical quantities,

$$[p,q] = -i\hbar\{p_{cl}, q_{cl}\} = -i\hbar \tag{22.11}$$

(in the case of canonical conjugate variables, there are no corrections in $\hbar$ to this formula, countrarely to our case). Similarly, at order $\hbar^0$, the matrix elements of matrices $T(\lambda)$ and $T(\mu)$ commute, i.e.,

$$\overset{(1)}{T}(\lambda)\overset{(2)}{T}(\mu) = \overset{(2)}{T}(\mu)\overset{(1)}{T}(\lambda) \qquad\qquad (\hbar = 0) \tag{22.12}$$

This is, in fact, the reason why the first term of expansion (22.9) should be proportional to the unit matrix E. At order $\hbar^1$, one should take matrices $\overset{(1)}{T}(\lambda)$ and $\overset{(2)}{T}(\mu)$ in the right hand-side of (22.10) at order $\hbar^0$, at which, indeed, they do commute, see (22.12). For the right-hand side one changes the commutator for the Poisson brackets, similarly to (22.11). So in the classical limit one gets from (22.10)

$$-i\hbar\Big\{ \overset{(1)}{T}_{cl}(\lambda) \underset{,}{\otimes} \overset{(2)}{T}_{cl}(\mu) \Big\} = \hbar \left[\overset{(1)}{T}_{cl}(\lambda)\overset{(2)}{T}_{cl}(\mu), \alpha(\lambda,\mu) \right] ,$$

and comparing with (22.7) one gets that $\alpha(\lambda,\mu) = -ir(\lambda,\mu)$. Thus the quasiclassical limit of the quantum R-matrix is

$$\begin{aligned} R(\lambda,\mu) &= f(\lambda,\mu)\Big(E - i\hbar r(\lambda,\mu)\Big), \qquad\qquad \hbar \to 0\ , \\ \widetilde{R}(\lambda,\mu) &= f(\lambda,\mu)\Big(\Pi - i\hbar\Pi r(\lambda,\mu)\Big), \qquad\qquad \hbar \to 0\,. \end{aligned} \tag{22.13}$$

The scalar factor $f(\lambda,\mu)$ is not, of course, essential and depends on the chosen normalization of the R-matrix. Let us derive now the classical Yang-Baxter equation (14.12) as the quasiclassical limit of the quantum equation,

$$R^{(12)}\,R^{(13)}\,R^{(23)} \;=\; R^{(23)}\,R^{(13)}\,R^{(12)}$$

To do this, one substitutes the quasiclassical expansion,

$$R(\lambda,\mu) = E - i\hbar r(\lambda,\mu) + \hbar^2 s(\lambda,\mu) + \dots \tag{22.14}$$

At orders $\hbar^0$ and $\hbar^1$ one has the identities,

$$E^{(12)}\,E^{(13)}\,E^{(23)} \;=\; E^{(23)}\,E^{(13)}\,E^{(12)}\,, \qquad\qquad r^{(12)} + r^{(13)} + r^{(23)} \;=\; r^{(23)} + r^{(13)} + r^{(12)}\ .$$

The nontrivial relation for $r(\lambda,\mu)$ appears at second order (the terms containing s in (22.14) are linear at this order and do not contribute !),

$$r^{(12)}\,r^{(13)} + r^{(12)}\,r^{(23)} + r^{(13)}\,r^{(23)} \;=\; r^{(23)}\,r^{(13)} + r^{(23)}\,r^{(12)} + r^{(13)}\,r^{(12)}\ , \tag{22.15}$$

which is just relation (14.12). The other way of obtaining it is to use the Jacobi identities for the Poisson brackets.

Let us also consider the quasiclassical limit of the XXX and XXZ R-matrices. Of course, the Plank constant $\hbar$ does not enter explicitly the expressions (21.4) and (21.5) for their matrix elements. It is not difficult to understand, however, that it should be in the parameters c and γ: $c \to \hbar c$, $\gamma \to \hbar\gamma$. Taking this into account, one obtains from (21.11) for the classical limit of the XXX R-matrix the rational r-matrix

$$r(\lambda,\mu) = \frac{c}{\lambda-\mu}\Pi\ , \tag{22.16}$$

which is exactly the classical r-matrix (14.13) of the classical nonlinear Schrödinger equation. Taking the quasiclassical limit of the trigonometrical R-matrix (21.5) results in obtaining another solution of the classical Yang-Baxter equation, namely, the trigonometrical r-matrix

$$r(\lambda,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u & v & 0 \\ 0 & v & u & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{22.17}$$

with

$$u = \gamma \coth(\lambda - \mu), \qquad v = -\frac{\gamma}{\sinh(\lambda - \mu)} \tag{22.18}$$

(remind that the classical r-matrix is defined up to adding any matrix proportional to the unit matrix !). This is the r-matrix for the sine-Gordon equation. The rational r-matrix is the limiting case ($\lambda \to \epsilon\lambda$, $\mu \to \epsilon\mu$, $\gamma \to \epsilon c$, $\epsilon \to 0$) of the trigonometrical one.

23 The algebraic Bethe Ansatz. Preliminaries. The generating state.

The algebraic Bethe Ansatz is a regular scheme of obtaining eigenvectors and eigenvalues of the transfer matrix $\tau(\lambda) = \mathrm{tr}\, T(\lambda)$ of an integrable quantum model. It is one of the central points of the quantum inverse scattering methods. Due to the commutativity of the transfer matrices at different values of the spectral parameter, $\tau(\lambda)\tau(\mu) = \tau(\mu)\tau(\lambda)$, the eigenvectors of $\tau(\lambda)$ are the same at all values of λ; they can be taken not depending on λ. The eigenvalues depend, of course, on λ. If the Hamiltonian of the model is expressed in terms of the transfer matrix by means of trace identities, the eigenvectors of it are the same, and the spectrum is easily obtained from the spectrum of the transfer matrix. The relation of the algebraic Bethe Ansatz to the coordinate Bethe Ansatz is, to some extent, analogous to the relation between the two ways of solving the quantum mechanics problem for the harmonic oscillator. One way is to solve the Schrödinger equation and to use the explicit form of eigenfunctions. The other way (which makes it possible to get many answers without bulky calculations) is to use the algebra of the creation and annihilation operators.

The construction of the algebraic Bethe Ansatz will be presented below in the simplest case for the models with the XXX or XXZ R-matrix (21.3)–(21.5). In this case, the monodromy matrix is a 2×2 matrix,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{23.1}$$

where A, B, C, and D are the quantum operators acting in the same quantum space of the model where the Hamiltonian acts. The commutation relations between these operators are given by the rational or trigonometrical R-matrix,

$$\widetilde{R}(\lambda,\mu)\Big(T(\lambda)\otimes T(\mu)\Big) = \Big(T(\mu)\otimes T(\lambda)\Big)\widetilde{R}(\lambda,\mu)\ , \tag{23.2}$$

which gives the following 16 commutation relations between the matrix elements of the monodromy matrices:

$$A(\lambda)A(\mu) = A(\mu)A(\lambda) \tag{23.3}$$

$$B(\lambda)B(\mu) = B(\mu)B(\lambda) \tag{23.4}$$

$$C(\lambda)C(\mu) = C(\mu)C(\lambda) \tag{23.5}$$

$$D(\lambda)D(\mu) = D(\mu)D(\lambda) \tag{23.6}$$

$$A(\mu)B(\lambda) = f(\mu,\lambda)B(\lambda)A(\mu) + g(\lambda,\mu)B(\mu)A(\lambda) \tag{23.7}$$

$$D(\mu)B(\lambda) = f(\lambda,\mu)B(\lambda)D(\mu) + g(\mu,\lambda)B(\mu)D(\lambda) \tag{23.8}$$

$$C(\lambda)A(\mu) = f(\mu,\lambda)A(\mu)C(\lambda) + g(\lambda,\mu)A(\lambda)C(\mu) \tag{23.9}$$

$$C(\lambda)D(\mu) = f(\lambda,\mu)D(\mu)C(\lambda) + g(\mu,\lambda)D(\lambda)C(\mu) \tag{23.10}$$

$$B(\mu)A(\lambda) = f(\mu,\lambda)A(\lambda)B(\mu) + g(\lambda,\mu)A(\mu)B(\lambda) \tag{23.11}$$

$$B(\lambda)D(\mu) = f(\mu,\lambda)D(\mu)B(\lambda) + g(\lambda,\mu)D(\lambda)B(\mu) \tag{23.12}$$

$$A(\lambda)C(\mu) = f(\mu,\lambda)C(\mu)A(\lambda) + g(\lambda,\mu)C(\lambda)A(\mu) \tag{23.13}$$

$$D(\mu)C(\lambda) = f(\mu,\lambda)C(\lambda)D(\mu) + g(\lambda,\mu)C(\mu)D(\lambda) \tag{23.14}$$

$$\Big[C(\lambda), B(\mu)\Big] = g(\lambda,\mu)\Big(A(\lambda)D(\mu) - A(\mu)D(\lambda)\Big) \tag{23.15}$$

$$\Big[B(\lambda), C(\mu)\Big] = g(\lambda,\mu)\Big(D(\lambda)A(\mu) - D(\mu)A(\lambda)\Big) \tag{23.16}$$

$$\Big[D(\lambda), A(\mu)\Big] = g(\lambda,\mu)\Big(B(\lambda)C(\mu) - B(\mu)C(\lambda)\Big) \tag{23.17}$$

$$\Big[A(\lambda), D(\mu)\Big] = g(\lambda,\mu)\Big(C(\lambda)B(\mu) - C(\mu)B(\lambda)\Big) \tag{23.18}$$

Functions f and g are here given by equations (21.4) in the rational case, and by equations (21.5) in the trigonometrical case. One should pay attention to the order of arguments λ, μ in these functions in equations (23.3)–(23.18)!

The scheme of the algebraic Bethe Ansatz can be applied to solve a model, if there exists the "generating" state $|0\rangle$ (it is also called the "Fock vacuum", the "bare vacuum", or simply the "vacuum state"). This is the state which at any λ is annihilated by the operator $C(\lambda)$, being also an eigenstate for the operators $A(\lambda)$ and $D(\lambda)$,

$$C(\lambda)|0\rangle = 0, \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad \forall\lambda\ . \tag{23.19}$$

Complex-valued functions $a(\lambda)$ and $d(\lambda)$ are usually called the "vacuum eigenvalues". The commutation relations (23.3)–(23.18) are completely defined by the given R-matrix, they are the same for the monodromy matrices of all the models with the same R-matrix. On the contrary, the vacuum eigenvalues $a(\lambda)$ and $d(\lambda)$ are different for different models with the same R-matrix; they are model dependent.

We assume also the existence of the dual vacuum state $\langle 0|$ annihilated by the operator $B(\lambda)$ acting from the right, with the properties

$$\langle 0|B(\lambda) = 0, \quad \langle 0|A(\lambda) = a(\lambda)\langle 0|, \quad \langle 0|D(\lambda) = d(\lambda)\langle 0|, \quad \forall\lambda; \tag{23.20}$$

$$\langle 0|0\rangle \equiv 1\,.$$

That this should be the "left" eigenstate of the operators $A(\lambda)$ and $D(\lambda)$, with the same eigenvalues $a(\lambda)$ and $d(\lambda)$, is evident from the equalities

$$\langle 0|A(\lambda)|0\rangle = a(\lambda)\langle 0|0\rangle, \quad \langle 0|D(\lambda)|0\rangle = d(\lambda)\langle 0|0\rangle.$$

The vacuum states are also the eigenstates of the transfer matrix $\tau(\lambda) = \mathrm{tr}\, T(\lambda) = A(\lambda) + D(\lambda)$,

$$\tau(\lambda)|0\rangle = (a(\lambda) + d(\lambda))|0\rangle, \quad \langle 0|\tau(\lambda) = (a(\lambda) + d(\lambda))\langle 0|\,. \tag{23.21}$$

Other Bethe eigenstates of the transfer matrix can be constructed from the vacuum state by applying the scheme of the algebraic Bethe Ansatz. It is done in the next Section.

How is the vacuum state in the model constructed? It is a very important question since usually operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ are not given explicitly enough, being expressed in a rather complicated way in terms of the local variables of the model. It appears, however, that in many cases the problem of constructing the vacuum state monodromy matrix is reduced to constructing vacuum states for local L-operators which is usually not difficult. It is done in the following way.

Consider the monodromy matrix $T(\lambda)$ of a lattice model given as a product of local L-operators,

$$T(\lambda) = L(M|\lambda)\dots L(m|\lambda)\dots L(1|\lambda) \equiv \prod_{m=1}^{\overleftarrow{M}} L(m|\lambda)\,, \tag{23.22}$$

the matrix elements of the L-operators at different sites commuting between themselves. In other words, each $L(m|\lambda)$ acts nontrivially only in its own local quantum space $\mathcal{H}_m$, the monodromy matrix $T(\lambda)$ acting nontrivially in the whole "global" quantum space $\mathcal{H}$,

$$\mathcal{H} = \bigotimes_{m=1}^{M} \mathcal{H}_m\,.$$

Assume that there exists the "local vacuum" state $|0\rangle_m \in \mathcal{H}_m$ for each $L(m|\lambda)$,

$$L(m|\lambda) = \begin{pmatrix} A_m(\lambda) & B_m(\lambda) \\ C_m(\lambda) & D_m(\lambda) \end{pmatrix}, \tag{23.23}$$

with the properties

$$C_m(\lambda)|0\rangle_m = 0, \quad A_m(\lambda)|0\rangle = a_m(\lambda)|0\rangle_m, \quad D_m(\lambda)|0\rangle_m = d_m(\lambda)|0\rangle_m\,. \tag{23.24}$$

Then, the state $|0\rangle \in \mathcal{H}$,

$$|0\rangle = \bigotimes_{m=1}^{M} |0\rangle_m\,, \tag{23.25}$$

is the vacuum state (23.19) for the monodromy matrix $T(\lambda)$ (23.22) with the vacuum eigenvalues

$$a(\lambda) = \prod_{m=1}^{M} a_m(\lambda), \quad d(\lambda) = \prod_{m=1}^{M} d_m(\lambda) \ . \tag{23.26}$$

Let us prove this statement. Denote by $T(\lambda)|0\rangle$ the matrix,

$$T(\lambda)|0\rangle \equiv \begin{pmatrix} A(\lambda)|0\rangle & B(\lambda)|0\rangle \\ C(\lambda)|0\rangle & D(\lambda)|0\rangle \end{pmatrix}.$$

For the monodromy matrix (23.22), acting on the state (23.24), one can write, using (23.24)

$$T(\lambda)|0\rangle = \left(\prod_{m=1}^{\overleftarrow{M}} L_0(m|\lambda) \right) |0\rangle \ ,$$

where $L_0(m|\lambda)$ is the 2×2 matrix,

$$L_0(m|\lambda) \equiv \begin{pmatrix} a_m(\lambda) & B_m(\lambda) \\ 0 & d_m(\lambda) \end{pmatrix}.$$

Now one should take into account the multiplication rule for the upper triangular matrices,

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \gamma_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 & \alpha_1\beta_2 + \beta_1\gamma_2 \\ 0 & \gamma_1\gamma_2 \end{pmatrix}.$$

It is then evident that

$$T(\lambda)|0\rangle = \begin{pmatrix} a(\lambda) & B(\lambda) \\ 0 & d(\lambda) \end{pmatrix} |0\rangle \ ,$$

with $a(\lambda)$ and $d(\lambda)$ given by (23.26), which finishes the proof.

An analogous construction can be done for the dual vacuum $\langle 0|$ (23.20). If it is possible to construct the local dual vacuum states ${}_m\langle 0|$ with the properties

$${}_m\langle 0|B_m(\lambda) = 0, \quad {}_m\langle 0|A_m(\lambda) = a_m(\lambda)\, {}_m\langle 0|, \quad {}_m\langle 0|D_m(\lambda) = d_m(\lambda)\, {}_m\langle 0|; \tag{23.27}$$

$${}_m\langle 0|0\rangle_m = 1,$$

then

$$\langle 0| = \bigotimes_{m=1}^{M} {}_m\langle 0|. \tag{23.28}$$

24 The algebraic Bethe Ansatz. Bethe eigenstates and Bethe equations.

The operator $C(\lambda)$ annihilating the vacuum state $|0\rangle$ can be regarded as an analog of the annihilation operator. Let us regard the operator $B(\lambda)$ as a kind of a creation operator and consider the states generated by it from the vacuum state,

$$B(\lambda_1)B(\lambda_2)\dots B(\lambda_N)|0\rangle = \left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle. \tag{24.1}$$

The order of the operators $B(\lambda_j)$ in the product is not essential since they commute, see (23.4). The commutation relations (23.15) between the operators B and C are not, of course, the canonical commutation relations between the creation and annihilation operators; they are more complicated. The commutation relations of the operators $A(\lambda)$ and $D(\lambda)$ with the operators $B(\lambda_j)$ are given by (23.7) and (23.8) as

$$A(\lambda)B(\lambda_j) = f(\lambda,\lambda_j)B(\lambda_j)A(\lambda) + g(\lambda_j,\lambda)B(\lambda)A(\lambda_j), \tag{24.2}$$

$$D(\lambda)B(\lambda_j) = f(\lambda_j,\lambda)B(\lambda_j)D(\lambda) + g(\lambda,\lambda_j)B(\lambda)D(\lambda_j). \tag{24.3}$$

If not for the second ("unwanted") terms in the right-hand sides of these relations, the state (24.1) would be an eigenstate of both operators $A(\lambda)$ and $D(\lambda)$ for any set of the parameters $\lambda_1,\lambda_2,\dots,\lambda_N$, with the eigenvalues $a(\lambda)\prod_{j=1}^{N} f(\lambda,\lambda_j)$ and $d(\lambda)\prod_{j=1}^{N} f(\lambda_j,\lambda)$, correspondingly. In fact, it is not the case due to the presence of the unwanted terms.

The idea of the algebraic Bethe Ansatz is that one can nevertheless construct eigenstates of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$ of the form (24.1) choosing the set of λ_j's in such a way that the unwanted terms arising under the action of operators $A(\lambda)$ and $D(\lambda)$ cancel each other.

Let us formulate the result. The state

$$|\Psi_N(\{\lambda_j\})\rangle = \left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle, \qquad \lambda_j \neq \lambda_k \quad (j \neq k) \tag{24.4}$$

is an eigenstate of the transfer matrix $\tau(\lambda)$,

$$\tau(\lambda)|\Psi_N(\{\lambda_j\})\rangle = \Theta(\lambda,\{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle, \tag{24.5}$$

if the set $\{\lambda_j;\ j = 1,2,\dots,N;\ \lambda_j \neq \lambda_k\ (j \neq k)\}$ satisfies the system of Bethe equations,

$$\frac{a(\lambda_j)}{d(\lambda_j)}\prod_{\substack{n=1\\ n\neq j}}^{N}\frac{f(\lambda_j,\lambda_n)}{f(\lambda_n,\lambda_j)} = 1, \qquad j = 1,2,\dots,N. \tag{24.6}$$

The eigenvalue Θ of the transfer matrix for this state is

$$\Theta(\lambda, \{\lambda_j\}) = a(\lambda) \prod_{j=1}^{N} f(\lambda, \lambda_j) + d(\lambda) \prod_{j=1}^{N} f(\lambda_j, \lambda). \tag{24.7}$$

Let us prove this statement. Consider first the action of the operator $A(\lambda)$ onto the state (24.1). By means of the commutation relations (24.2), one can commute the operator $A(\lambda)$ to the utmost right position, using then the fact (23.19) that the vacuum state is an eigenstate

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad A(\lambda_j)|0\rangle = a(\lambda_j)|0\rangle. \tag{24.8}$$

It is not difficult to see that the structure of the answer thus obtained should be as follows:

$$\begin{aligned} A(\lambda) \left(\prod_{j=1}^{N} B(\lambda_j) \right) |0\rangle = a(\lambda)\Lambda(\lambda, \{\lambda_j\}) \left(\prod_{j=1}^{N} B(\lambda_j) \right) |0\rangle \\ + \sum_{n=1}^{N} a(\lambda_n)\Lambda_n(\lambda, \{\lambda_j\}) B(\lambda) \left(\prod_{\substack{j=1 \\ j \neq n}}^{N} B(\lambda_j) \right) |0\rangle. \end{aligned} \tag{24.9}$$

Indeed, while moving the operator A to the right, one can use in commuting with each operator $B(\lambda_j)$ either the "wanted" term (the first one in the right-hand side) or the "unwanted" term (the second one) in equation (24.2), obtaining thus the sum of 2^N terms after putting the operator A to the utmost right position. The first term in the right-hand side of equation (24.9) is obtained, if only the wanted terms are used in the commutation. In all the other cases one should also use some number of the unwanted commutations, under which the operator $A(\lambda)$ exchanges its argument λ with some operator $B(\lambda_k)$ (it is the operator with which the unwanted commutator is used for the first time),

$$A(\lambda)B(\lambda_k) \to g(\lambda_k, \lambda)B(\lambda)A(\lambda_k).$$

After this, $A(\lambda_k)$ appears instead of $A(\lambda)$; it can exchange its argument λ_k only with some of $B(\lambda_l)$ situated to the right of $B(\lambda)$ (remind that $B(\lambda)$ has "replaced" $B(\lambda_k)$). Finally, the operator A comes to the utmost right position, as $A(\lambda_n)$ (since $B(\lambda_n)$ is the operator with which one used the unwanted commutator for the last time), and it acts on the vacuum state as $A(\lambda_n)|0\rangle = a(\lambda_n)|0\rangle$. It is clear that all the operators $B(\lambda_j)$ with $j \neq n$ are present in what remains, and also $B(\lambda)$. Picking up all the terms of this kind together, one gets just the n-th term of the sum in (24.9).

To calculate the coefficient Λ is very easy,

$$\Lambda(\lambda, \{\lambda_j\}) = \prod_{j=1}^{n} f(\lambda, \lambda_j). \tag{24.10}$$

The expressions for the coefficients Λ_n obtained by applying the scheme of the calculation described above are rather complicated, and they seem to be depending on the position which the operator $B(\lambda_n)$ is placed on in the expression (24.1). The state (24.1) is, however, symmetrical under the permutations of λ_j's due to the commutativity of operators B,

$$B(\lambda_j)B(\lambda_k) = B(\lambda_k)B(\lambda_j).$$

So the coefficients Λ_n and Λ_k should be obtained from each other by the change $\lambda_n \leftrightarrow \lambda_k$, and it is sufficient to calculate any one of them, say, Λ_1.

To do this, let us put the operator $B(\lambda_1)$ to the very left place (next to the operator $A(\lambda)$),

$$A(\lambda)B(\lambda_1)B(\lambda_2)\cdots B(\lambda_N)|0\rangle.$$

We are interested in the coefficient Λ_1 at the term in the right-hand side of (24.9), which does not contain the operator $B(\lambda_1)$. The only possibility to "eliminate" $B(\lambda_1)$ is to use the unwanted commutator,

$$A(\lambda)B(\lambda_1) \to g(\lambda_1, \lambda)B(\lambda)A(\lambda_1),$$

at the first step, otherwise the operator $B(\lambda)$ would remain. In commuting the operator $A(\lambda_1)$ which appears after the first commutation with all the other operators $B(\lambda_j)$ $(j = 2, 3, \ldots, N)$, one should use the wanted commutator, otherwise $B(\lambda_1)$ will appear again. Hence,

$$\Lambda_1(\lambda, \{\lambda_j\}) = g(\lambda_1, \lambda) \prod_{j=2}^{N} f(\lambda_1, \lambda_j).$$

Due to the symmetry, Λ_n is obtained simply by the change $\lambda_1 \leftrightarrow \lambda_n$, so that

$$\Lambda_n(\lambda, \{\lambda_j\}) = g(\lambda_n, \lambda) \prod_{\substack{j=1 \\ j\neq n}}^{N} f(\lambda_n, \lambda_j). \tag{24.11}$$

In other words, the simplest way to get Λ_n is to write the state (24.1) as

$$\left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle = B(\lambda_n)\left(\prod_{\substack{j=1 \\ j\neq n}}^{N} B(\lambda_j)\right)|0\rangle,$$

and then to perform the computation (a nice trick suggested by L.Faddeev and L.Takhtadjan in the early days of the quantum inverse scattering method).

It is to be noted that different expressions for Λ_n could be obtained depending on the place on which the operator $B(\lambda_n)$ is situated in the "chain" of operators $B(\lambda_j)$ (24.1). They are all equivalent, of course (due to a lot of identities involving functions f and g), though it is rather difficult to check this equivalence directly.

The action of the operator $A(\lambda)$ on the state (24.1) is calculated. In quite a similar way, one also calculates the action of the operator $D(\lambda)$:

$$D(\lambda)\left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle = d(\lambda)\tilde{\Lambda}(\lambda,\{\lambda_j\})\left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle + \sum_{n=1}^{N} d(\lambda_n)\tilde{\Lambda}_n(\lambda,\{\lambda_j\})B(\lambda)\left(\prod_{\substack{j=1\\ j\neq n}}^{N} B(\lambda_j)\right)|0\rangle \tag{24.12}$$

with

$$\tilde{\Lambda}(\lambda,\{\lambda_j\}) = \prod_{j=1}^{N} f(\lambda_j,\lambda), \tag{24.13}$$

$$\tilde{\Lambda}_n(\lambda,\{\lambda_j\}) = g(\lambda,\lambda_n)\prod_{\substack{j=1\\ j\neq n}}^{N} f(\lambda_j,\lambda_n). \tag{24.14}$$

The state (24.1) in the case of a general set $\{\lambda_j\}$ is not an eigenstate of the operators $A(\lambda)$ or $D(\lambda)$. It can be, however, made an eigenstate of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$, if one demands that the "unwanted" terms in the right-hand sides of equations (24.9) and (24.12) cancel one another (for all given structures in operators B) in the action of $\tau(\lambda)$ on the state (24.1), i.e.,

$$a(\lambda_n)\Lambda_n(\lambda,\{\lambda_j\}) + d(\lambda_n)\tilde{\Lambda}_n(\lambda,\{\lambda_j\}) = 0, \quad n = 1,2,\ldots,N,$$

which thus gives the system (24.6) of Bethe equations (since $g(\lambda_n,\lambda) = -g(\lambda,\lambda_n)$). The corresponding eigenvalue of the operator $\tau(\lambda)$ is equal to

$$a(\lambda)\Lambda(\lambda,\{\lambda_j\}) + d(\lambda)\tilde{\Lambda}(\lambda,\{\lambda_j\})$$

or, equivalently, to the expression (24.7).

The statement (24.4)–(24.7) is thus proved.

25 The algebraic Bethe Ansatz. Some remarks.

(a) We constructed eigenvectors of the transfer matrix starting from the generating (vacuum) state $|0\rangle$ (23.19). If the dual vacuum state $\langle 0|$ (23.20) exists, then one can construct the dual eigenvectors in quite a similar way. Let us formulate the corresponding statement.

The dual state

$$\langle\Psi_N(\{\lambda_j\})| = \langle 0|\left(\prod_{j=1}^{N} C(\lambda_j)\right), \qquad \lambda_j \neq \lambda_k \quad (j \neq k), \tag{25.1}$$

is the left eigenstate of the transfer matrix,

$$\langle\Psi_N(\{\lambda_j\})|\tau(\lambda) = \Theta(\lambda,\{\lambda_j\})\langle\Psi_N(\{\lambda_j\})|, \tag{25.2}$$

if the λ_j's satisfy the same system of Bethe equations (24.6) as for the right eigenstate (24.4). The eigenvalue Θ is also the same (24.7).

(b) If the sets $\{\lambda_j^B;\, j=1,\ldots,N_B\}$ and $\{\lambda_j^C;\, j=1,\ldots,N_C\}$ are different (and this is evidently true if if $N_B \neq N_C$), the corresponding states are "orthogonal", i.e.,

$$\langle \Psi_{N_C}(\{\lambda_j^C\})|\Psi_{N_B}(\{\lambda_j^B\})\rangle = 0 \qquad \left(\{\lambda^C\} \neq \{\lambda^B\}\right). \tag{25.3}$$

If $N_B \neq N_C$, the orthogonality is easily seen from the structure of commutation relations (23.3)–(23.18), by moving, in expression

$$\langle 0| \left(\prod_{j=1}^{N_C} C(\lambda_j^C)\right) \left(\prod_{j=1}^{N_B} B(\lambda_j^B)\right) |0\rangle \ ,$$

the operators C to the right, and the operators B to the left. It should be emphasized that if $N_B \neq N_C$, the quantity above equals zero also if the sets λ^B and λ^C do not satisfy the Bethe equations.

If $N^B = N^C$, the states are orthogonal only if the sets satisfy the Bethe equations. To prove the orthogonality, one calculates the matrix element of the transfer matrix between the Bethe eigenstates acting by $\tau(\lambda)$ to the right and to the left,

$$\begin{aligned}\langle \Psi_{N_C}(\{\lambda_j^C\})|\tau(\lambda)|\Psi_{N_B}(\{\lambda_j^B\})\rangle &= \Theta(\lambda,\{\lambda_j^B\})\langle \Psi_{N_C}(\{\lambda_j^C\})|\Psi_{N_B}(\{\lambda_j^B\})\rangle \\ &= \Theta(\lambda,\{\lambda_j^C\})\langle \Psi_{N_C}(\{\lambda_j^C\})|\Psi_{N_B}(\{\lambda_j^B\})\rangle,\end{aligned}$$

so that the eigenstates are orthogonal if $\Theta(\lambda,\{\lambda_j^B\}) \neq \Theta(\lambda,\{\lambda_j^C\})$, $\forall\lambda$, which is so if $\{\lambda_j^C\} \neq \{\lambda_j^B\}$.

(c) A very interesting and important point is to calculate the "norm" of a Bethe eigenstate (in our algebraic formulation, the "norm" should not be obligatorily positive), i.e., the quantity

$$\mathcal{N}(\{\lambda_j\}) \equiv \langle \Psi_N(\{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle$$

where the momenta λ_j satisfy the Bethe equations (24.6). The hypothetical answer was given long ago by M.Gaudin, but the proof was provided considerably later only within the frame of the quantum inverse scattering method (V.Korepin).

In the case of the rational (XXX) R-matrix, the answer is given as

$$\mathcal{N}(\{\lambda_j\}) = c^N \left(\prod_{j=1}^{N} a(\lambda_j)d(\lambda_j)\right) \det{}_N\left(\Phi\right) \tag{25.4}$$

where $\det_N(\Phi)$ denotes the determinant of the $N\times N$ dimensional matrix with the matrix elements Φ_{jk} given as

$$\Phi_{jk} = \delta_{jk}\left(z_k + \sum_{l=1}^{N} K(\lambda_k,\lambda_l)\right) - K(\lambda_j,\lambda_k) \tag{25.5}$$

with

$$z_k = i\frac{\partial}{\partial\lambda}\left(\frac{a(\lambda)}{d(\lambda)}\right)\Big|_{\lambda=\lambda_k}, \tag{25.6}$$

and

$$K(\lambda,\mu) = i\frac{\partial}{\partial\lambda}\ln\left(\frac{f(\lambda,\mu)}{f(\mu,\lambda)}\right) = \frac{2c}{(\lambda-\mu)^2+c^2} \tag{25.7}$$

is exactly the kernel entering the Lieb equation (see (38.8) further). The analogous formula can be written also for the models with the XXZ R-matrix.

(d) In our derivations, we assume that all λ_j's entering the set $\{\lambda_j\}$ are different, $\lambda_j \neq \lambda_k$ if $j \neq k$. It is essential. Why should not equal λ's be taken in the Bethe states? Let us explain it. Assume that, e.g., $\lambda_1 = \lambda_2$, all the other λ_j's ($j \geq 3$) being different. Consider the state (24.1) in this case,

$$B^2(\lambda_1)\left(\prod_{j=3}^{N} B(\lambda_j)\right)|0\rangle \qquad (\lambda_j \neq \lambda_k,\ j \neq k)\,.$$

Producing the calculation along the line of Section 24, one gets a system of equations for quantities λ_j, demanding that this state be an eigenstate of the transfer matrix. It is a system of N equations for $N-1$ values λ_j ($j = 1, 3, 4, \ldots, N$). Even if this system possesses solutions, they are much more "restricted" than in the case where all λ's are different. But for many systems (e.g., for the nonrelativistic Bose gas in the repulsive case) this new system has no solutions at all. This fact is closely related to the "Pauli principle" for interacting one dimensional bosons discussed earlier.

(e) In constructing the scheme of the algebraic Bethe Ansatz, an assumption was made that the generating states, $|0\rangle$ and $\langle 0|$, exist with the vacuum eigenvalues $a(\lambda)$ and $d(\lambda)$. The examples of constructing such states in a concrete model will be given further, the corresponding functions $a(\lambda)$ and $d(\lambda)$ will be also found. But one can put an important question: does the compatibility of the commutation relations between the matrix elements of the monodromy matrix (given by the R-matrix) result in some restrictions for the functions $a(\lambda)$ and $d(\lambda)$ within the frame of a general algebraic scheme? The answer is that one can formally construct the "abstract" monodromy matrix with any given functions $a(\lambda)$ and $d(\lambda)$, which may be thus considered as "free functional parameters" of some generalized model with XXX or XXZ R-matrix (in the XXZ case, there are some unessential restrictions due to the periodicity in the spectral parameter). This result, in particular, allows one to consider the $N+1$ terms in the right-hand side of equation (24.9) as linearly independent.

26 The quantum determinant.

The existence of the R-matrix for the quantum model allows one to generalize the notion of the determinant for the monodromy matrix which is nontrivial because its matrix elements do not commute with each other. Consider, as an example, a model with the rational R-matrix (21.3), (21.4),

$$R(\lambda,\mu) = (\lambda-\mu)E + ic\Pi. \tag{26.1}$$

At point $\lambda-\mu=-ic$, this R-matrix is proportional to the projector,

$$R(\lambda,\mu=\lambda+ic) = -ic(E-\Pi), \quad (E-\Pi)^2 = 2(E-\Pi).$$

Correspondingly, the functions f and g entering the commutation relations (23.3)–(23.18) become very simple,

$$f(\lambda,\mu=\lambda+ic)=0, \quad g(\lambda,\mu=\lambda+ic)=1.$$

Using this fact, one can easily check that the "scalar" quantum operator $\det\limits_q T(\lambda)$

$$\begin{aligned}\det_q T(\lambda) &\equiv A(\lambda-ic/2)D(\lambda+ic/2) - B(\lambda-ic/2)C(\lambda+ic/2)\\ &= D(\lambda-ic/2)A(\lambda+ic/2) - C(\lambda-ic/2)B(\lambda+ic/2)\end{aligned} \tag{26.2}$$

is commuting with the quantum operators $A(\mu)$, $B(\mu)$, $C(\mu)$, and $D(\mu)$, i.e., with all the matrix elements of the monodromy matrix,

$$\Big[\det_q T(\lambda), T_{i_1,i_2}(\mu)\Big] = 0, \quad \forall\lambda,\mu. \tag{26.3}$$

This means that the quantum determinant is a "c-number", being in the center of the "algebra" defined by the commutation relations (23.3)–(23.18).

If there exist the vacuum states $|0\rangle$ and $\langle 0|$ in a model, one can easily compute the value of the quantum determinant,

$$\langle 0|\det_q T(\lambda)|0\rangle = a(\lambda-ic/2)d(\lambda+ic/2)\langle 0|0\rangle. \tag{26.4}$$

A remarkable property of the quantum determinants generalizing the corresponding property of the ordinary determinants is their multiplicativity. Assume that there are two monodromy matrices, $T_1(\lambda)$ and $T_2(\lambda)$, the matrix elements of different matrices commuting,

$$\Big[(T_1(\lambda))_{i_1 i_2}, (T_2(\mu))_{k_1 k_2}\Big] = 0, \tag{26.5}$$

and for the same monodromy matrix, the commutation relations are given by the rational R-matrix,

$$\tilde{R}(\lambda,\mu)\,(T_i(\lambda)\otimes T_i(\mu)) = (T_i(\mu)\otimes T_i(\lambda))\,\tilde{R}(\lambda,\mu) \quad (i=1,2). \tag{26.6}$$

Then also the monodromy matrix $T(\lambda)$,

$$T(\lambda) \equiv T_1(\lambda)T_2(\lambda), \quad (T(\lambda))_{i_1 i_2} = (T_1(\lambda))_{i_1 k} (T_2(\lambda))_{k i_2}$$

(the sum over k is assumed) satisfies the relation with the same R-matrix,

$$\tilde{R}(\lambda, \mu) (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) \tilde{R}(\lambda, \mu).$$

It is not very difficult to check that then

$$\det_q T(\lambda) = \det_q T_1(\lambda) \det_q T_2(\lambda). \tag{26.7}$$

In particular, the quantum determinant of the monodromy matrix constructed from the L-operators,

$$T(\lambda) = \overleftarrow{\prod_{m=1}^{M}} L(m|\lambda),$$

is equal to the product of the quantum determinants of the L-operators,

$$\det_q T(\lambda) = \prod_{m=1}^{M} \det_q L(m|\lambda). \tag{26.8}$$

The notion of a quantum determinant is easily generalized in the case of the XXZ R-matrix; it can also be done for the models with other R-matrices.

27 The quantum inverse scattering method for the quantum nonlinear Schrödinger equation. The transfer matrix and the trace identities.

The quantum nonlinear Schrödinger (NS) equation describes the nonrelativistic Bose gas with the δ-function pair potential between the particles. It was considered in Sections 2–8 within the frame of the coordinate Bethe Ansatz. Let us now apply the quantum inverse scattering method to it. As discussed in Section 17, for quantization it is convenient to put even a continuous system onto a lattice, with M sites and lattice spacing Δ, going then to the continuous limit,

$$\Delta \longrightarrow 0, \qquad M \longrightarrow \infty, \qquad L = M\Delta \quad \text{fixed}, \tag{27.1}$$

where L is the length of the space box. The periodic boundary conditions are assumed to be imposed. The infinitesimal L-operator for the quantum NS model is given, analogously to the classical case (see (17.13)) as

$$L(m|\lambda) = I - V(x_m|\lambda)\Delta + O(\Delta^2) \tag{27.2}$$

where I is the unit 2×2 matrix, and the potential $V(x_m|\lambda)$,

$$V(x_m|\lambda) = \begin{pmatrix} i\lambda/2 & i\sqrt{c}\,\psi_n^+ \\ -i\sqrt{c}\,\psi_n & -i\lambda/2 \end{pmatrix}, \tag{27.3}$$

contains now the quantum boson fields (quantum operators) $\psi_m \approx \psi(x_m)$, $\psi_m^+ \approx \psi^+(x_m)$ ($x_m = m\Delta$) with the commutation relations (compare with (17.4))

$$[\psi_n, \psi_m] = [\psi_n^+, \psi_m^+] = 0, \qquad [\psi_n, \psi_m^+] = \frac{1}{\Delta}\delta_{nm}. \tag{27.4}$$

The lattice monodromy matrix is defined as a matrix product

$$T(M, 1|\lambda) \equiv T_M(\lambda) \equiv L(M|\lambda)L(M-1|\lambda)\dots L(1|\lambda), \tag{27.5}$$

and the monodromy matrix $T(\lambda)$ of the continuous NS equation is obtained after taking the limit (27.1),

$$T(\lambda) \equiv T(L, 0|\lambda) = \lim T_M(\lambda). \tag{27.6}$$

Of course, this agrees completely with the possibility of writing the solution in the continuous case as

$$T(\lambda) = \mathcal{P} \exp\left\{ -\int_0^L V(z|\lambda) dz \right\} \tag{27.7}$$

where $\mathcal{P}$ denotes the usual space ordering of the exponential. Already in the classical case, this space ordering should be present due to the noncommutativity of matrices $V(z|\lambda)$. In the quantum case, it is related also to the noncommutativity of quantum operators. From the definition of the transition matrix $T(x, y|\lambda)$ and of the monodromy matrix $T(\lambda)$, it is obvious that the expression for the matrix $T_q(\lambda)$ in the quantum case is obtained from the expression for the classical monodromy matrix $T_{cl}(\lambda)$ by substituting the quantum operators instead of the classical fields and then by the normal ordering of the local quantum operators (all the operators ψ^+ should be put to the left of the operators ψ),

$$T_q(\lambda) =: T_{cl}(\lambda) :, \tag{27.8}$$

giving, of course, nothing new in the expression for $T_q(\lambda)$ at this stage.

Let us discuss now the trace identities in the quantum case, i.e., the expressions for the Hamiltonian and other conservation laws in terms of the transfer matrix $\tau(\lambda) = \operatorname{tr} T(\lambda)$. In the classical case, we had the expansion (12.2),

$$\ln \tau_{cl}(\lambda)\Big|_{\lambda \to +i\infty} = -\frac{i\lambda L}{2} + ic \sum_{n=1}^{\infty} \lambda^{-n} I_n^{(cl)} \tag{27.9}$$

where the quantities $I_n^{(cl)}$ are the commuting conservation laws, the first three of them being the charge, the momentum and the Hamiltonian,

$$I_1^{(cl)} = Q, \quad I_2^{(cl)} = P, \quad I_3^{(cl)} = H, \quad \dots \ ; \quad \left\{ I_n^{(cl)}, I_m^{(cl)} \right\} = 0. \tag{27.10}$$

The derivation of Section 12, until equation (12.11), remains valid also in the quantum case (after taking into account the normal ordering : :). But in equation (12.11), though $W_n(x|\lambda)$ (and $D(x,y|\lambda)$) are diagonal matrices, they are not commuting since their matrix elements contain quantum operators $\psi(z)$, $\psi^+(z)$. Hence, the correct expression for $D_q(L,0|\lambda)$ is

$$D_q(L,0|\lambda) = :\mathcal{P}\exp\left\{-\frac{i\lambda L}{2}\sigma_3 + \sum_{n=1}^{\infty}\lambda^{-n}\int_0^{\infty} W_n^{(cl)}(z|\lambda)dz\right\}: \tag{27.11}$$

(here, of course, the quantum operators $\psi(z), \psi^+(z)$ should be inserted instead of the classical fields into the functions $W_n^{(cl)}(z|\lambda)$). So, one has the following expansion for D_q

$$D_q(L,0|\lambda) = \mathrm{e}^{-\frac{i\lambda L}{2}\sigma_3}\left(I + \alpha_1\lambda^{-1} + \alpha_2\lambda^{-2} + \alpha_3\lambda^{-3} + \dots\right). \tag{27.12}$$

The expressions for the 2×2 matrices α_n are obtained by iterating the equation (12.11), taking into account that due to the presence of quantum fields the diagonal matrices W_n (12.17) do not commute in the quantum case:

$$\begin{aligned}
\alpha_1 &= -:\int_0^L W_1(z)dz:,\\
\alpha_2 &= -:\int_0^L W_2(z)dz: + :\int_0^L W_1(z)dz\int_0^z W_1(z_1)dz_1:,\\
\alpha_3 &= -:\int_0^L W_3(z)dz: + :\int_0^L W_2(z)dz\int_0^z W_1(z_1)dz_1:\\
&\quad + :\int_0^L W_1(z)dz\int_0^z W_2(z_1)dz_1: - :\int_0^L W_1(z)dz\int_0^z W_1(z_1)dz_1\int_0^{z_1} W_1(z_2)dz_2:.
\end{aligned} \tag{27.13}$$

As in the classical case, $(D_q)_{11} \gg (D_q)_{22}$ at $\lambda\to\infty$, so that

$$\tau_q(\lambda) = \mathrm{e}^{-\frac{i\lambda L}{2}}\left(1 + a_1\lambda^{-1} + a_2\lambda^{-2} + a_3\lambda^{-3} + \dots\right), \quad \lambda\to\infty \tag{27.14}$$

where the "scalar" a_i denotes the matrix element 11 of the 2×2 matrix α_i, $a_i \equiv (\alpha_i)_{11}$. Taking the logarithm, one obtains in the quantum case the expansion

$$\ln\left(\mathrm{e}^{\frac{i\lambda L}{2}}\tau_q(\lambda)\right) = \lambda^{-1}b_1 + \lambda^{-2}b_2 + \lambda^{-3}b_3 + \dots \tag{27.15}$$

where

$$b_1 = a_1, \quad b_2 = a_2 - \frac{a_1^2}{2}, \quad b_3 = a_3 - \frac{a_1a_2 + a_2a_1}{2} + a_1^3, \quad \dots\ . \tag{27.16}$$

Quantities a_1, a_2, a_3 (and b_1) are normally ordered (see(27.13)), but the products a_1^2, a_1a_2, a_2a_1, a_1^3 are not. Putting them into the normally ordered form one gets

$$b_1 = icQ, \quad b_2 = icP + \frac{c^2}{2}Q, \quad b_3 = icH + c^2P - \frac{ic^3}{3}Q, \quad \dots \tag{27.17}$$

where Q, P, and H are the operator of number of particles, the momentum, and the Hamiltonian of the quantum NS equation. Finally, one produces the quantum trace identities

$$\ln \tau_q(\lambda)\Big|_{\lambda \leftarrow +i\infty} = -\frac{i\lambda L}{2} + ic \sum_{n=1}^{\infty} \lambda^{-n} I_n^{(q)} \tag{27.18}$$

with

$$I_1 = Q, \quad I_2 = P - \frac{icQ}{2}, \quad I_3 = H - icP - \frac{c^2}{3}Q, \quad \ldots, \tag{27.19}$$

which differ by the "quantum corrections" from the classical trace identities (27.9), (27.10).

Let us emphasize again that the trace identities are an "external object" with respect to the inverse scattering method, in the sense that one does not know in advance if a given Hamiltonian can be related to some transfer matrix. Usually, one has to make considerable effort to get them for a given model.

28 The quantum inverse scattering method for the quantum nonlinear Schrödinger equation. The R-matrix.

Now we need to get the quantum R-matrix for the quantum NS model. It can be computed, of course, by means of the infinitesimal operator (27.2) using relation (19.9), but it may be not the best way, especially that the infinitesimal L-operator is defined up to the terms of the order $O(\Delta^2)$. It is much more reliable to start from the Yang-Baxter relation (20.2), finding the solution of it corresponding to the quantum NS equation. How is this done?

One looks for the 4×4 dimensional matrix $R(\lambda, \mu) = R(\lambda - \mu)$ with the rational dependence on the spectral parameters (due to the possibility of multiplying the R-matrix by any scalar function $\phi(\lambda, \mu)$ the dependence can be made, in fact, polynomial). Also, this R-matrix should have the correct quasiclassical limit (22.13), with r being the classical rational r-matrix (14.13) for the classical NS equation. It is natural to demand that the symmetry relation (10.8), (12.8) takes place also in the quantum case (the star there should be understood as the hermitean conjugation of the quantum operators ψ, ψ^+ and the complex conjugation of the c-number parameters, without transposing the matrix elements of the matrices involved). This relation can be seen to give the analogous symmetry relation for the R-matrix (one should use the bilinear relation (19.11) to get it).

As a result, one comes just to the rational (or XXX) R-matrix (21.3), (21.4).

This approach to calculate the quantum R-matrices, based on constructing the solution of the Yang-Baxter equation with a known quasiclassical limit preserving the symmetries of the classical R-matrix, is of special importance in quantizing, e.g.,

relativistic field theory models. In this case, due to the ultaviolet divergencies, the construction of the quantum Hamiltonian operator and of the quantum R-matrix is very difficult to perform without constructing the quantum R-matrix first.

To conclude this small but important Section, it is worth mentioning that for the XXX R-matrix (21.3), (21.4) one can construct, similarly to the classical case (see (17.16)), the exact L-operator for the lattice NS model. It is

$$L(m|\lambda) = \begin{pmatrix} 1 - \dfrac{i\lambda\Delta}{2} + \dfrac{c\Delta^2}{2}\psi_m^+\psi_m & -i\sqrt{c}\,\Delta\psi_m^+\tilde{\rho}_m \\ i\sqrt{c}\,\Delta\rho_m\psi_m & 1 + \dfrac{i\lambda\Delta}{2} + \dfrac{c\Delta^2}{2}\psi_m^+\psi_m \end{pmatrix}, \tag{28.1}$$

where ρ_m and $\tilde{\rho}_m$ are functions of the operator $\psi_m^+\psi_m$, of the number of particles at the m-th site, satisfying the relation

$$\rho_m\tilde{\rho}_m = 1 + \frac{c\Delta^2}{4}\psi_m^+\psi_m. \tag{28.2}$$

29 The quantum inverse scattering method for the quantum nonlinear Schrödinger equation. The algebraic Bethe Ansatz.

To quantize the NS equation, one can use the exact lattice L-operator (28.1). It is not difficult to see, however, that in the case of the NS equation it is quite sufficient to use the infinitesimal L-operator (27.2) which is written explicitly as

$$L(m|\lambda) = \begin{pmatrix} 1 - \dfrac{i\lambda\Delta}{2} & -i\sqrt{c}\psi_m^+\Delta \\ i\sqrt{c}\psi_m\Delta & 1 + \dfrac{i\lambda\Delta}{2} \end{pmatrix} + O(\Delta^2). \tag{29.1}$$

The monodromy matrix $T(\lambda)$ (27.6) of the continuous quantum model was discussed in detail in Section 27. It is given as the limit (27.1) of the monodromy matrix $T^{(M)}(\lambda)$ which is a matrix product of the L-operators along the infinitesimal lattice,

$$T(\lambda) = \lim T^{(M)}(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{29.2}$$

where

$$T^{(M)}(\lambda) = \prod_{m=1}^{\overleftarrow{M}} L(m|\lambda) \equiv \begin{pmatrix} A^{(M)}(\lambda) & B^{(M)}(\lambda) \\ C^{(M)}(\lambda) & D^{(M)}(\lambda) \end{pmatrix}.$$

Let us construct now, following the scheme of Section 23, the vacuum state $|0\rangle$ for the continuous model. One begins with constructing the local vacua $|0\rangle_m$ ($m = 1, 2, \ldots, M$). At the site m, it is a local Fock vacuum annihilates by the operator ψ_m, $\psi_m|0\rangle_m = 0$. This local vacuum is evidently an eigenstate of the diagonal

elements of the L-operator (29.1), the corresponding eigenvalues (see (23.24)) being equal to

$$a_m(\lambda) = 1 - \frac{i\lambda\Delta}{2}; \qquad d_m(\lambda) = 1 + \frac{i\lambda\Delta}{2}. \tag{29.3}$$

So the global vacuum $|0\rangle^{(M)}$ for the infinitesimal lattice is given (see(23.25)) as

$$|0\rangle^{(M)} = \bigotimes_{m=1}^{M} |0\rangle_m, \qquad \psi_m|0\rangle^{(M)} = 0 \quad (m = 1, 2, \ldots, M),$$
$$C^{(M)}(\lambda)|0\rangle^{(M)} = 0, \tag{29.4}$$

the eigenvalues of the diagonal elements $A^{(M)}(\lambda)$ and $D^{(M)}(\lambda)$ of the monodromy matrix $T^{(M)}(\lambda)$ being equal to

$$a^{(M)}(\lambda) = \left(1 - \frac{i\lambda\Delta}{2}\right)^M, \qquad d^{(M)}(\lambda) = \left(1 + \frac{i\lambda\Delta}{2}\right)^M.$$

In the continuous limit, the global vacuum $|0\rangle$ (the limit of the $|0\rangle^{(M)}$) is the Fock vacuum annihilated by the operators $\psi(x)$,

$$\psi(x)|0\rangle = 0, \quad 0 \le x \le L; \qquad C(\lambda)|0\rangle = 0. \tag{29.5}$$

The state $|0\rangle$ is an eigenstate of the operators $A(\lambda)$ and $D(\lambda)$,

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle, \qquad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \tag{29.6}$$

where the eigenvalues are the limits $a(\lambda) = \lim a^{(M)}(\lambda)$ and $d(\lambda) = \lim d^{(M)}(\lambda)$, so that

$$a(\lambda) = \exp\left\{-\frac{i\lambda L}{2}\right\}, \qquad d(\lambda) = \exp\left\{\frac{i\lambda L}{2}\right\}. \tag{29.7}$$

Now one constructs the Bethe eigenstates of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$. They are of the form (24.4),

$$|\Psi_N(\{\lambda_j\})\rangle = \left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle. \tag{29.8}$$

The quasimomenta λ_j $(j = 1, 2, \ldots, N)$ should satisfy the system of Bethe equations (24.6), with functions $a(\lambda)$, $d(\lambda)$ defined by (29.7), and functions $f(\lambda, \mu)$, $g(\lambda, \mu)$ being the matrix elements of the rational R-matrix given in (21.4). Thus, one obtains the Bethe equations of the form

$$\exp\{i\lambda_j L\} = \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic} \qquad (j = 1, 2, \ldots, N), \tag{29.9}$$

which is just the system (7.5) obtained by means of the coordinate Bethe Ansatz.

The eigenvalues Θ of the transfer matrix $\tau(\lambda)$ on the Bethe states,

$$\tau(\lambda)|\Psi_N(\{\lambda_j\})\rangle = \Theta(\lambda,\{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle, \tag{29.10}$$

are (see equation (24.7))

$$\Theta(\lambda,\{\lambda_j\}) = \exp\left\{-\frac{i\lambda L}{2}\right\}\prod_{j=1}^{N}\frac{\lambda-\lambda_j+ic}{\lambda-\lambda_j} + \exp\left\{\frac{i\lambda L}{2}\right\}\prod_{j=1}^{N}\frac{\lambda-\lambda_j-ic}{\lambda-\lambda_j}. \tag{29.11}$$

It seems at first that the function $\Theta(\lambda,\{\lambda_j\})$ as a function of λ has poles at points $\lambda=\lambda_j$. It is not, however, difficult to see that due to the Bethe equations (29.9) the residues at these poles are equal to zero, i.e., in fact, there are no poles. It means also that the system of Bethe equations itself can be obtained by the requirement that the eigenvalue of the transfer matrix has no poles at points $\lambda=\lambda_j$ (the "Manakov's principle").

Let us explain now how the eigenvalues Q_N, P_N, and E_N of the number of particles operator Q, of the momentum operator P, and of the Hamiltonian H,

$$\begin{aligned} Q|\Psi_N(\{\lambda_j\})\rangle &= Q_N|\Psi_N(\{\lambda_j\})\rangle,\\ P|\Psi_N(\{\lambda_j\})\rangle &= P_N(\{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle,\\ H|\Psi_N(\{\lambda_j\})\rangle &= E_N(\{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle, \end{aligned} \tag{29.12}$$

can be obtained from the trace identities. From the equations (27.18), (27.19), expanding both sides of equation (29.10) in λ at $\lambda\to+i\infty$, one gets that the following expansion is valid,

$$\ln\Theta(\lambda,\{\lambda_j\})\Big|_{\lambda\to+i\infty} = -\frac{i\lambda L}{2} + ic\sum_{n=1}^{\infty}\lambda^{-n}J_n, \tag{29.13}$$

where

$$J_1 = Q_N,\quad J_2 = P_N - \frac{ic}{2},\quad J_3 = E_N - icP_N - \frac{c^2}{3}Q_N.$$

It should be mentioned that at $\lambda\to+i\infty$ the second term in the right hand side of equation (29.11) is of order $O(\lambda^{-\infty})$ and does not contribute to the expansion. Taking this into account, one readily computes the eigenvalues

$$Q_N = N,\quad P_N = \sum_{j=1}^{N}\lambda_j,\quad E_N = \sum_{j=1}^{N}\lambda_j^2, \tag{29.14}$$

reproducing the corresponding formulae (7.9)–(7.11) of the coordinate Bethe Ansatz.

Thus, we have demonstrated how the quantum inverse scattering method works for the quantum NS equation.

30 Spin models on a one-dimensional lattice. The fundamental L-operator.

Any R-matrix satifying the Yang-Baxter equation can be regarded as an L-operator of a "spin model" on a one dimensional lattice. A peculiar feature of this L-operator is that the dimension of the quantum space is equal to the dimension of the matrix space. Let us formulate this statement more precisely.

Consider an R-matrix satisfying the Yang-Baxter equation,

$$\overset{(12)}{R}(\lambda,\mu)\overset{(13)}{R}(\lambda,\nu)\overset{(23)}{R}(\mu,\nu)=\overset{(23)}{R}(\mu,\nu)\overset{(13)}{R}(\lambda,\nu)\overset{(12)}{R}(\lambda,\mu). \tag{30.1}$$

In indices, this relation is written as

$$(R(\lambda,\mu))^{i_1 i'}_{k_1 k'}\,(R(\lambda,\nu))^{i' i_2}_{l_1 l'}\,(R(\mu,\nu))^{k' k_2}_{l' l_2}=(R(\mu,\nu))^{k_1 k'}_{l_1 l'}\,(R(\lambda,\nu))^{i_1 i'}_{l' l_2}\,(R(\lambda,\mu))^{i' i_2}_{k' k_2}\,. \tag{30.1a}$$

Here, indices i_1, i_2, i', k_1, k_2, k', and l_1, l_2, l' are related to the first, second, and third vector spaces, correspondingly. The sums over the repeated indices i', k', l' should be taken. Both the left hand side and the right hand side of the equation (30.1) are $N^3\times N^3$ dimensional matrices acting in the tensor product, $V\otimes V\otimes V$, of the three N-dimensional spaces V (see Section 20).

Let us compare the Yang-Baxter equation with the "bilinear" equation (19.2), (19.9) for L-operator,

$$\overset{(12)}{R}(\lambda,\mu)\overset{(1)}{L}(\lambda)\overset{(2)}{L}(\mu)=\overset{(2)}{L}(\mu)\overset{(1)}{L}(\lambda)\overset{(12)}{R}(\lambda,\mu), \tag{30.2}$$

or in indices,

$$(R(\lambda,\mu))^{i_1 i'}_{k_1 k'}\,(L(\lambda))^{\alpha_1\alpha'}_{i' i_2}\,(L(\mu))^{\alpha'\alpha_2}_{k' k_2}=(L(\mu))^{\alpha_1\alpha'}_{k_1 k'}\,(L(\lambda))^{\alpha'\alpha_2}_{i_1 i'}\,(R(\lambda,\mu))^{i' i_2}_{k' k_2}\,. \tag{30.2a}$$

The left-hand side and the right-hand side of equation (30.2) act both in the tensor product $V\otimes V\otimes\mathcal{H}_L$ of two matrix spaces V and of the third, "quantum", space $\mathcal{H}_L$, where the matrix elements $(L(\lambda))_{i_1 i_2}$ of the $N\times N$ matrix L act. The indices α_1, α_2, α' are related to this quantum space.

It is easily seen that the Yang-Baxter equation (30.1) is just the bilinear relation (30.2) for the L-operator defined by the R-matrix itself,

$$(L(\lambda))^{\alpha_1\alpha_2}_{i_1 i_2}\equiv(R(\lambda,\nu))^{i_1 i_2}_{\alpha_1\alpha_2}\,. \tag{30.3}$$

(the spectral parameter ν here can be regarded, from the point of view of the bilinear relation, as a fixed parameter, and the quantum indices α_1,α_2 are identified with the matrix indices of the R-matrix in the third vector space).

This L-operator is usually called the fundamental L-operator. Its peculiar feature is that the dimensions of the quantum space and of the matrix space are the same.

In other words, one regards the third matrix space V in the Yang-Baxter equation (30.1) as a quantum space $\mathcal{H}_L$ in the bilinear relation (30.2).

Remark. The following analogy with the representations of the Lie groups can be given. The bilinear relation

$$\overset{(12)}{R}\ \overset{(1)}{T}\ \overset{(2)}{T} = \overset{(2)}{T}\ \overset{(1)}{T}\ \overset{(12)}{R}$$

gives a set of commutation relations between the matrix elements of the monodromy matrix, the R-matrix defining the set of "structure constants" for these relations. Different monodromy matrices $T(\lambda)$ (with different quantum spaces) correspond to different "representations" of these commutation relations. The Yang-Baxter relation (where the third matrix space is identified with the quantum space) gives the "fundamental" representation of the commutation relations (by analogy with the group theory, it would probably be better to call it "adjoint representation"). The Yang-Baxter relation is thus analogous to the Jacobi identities for the structure constants of the Lie group.

31 Fundamental spin models on a one-dimensional lattice. The local trace identities.

The fundamental L-operator generates a spin model on a one-dimensional lattice which can be called the fundamental spin model for a given R-matrix. A distinctive property of this model is the equality of dimensions of the matrix space to the dimension of the local "quantum" space (the local "spin" space at the site of the lattice).

Remind that the R-matrix $R(\lambda,\nu)$ is defined by the Yang-Baxter equation up to a scalar factor $\varphi(\lambda,\nu)$ (see (22.5)). Let us assume in this Section that this factor can be chosen so that

$$\varphi(\lambda,\nu)\ R(\lambda,\nu)|_{\lambda=\nu} = \Pi\,, \tag{31.1}$$

where Π is the permutation matrix (13.6).

Consider a one-dimensional lattice with M sites, and relate to the site number m ($m = 1, 2, \ldots, M$) the fundamental L-operator

$$(L(m|\lambda))^{\alpha_m\beta_m}_{i_1 i_2} \equiv \varphi(\lambda,\nu)\,(R(\lambda,\nu))^{i_1 i_2}_{\alpha_m\beta_m}\,, \tag{31.2}$$

$$L(m|\lambda)|_{\lambda=\nu} = \Pi, \tag{31.2a}$$

regarding ν as some fixed parameter.

With each site m we associate its own local quantum space $\mathcal{H}_m = V$, where the matrix elements $L(m|\lambda)_{i_1 i_2}$ act, the quantum indices α_m, β_m corresponding to the action in $\mathcal{H}_m$. Due to the Yang-Baxter equation,

$$\overset{(12)}{R}\ (\lambda,\mu)\ \overset{(1)}{L}\ (m|\lambda)\ \overset{(2)}{L}\ (m|\mu) = \overset{(2)}{L}\ (m|\mu)\ \overset{(1)}{L}\ (m|\lambda)\ \overset{(12)}{R}\ (\lambda,\mu),$$

and the matrix elements of $L(m|\lambda)$ and $L(n|\mu)\,(m \neq n)$ are commuting operators.

Construct now the monodromy matrix

$$T(\lambda) = L(M|\lambda)L(M-1|\lambda)\dots L(2|\lambda)L(1|\lambda), \tag{31.3}$$

or, in indices,

$$(T(\lambda))^{\{\alpha\}\{\beta\}}_{i_1 i_2} = (L(M|\lambda))^{\alpha_M \beta_M}_{i_1 i'_M} \, (L(M-1|\lambda))^{\alpha_{M-1}\beta_{M-1}}_{i'_M i'_{M-1}} \dots (L(2|\lambda))^{\alpha_2\beta_2}_{i'_3 i'_2} \, (L(1|\lambda))^{\alpha_1\beta_1}_{i'_2 i_2} \tag{31.3a}$$

(one performs summing up over the repeated (primed) indices). We denote by $\{\alpha\}$, $\{\beta\}$ the "multiindices", e.g., $\{\alpha\} \equiv \alpha_1, \alpha_2, \dots, \alpha_M$. The monodromy matrix is the operator acting as

$$T(\lambda): \quad V \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_M \longrightarrow V \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_M$$

(in our case, $\mathcal{H}_m \equiv V$, being a vector space of dimension N), or

$$T(\lambda): \; V \otimes \mathcal{H} \longrightarrow V \otimes \mathcal{H}, \qquad\qquad \mathcal{H} = \bigotimes_{m=1}^{M} \mathcal{H}_m,$$

$\mathcal{H}$ being the global quantum space of dimension N^M.

The commutation relations between the matrix elements of the monodromy matrix are the same as for the L-operator,

$$\overset{(12)}{R}(\lambda,\mu) \overset{(1)}{T}(\lambda) \overset{(2)}{T}(\mu) = \overset{(2)}{T}(\mu) \overset{(1)}{T}(\lambda) \overset{(12)}{R}(\lambda,\mu). \tag{31.4}$$

The transfer matrix $\tau(\lambda)$,

$$\tau(\lambda) = \operatorname{tr} T(\lambda), \qquad\qquad \tau(\lambda)^{\{\alpha\}\{\beta\}} = (T(\lambda))^{\{\alpha\}\{\beta\}}_{ii}, \qquad\qquad \tau(\lambda): \; \mathcal{H} \longrightarrow \mathcal{H} \tag{31.5}$$

(the trace is taken in the matrix space), is an operator in the global quantum space. The transfer matrix gives a family of commuting quantum operators:

$$\tau(\lambda)\tau(\mu) = \tau(\mu)\tau(\lambda), \qquad\qquad \tau(\lambda)^{\{\alpha\}\{\alpha'\}}\tau(\mu)^{\{\alpha'\}\{\beta\}} = \tau(\mu)^{\{\alpha\}\{\alpha'\}}\tau(\lambda)^{\{\alpha'\}\{\beta\}} \tag{31.6}$$

(the sum over $\{\alpha'\}$ is implied), generating the conservation laws. It appears that in the fundamental model it is possible to construct a family of local conservation laws expanding the function $\ln\tau(\lambda)$ at the point $\lambda = \nu$ where the L-operator is equal to the permutation matrix. Let us demonstrate it.

Calculate first the value $\tau(\nu)$. Let us prescribe the index "0" to the matrix space, and the indices m, $(m = 1, \dots, M)$ to the local quantum space. Then one has due to (31.1)–(31.3)

$$T(\nu) = \overset{(0,M)}{\Pi} \overset{(0,M-1)}{\Pi} \dots \overset{(0,2)}{\Pi} \overset{(0,1)}{\Pi}. \tag{31.7}$$

The entries of the permutation matrix between the matrix space and the m-th quantum space are

$$\left(\overset{(0,m)}{\Pi}\right)^{\alpha_m\beta_m}_{i_1 i_2} = \delta_{i_1\beta_m}\delta_{\alpha_m i_2},$$

so that one has for the components of $T(\nu)$

$$\begin{aligned}
&(T(\nu))^{\{\alpha\}\{\beta\}}_{i_1 i_2} \\
&= \left(\delta_{i_1\beta_M}\delta_{\alpha_M i'_M}\right)\left(\delta_{i'_M\beta_{M-1}}\delta_{\alpha_{M-1}i'_{M-1}}\right)\dots\left(\delta_{i'_3\beta_2}\delta_{\alpha_2 i'_2}\right)\left(\delta_{i'_2\beta_1}\delta_{\alpha_1 i_2}\right) \qquad (31.7a)\\
&= \delta_{i_1\beta_M}\delta_{\alpha_M\beta_{M-1}}\delta_{\alpha_{M-1}\beta_{M-2}}\dots\delta_{\alpha_2\beta_1}\delta_{\alpha_1 i_2}.
\end{aligned}$$

To get the matrix elements of the transfer matrix in the quantum space, one should take the trace tr_0 in the matrix space, obtaining

$$\tau(\nu)^{\{\alpha\}\{\beta\}} = (\mathrm{tr}_0\, T(\nu))^{\{\alpha\}\{\beta\}} = \delta_{\alpha_M\beta_{M-1}}\delta_{\alpha_{M-1}\beta_{M-2}}\dots\delta_{\alpha_2\beta_1}\delta_{\alpha_1\beta_M}, \qquad (31.8)$$

i.e., the operator $\tau(\nu)$ is the cyclic shift operator along the chain. Obviously, the inverse operator $\tau^{-1}(\nu)$ gives the shift in the opposite direction,

$$\tau^{-1}(\nu) = \delta_{\alpha_M\beta_1}\delta_{\alpha_{M-1}\beta_M}\dots\delta_{\alpha_2\beta_3}\delta_{\alpha_1\beta_2}. \qquad (31.9)$$

Let us now compute the first derivative $\tau'(\lambda)|_{\lambda=\nu} \equiv \tau'(\nu)$:

$$\begin{aligned}
\tau'(\nu) &= \mathrm{tr}_0\left(\sum_{m=1}^{M} \overset{(0,M)}{\Pi}\dots\overset{(0,m+1)}{\Pi}\, L'(m|\nu)\, \overset{(0,m-1)}{\Pi}\dots\overset{(0,1)}{\Pi}\right) \qquad (31.10)\\
&= \sum_{m=1}^{M} \delta_{\alpha_M\beta_{M-1}}\dots\delta_{\alpha_{m+2}\beta_{m+1}}\,(L'(m|\nu))^{\alpha_m\beta_m}_{\alpha_{m+1}\beta_{m-1}}\,\delta_{\alpha_{m-1}\beta_{m-2}}\dots\delta_{\alpha_1\beta_M}
\end{aligned}$$

(the notation $L'(m|\nu) \equiv L'(m|\lambda)\,|_{\lambda=\nu}$ is used).

Consider the first logarithmic derivative,

$$\frac{d}{d\lambda}\ln\tau(\lambda)\Big|_{\lambda=\nu} \equiv \tau^{-1}(\nu)\tau'(\nu).$$

One has

$$\begin{aligned}
(\tau^{-1}(\nu)\tau'(\nu))^{\{\alpha\}\{\beta\}} &= \sum_{\{\gamma\}}\left(\tau^{-1}(\nu)\right)^{\{\alpha\}\{\gamma\}}(\tau'(\nu))^{\{\gamma\}\{\beta\}} \\
&= \sum_{m=1}^{M}\delta_{\alpha_M\beta_M}\dots\delta_{\alpha_{m+1}\beta_{m+1}}\,(L'(m|\nu))^{\alpha_{m-1}\beta_m}_{\alpha_m\beta_{m-1}}\,\delta_{\alpha_{m-2}\beta_{m-2}}\dots\delta_{\alpha_1\beta_1} \qquad (31.11)
\end{aligned}$$

It should me mentioned that, formally, the indices $\alpha_{M+1}, \alpha_0, \beta_0$ are present in the sums in (31.10), (31.11) (in the terms of the sums containing an L-operator at

the first or at the last place). It is clear from the calculation that one should put $\alpha_{M+1} \equiv \alpha_1$, and $\alpha_0 \equiv \alpha_M, \beta_0 \equiv \beta_M$.

The equation (31.11) can be written in the form

$$H \equiv \frac{d}{d\lambda} \ln \tau(\lambda) \mid_{\lambda=\nu} = \sum_{m=1}^{M} H_{m-1,m} \quad (H_{0,1} \equiv H_{M,1}) \tag{31.12}$$

where the operators $H_{m-1,m} : \mathcal{H} \longrightarrow \mathcal{H}$ are given as

$$\begin{aligned} (H_{m-1,m})^{\{\alpha\}\{\beta\}} &= \delta_{\alpha_1\beta_1} \dots \delta_{\alpha_{m-2}\beta_{m-2}} h_{\alpha_{m-1}\alpha_m|\beta_{m-1}\beta_m} \delta_{\alpha_{m+1}\beta_{m+1}} \dots \delta_{\alpha_M\beta_M}, \\ h_{\alpha_{m-1}\alpha_m|\beta_{m-1}\beta_m} &= (L'(\lambda) \mid_{\lambda=\nu})^{\alpha_{m-1}\beta_m}_{\alpha_m\beta_{m-1}}, \end{aligned} \tag{31.13}$$

and (see (31.2))

$$(L'(\lambda) \mid_{\lambda=\nu})^{\alpha_{m-1}\beta_m}_{\alpha_m\beta_{m-1}} = \frac{d}{d\lambda} (\varphi(\lambda,\nu) R(\lambda,\nu))^{\alpha_m\beta_{m-1}}_{\alpha_{m-1}\beta_m} . \tag{31.14}$$

Operator $H_{m-1,m}$ acts nontrivially only in the local spaces $\mathcal{H}_{m-1}$ and $\mathcal{H}_m$ of the global quantum space $\mathcal{H}$, i.e., it has the structure

$$H_{m-1,m} = \overset{(1)}{I} \otimes \dots \otimes \overset{(m-2)}{I} \otimes \overset{(m-1,m)}{h} \otimes \overset{(m+1)}{I} \otimes \dots \otimes \overset{(M)}{I}, \tag{31.15}$$

where $\overset{(k)}{I}$ denotes the unit matrix in the local space $\mathcal{H}_k$, and the operator $\overset{(m-1,m)}{h}$ is an $N^2 \times N^2$ dimensional matrix acting in the local spaces $\mathcal{H}_{m-1}$ and $\mathcal{H}_m$.

So operator H (31.12) can be considered as an $N^M \times N^M$ dimensional matrix Hamiltonian describing the interactions of two nearest neighbour "spins" on a one-dimensional lattice with M sites.

The Taylor series expansion of the logarithm of the monodromy matrix at point ν is of the form

$$\ln \tau(\lambda) = \sum_{n \geq 0} (\lambda - \nu)^n I_n \tag{31.16}$$

where the operators I_n,

$$I_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (\ln \tau(\lambda)) \Big|_{\lambda=\nu},$$

due to (31.6), are commuting quantum operators (integrals of motion),

$$[I_m, I_n] = 0. \tag{31.17}$$

It is not difficult to show that the operator I_n, $n \geq 1$ describes the interaction of $n+1$ nearest neighbours on a lattice. In principle, any of the operators I_n can be taken as a Hamiltonian of a quantum system. The usual choice is the Hamiltonian $I_1 \equiv H$ (31.12) describing the interaction of two nearest spins with the periodic boundary conditions (on a closed chain). It is often written by physicists just as

$$H = \sum_{m=1}^{M} h_{m-1,m} \qquad (h_{0,1} \equiv h_{M,1}), \tag{31.12a}$$

i.e., for brevity of notations, one writes explicitly only the "part" $h_{m-1,m}$ of the operator $H_{m-1,m}$. We use, as a rule, these notations in what follows.

32 The Heisenberg spin chains. The trace identities.

The Heisenberg XYZ spin chain is the model on the one-dimensional lattice defined by the Hamiltonian

$$H_{XYZ} = \sum_{m=1}^{M} \left(J_x \sigma_1^{(m-1)} \sigma_1^{(m)} + J_y \sigma_2^{(m-1)} \sigma_2^{(m)} + J_z \sigma_3^{(m-1)} \sigma_3^{(m)} \right), \tag{32.1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices, so that $\vec{s}^{(m)} = \frac{1}{2}\vec{\sigma}^{(m)}$ is the spin $\frac{1}{2}$ sitting at the m-th site of the lattice. The matrices

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

will be also often used. The commutation relations are

$$\left[\sigma_p^{(m)}, \sigma_q^{(n)} \right] = 2i \delta_{mn} \epsilon_{pqr} \sigma_r^{(m)},$$

and we assume the periodic boundary conditions, $\vec{\sigma}^{(0)} \equiv \vec{\sigma}^{(M)}$ (and $\vec{\sigma}^{(M+1)} \equiv \vec{\sigma}^{(1)}$). This model describes the interaction of the nearest neighbouring spins $\frac{1}{2}$ on the one-dimensional lattice. If $J_x = J_y \neq J_z$, the model is called the XXZ spin chain, and if $J_x = J_y = J_z$, it is called the XXX (isotropic) chain.

The XYZ chain, XXZ chain, and XXX chain are the fundamental spin models for the XYZ, XXZ, and XXX R-matrices, correspondingly, hence the names of the R-matrices. Below, the XXZ and XXX spin chains will be considered in detail.

One of the constants J, say J_x in the Hamiltonian can be taken equal to ± 1 without loss of generality. So, we write the Hamiltonian of the XXZ model as

$$H_{XXZ} = \pm \sum_{m=1}^{M} \left(\sigma_1^{(m-1)} \sigma_1^{(m)} + \sigma_2^{(m-1)} \sigma_2^{(m)}) + \Delta \sigma_3^{(m-1)} \sigma_3^{(m)} \right), \tag{32.2}$$

and the Hamiltonian of the XXX model ($\Delta = 1$) as

$$H_{XXX} = \pm \sum_{m=1}^{M} \vec{\sigma}^{(m-1)} \vec{\sigma}^{(m)}. \tag{32.3}$$

Both these Hamiltonians commute with the operator S_3 of the third component of the total spin,

$$S_3 = \frac{1}{2} \sum_{m=1}^{M} \sigma_3^{(m)}.$$

The Hamiltonian of the XXX model commutes also with the total spin operator $\vec{S}$.

Let us consider first the XXX chain. The Hamiltonian (32.3) with the minus sign possesses the ferromagnetic ground states (all spins up and all spins down),

$$|0\rangle \equiv \overset{M}{\underset{m=1}{\otimes}} |\uparrow\rangle_m, \qquad |0'\rangle \equiv \overset{M}{\underset{m=1}{\otimes}} |\downarrow\rangle_m, \tag{32.4}$$

$$\sigma_+^{(m)}|\uparrow\rangle_m = 0, \quad \sigma_3^{(m)}|\uparrow\rangle_m = |\uparrow\rangle_m, \quad \sigma_-^{(m)}|\downarrow\rangle_m = 0, \quad \sigma_3^{(m)}|\downarrow\rangle_m = -|\downarrow\rangle_m.$$

For the Hamiltonian (32.3) with the plus sign, the ferromagnetic states are not the ground states of the system.

As already mentioned, the XXX spin $\frac{1}{2}$ chain is the fundamental spin model for the XXX (rational) 4×4-dimensional R-matrix

$$R(\lambda, \mu) = E - \frac{ic}{\lambda - \mu}\Pi,$$

This R-matrix acts in the tensor product $V \otimes V$ of the two two-dimensional vector spaces V; E is the unit matrix in $V \otimes V$, and Π is the permutation matrix,

$$E = I \otimes I \qquad \text{or} \qquad \overset{(12)}{E} = \overset{(1)(2)}{I\ I},$$

$$\Pi = \frac{1}{2}\left(I \otimes I + \vec{\sigma} \otimes \vec{\sigma}\right) \qquad \text{or} \qquad \overset{(12)}{\Pi} = \frac{1}{2}\left(\overset{(1)(2)}{I\ I} + \overset{(1)(2)}{\vec{\sigma}\ \vec{\sigma}}\right).$$

Note that

$$\overset{(1)}{\vec{\sigma}}\ \overset{(2)}{\vec{\sigma}} = \overset{(1)}{\sigma_1}\overset{(2)}{\sigma_1} + \overset{(1)}{\sigma_2}\overset{(2)}{\sigma_2} + \overset{(1)}{\sigma_3}\overset{(2)}{\sigma_3} = \overset{(1)}{\sigma_3}\overset{(2)}{\sigma_3} + 2\left(\overset{(1)}{\sigma_+}\overset{(2)}{\sigma_-} + \overset{(1)}{\sigma_-}\overset{(2)}{\sigma_+}\right).$$

For the XXX chain, there is no need to keep an arbitrary parameter c. It can be seen that in all further calculations the rescaling of the spectral parameters according to the rule $\lambda \to c\lambda$ makes the dependence on c trivial. So in what follows we put $c = 1$ writing the R-matrix in the form

$$R(\lambda, \mu) = E - \frac{i}{\lambda - \mu}\Pi, \tag{32.5}$$

Define now following (30.3), (31.2) the fundamental L-operator for the XXX spin $\frac{1}{2}$ chain as

$$(L(\lambda))^{\alpha,\beta}_{i_1 i_2} \equiv \left((\lambda - \nu)E^{i_1 i_2}_{\alpha,\beta} - i\Pi^{i_1 i_2}_{\alpha,\beta}\right)\Big|_{\nu=-\frac{i}{2}}, \tag{32.6}$$

$$L(\lambda)\Big|_{\lambda=-\frac{i}{2}} = -i\Pi. \tag{32.6a}$$

Let us make some comments. We put the arbitrary parameter ν in (30.3), (31.2) equal to $-\frac{i}{2}$. For convenience, the normalization (31.2a) is changed for (32.6a), which changes further formulae in an obvious way. The indices $\alpha, \beta = 1, 2$ are

related to the local quantum space, the indices $i_1, i_2 = 1, 2$ being related to the matrix space. The L-operator can also be written as

$$L(\lambda) = \lambda\,(I \otimes I) - \frac{i}{2}\Big(\sigma_3 \otimes \sigma_3 + 2(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)\Big) \qquad (32.6b)$$

It is a 2×2 matrix whose matrix elements are also 2×2 matrices (acting in another quantum) space. More explicitly, one can write

$$L(\lambda) = \begin{pmatrix} \lambda - \frac{i}{2}\sigma_3 & -i\sigma_- \\ -i\sigma_+ & \lambda + \frac{i}{2}\sigma_3 \end{pmatrix}, \qquad L(\lambda)\Big|_{\lambda=-\frac{i}{2}} = -i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (32.6c)$$

Consider now one dimensional lattice. The total number of sites M is assumed to be even. Relate to each site m ($m = 1, 2, \ldots, M$) of the lattice the local quantum space $\mathcal{H}_m = \mathbf{C}^2$ where the spin operators $\vec{\sigma}^{(m)}$ act. Define the corresponding L-operators,

$$L(m|\lambda) = \begin{pmatrix} \lambda - \frac{i}{2}\sigma_3^{(m)} & -i\sigma_-^{(m)} \\ -i\sigma_+^{(m)} & \lambda + \frac{i}{2}\sigma_3^{(m)} \end{pmatrix}. \qquad (32.7)$$

The monodromy matrix $T(\lambda)$ is

$$T(\lambda) = \prod_{m=1}^{\overleftarrow{M}} L(m|\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \qquad (32.8)$$

Its matrix elements act in the "big" global quantum space $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m = \mathbf{C}^{2M}$ which is $2^M \times 2^M$ dimensional.

The transfer matrix $\tau(\lambda) = \operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda)$ is an operator in the big quantum space with the property $\tau(\lambda)\tau(\mu) = \tau(\mu)\tau(\lambda)$. From the general formula (31.12), one gets the following expression for the first logarithmic derivative (the extra factors i due to the normalization (32.6a) instead of (31.2a) should be taken into account):

$$\frac{d}{d\lambda} \ln \tau(\lambda)\Big|_{\lambda=-\frac{i}{2}} = i \sum_{m=1}^{M} h_{m-1,m} \qquad (32.9)$$

with

$$(h_{m-1,m})_{\alpha_{m-1}\alpha_m|\beta_{m-1}\beta_m} = \Big(L'(m|\lambda)^{\alpha_{m-1}\beta_m}_{\alpha_m\beta_{m-1}}\Big)\Big|_{\lambda=-\frac{i}{2}} = \delta_{\alpha_{m-1}\beta_m}\delta_{\alpha_m\beta_{m-1}} = (\Pi)^{\alpha_{m-1}\beta m-1}_{\alpha_m\beta_m}$$

so that

$$h_{m-1,m} = \frac{1}{2}\begin{pmatrix} {\scriptstyle (m-1)(m)} & {\scriptstyle (m-1)(m)} \\ I\quad I & + \quad \vec{\sigma}\quad \vec{\sigma} \end{pmatrix}. \qquad (32.10)$$

Hence, one gets the following trace identities for the Hamiltonian (32.3) of the XXX chain:

$$H_{XXX} = \pm \sum_{m=1}^{M} \vec{\sigma}^{(m-1)}\vec{\sigma}^{(m)} = \mp 2i \frac{d}{d\lambda} \ln \tau(\lambda)\Big|_{\lambda=-\frac{i}{2}} \mp M, \qquad (32.11)$$

where $\vec{\sigma}^{(0)} \equiv \vec{\sigma}^{(M)}$, i.e., the periodic boundary conditions are imposed.

Quite analogously, calculating the higher logarithmic derivatives in the transfer matrix, one produces the higher conservation laws which commute with the Hamiltonian and between themselves.

33 The algebraic Bethe Ansatz for the Heisenberg spin $\frac{1}{2}$ XXX chain.

Having obtained the trace identities, we can apply the algebraic Bethe Ansatz for solving the XXX model. First, one constructs the vacuum eigenstate $|0\rangle$ for the monodromy matrix (32.8),

$$C(\lambda)|0\rangle = 0, \qquad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \qquad D(\lambda)|0\rangle = d(\lambda)|0\rangle.$$

This is just the ferromagnetic state (32.4) (all spins up)

$$|0\rangle \equiv \bigotimes_{m=1}^{M} |0\rangle_m, \qquad |0\rangle_m \equiv |\uparrow\rangle_m, \qquad \sigma_+^{(m)}|0\rangle_m = 0. \tag{33.1}$$

Using the explicit form of the L-operator (32.6) one has

$$L(m|\lambda)|0\rangle_m = \begin{pmatrix} a_m(\lambda) & -i\sigma_-^{(m)} \\ 0 & d_m(\lambda) \end{pmatrix},$$

with $a_m(\lambda) = \left(\lambda - \frac{i}{2}\right)$, $d_m(\lambda) = \left(\lambda + \frac{i}{2}\right)$, so that

$$a(\lambda) = \left(\lambda - \frac{i}{2}\right)^M, \qquad d(\lambda) = \left(\lambda + \frac{i}{2}\right)^M. \tag{33.2}$$

It follows then from the general scheme of the algebraic Bethe Ansatz (see Section 24) that the states (24.4),

$$|\Psi_N(\{\lambda_j\})\rangle = \left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle, \qquad \lambda_j \neq \lambda_k \ (j \neq k),$$

are Bethe eigenstates, if the parameters λ_j satisfy the system of the Bethe equations (24.6),

$$\frac{a(\lambda_j)}{d(\lambda_j)} \prod_{\substack{n=1 \\ n\neq j}}^{N} \frac{f(\lambda_j, \lambda_n)}{f(\lambda_n, \lambda_j)} = 1, \qquad j = 1, 2, \ldots, N$$

or

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^M = \prod_{\substack{n=1 \\ n\neq j}}^{N} \frac{\lambda_j - \lambda_n + i}{\lambda_j - \lambda_n - i}, \qquad j = 1, 2, \ldots, N. \tag{33.3}$$

The eigenvalues Θ of the transfer matrix,

$$\tau(\lambda)|\Psi_N(\{\lambda_j\})\rangle = \Theta(\lambda,\{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle\,, \tag{33.4}$$

are given (see 24.7)) as

$$\Theta(\lambda,\{\lambda_j\}) = \left(\lambda-\frac{i}{2}\right)^M \prod_{j=1}^N \frac{\lambda-\lambda_j+i}{\lambda-\lambda_j} + \left(\lambda+\frac{i}{2}\right)^M \prod_{j=1}^N \frac{\lambda-\lambda_j-i}{\lambda-\lambda_j}. \tag{33.5}$$

From the trace identities (32.11), one gets the eigenvalues E_N of the Hamiltonian (32.3) on the Bethe eigenstates

$$\begin{aligned} H_{XXX}|\Psi_N(\{\lambda_j\})\rangle &= E_N(\{\lambda_j\})|\Psi_N(\{\lambda_j\})\rangle, \\ E_N(\{\lambda_j\}) &= \mp 2i\frac{d}{d\lambda}\ln\Theta(\lambda,\{\lambda_j\})\Big|_{\lambda=-\frac{i}{2}} \mp M. \end{aligned} \tag{33.6}$$

The explicit calculation gives for the eigenvalues

$$E_N(\{\lambda_j\}) = \mp\sum_{j=1}^N \frac{2}{\lambda_j^2+\frac{1}{4}} \pm M. \tag{33.7}$$

Let us note that for the vacuum state $|0\rangle$ one has

$$H_{XXX}|0\rangle = \pm M|0\rangle$$

The signs $+,-$ here are related to the antiferromagnetic case and to the ferromagnetic case, correspondingly. In the ferromagnetic case, the state $|0\rangle$ (all spins up) together with the state $|0'\rangle$ (all spins down) are the degenerate ground states of the magnet.

34 The Heisenberg spin $\frac{1}{2}$ XXZ chain.

The case of the XXZ spin chain can be considered quite similarly, the calculations being only a little bit more complicated. The XXZ spin $\frac{1}{2}$ chain is generated by the XXZ R-matrix (21.5). Let us put $\Delta = \cos\gamma$ in the Hamiltonian (32.2) of the XXZ chain, where γ is identified with the coupling constant entering the XXZ (trigonometric) R-matrix (21.3), (21.5). For the Hamiltonian to be a Hermitean operator, the constant γ should be real ($0 \le \gamma \le \pi$, $|\Delta| \le 1$), or pure imaginary ($\Delta \ge 1$), or $\mathrm{Re}\,\gamma = \pi$ ($\Delta \le -1$).

The fundamental L-operator can be written in the form

$$L(m|\lambda) = -i\begin{pmatrix} \cosh(\lambda-\frac{i\gamma}{2}\sigma_z^{(m)}) & \sigma_-^{(m)}\sin\gamma \\ \sigma_+^{(m)}\sin\gamma & \cosh(\lambda+\frac{i\gamma}{2}\sigma_z^{(m)}) \end{pmatrix}, \tag{34.1}$$

so that at the point $\lambda = \frac{i\pi}{2} - \frac{i\gamma}{2}$ it is proportional to the permutation matrix Π,

$$L\Big(m\Big|\lambda = \frac{i\pi}{2} - \frac{i\gamma}{2}\Big) = -i\sin\gamma\,\Pi. \tag{34.2}$$

The trace identities expressing the Hamiltonian in terms of the monodromy matrix are now given as

$$\begin{aligned} H_{XXZ} &= \sum_{m=1}^{M} \left(\sigma_1^{(m-1)}\sigma_1^{(m)} + \sigma_2^{(m-1)}\sigma_2^{(m)} + \Delta\sigma_3^{(m-1)}\sigma_3^{(m)}\right) \\ &= -2i\sin\gamma\frac{d}{d\lambda}\ln\tau(\lambda)\Big|_{\lambda=\frac{i\pi}{2}-\frac{i\gamma}{2}} - M\cos\gamma. \end{aligned} \tag{34.3}$$

The vacuum state $|0\rangle$ is given by the same formula (33.1) as in the XXX case,

$$|0\rangle \equiv \bigotimes_{m=1}^{M} |0\rangle_m, \qquad |0\rangle_m \equiv |\uparrow\rangle_m, \qquad \sigma_+^{(m)}|0\rangle_m = 0, \tag{34.4}$$

the vacuum eigenvalues being now

$$a(\lambda) = (-i)^M\left(\cosh\left(\lambda - \frac{i\gamma}{2}\right)\right)^M, \qquad d(\lambda) = (-i)^M\left(\cosh\left(\lambda + \frac{i\gamma}{2}\right)\right)^M. \tag{34.5}$$

The Bethe equations read

$$\left(\frac{\cosh\left(\lambda_j - \frac{i\gamma}{2}\right)}{\cosh\left(\lambda_j + \frac{i\gamma}{2}\right)}\right)^M = \prod_{\substack{k=1\\k\neq j}} \frac{\sinh\left(\lambda_j - \lambda_k - i\gamma\right)}{\sinh\left(\lambda_j - \lambda_k + i\gamma\right)}, \tag{34.6}$$

and the eigenvalues of the Hamiltonian obtained from the trace identities (34.3) are

$$E_N(\{\lambda_j\}) = \sum_{j=1}^{N}\varepsilon_0(\lambda_j) + M\cos\gamma, \qquad \varepsilon_0(\lambda) = \frac{2\sin^2\gamma}{\cosh\left(\lambda + \frac{i\gamma}{2}\right)\cosh\left(\lambda - \frac{i\gamma}{2}\right)}. \tag{34.7}$$

To conclude, let us also make some comments about the Heisenberg XYZ chain. It is a fundamental spin model generated by the R-matrix (21.6), (21.7). Due to the property (21.8), the standard scheme of Section 31 can be used to derive the local integrals of motion, the Hamiltonian (32.1) being expressed via the first logarithmic derivative of the transfer matrix. However, the fundamental L-operator of the XYZ chain generated by the R-matrix (21.6) has the same structure as the R-matrix,

$$L = \begin{pmatrix} \mathrm{a} & 0 & 0 & \mathrm{d} \\ 0 & \mathrm{c} & \mathrm{b} & 0 \\ 0 & \mathrm{b} & \mathrm{c} & 0 \\ \mathrm{d} & 0 & 0 & \mathrm{a} \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, and D are 2×2 matrices. It is easy to see that entry 21 of this L-operator has no vacuum state, i.e., the equation

$$Cx \equiv \begin{pmatrix} 0 & \mathrm{b} \\ \mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

has no solutions for $\mathrm{b} \neq 0$, $\mathrm{d} \neq 0$. It means that one cannot apply the standard scheme of the Bethe Ansatz which should be considerably modified (Faddeev, Takhtadjan, 1989), so that the solution of the XYZ chain is obtained in a considerably more complicated way.

35 Fundamental vertex models of classical statistical physics on a two-dimensional lattice.

Any R-matrix satisfying the Yang-Baxter equation generates also an exactly solvable model of classical statistical physics on a two-dimensional lattice. This model is closely related to the corresponding fundamental spin model, the partition function of the classical model being expressed in terms of the monodromy matrix of the spin model.

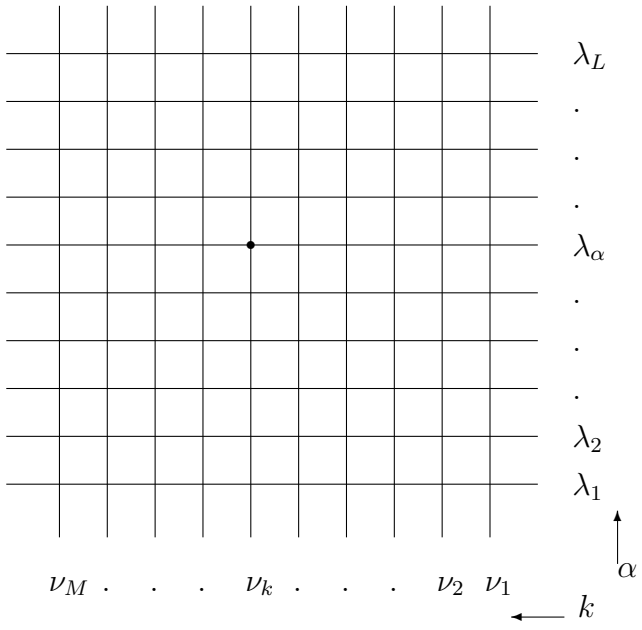

Fig. 35.1: The rectangular $M \times L$ lattice.

Consider a $M \times L$ rectangular lattice presented in Fig. 35.1. The vertical lines of the lattice are labeled by Latin indices k, $k = 1, 2, \ldots, M$, and the horizontal lines by Greek indices α, $\alpha = 1, 2, \ldots, L$. Associate spectral parameters λ_α to the horizontal lines, and spectral parameters ν_k to the vertical lines. Let us define "the

matrix of statistical weights" $L_{\alpha,k}$ at the vertex corresponding to the intersection of the α-th and k-th lines by means of the fundamental L-operator (see (31.2)),

$$(L_{\alpha,k}(\lambda_\alpha,\nu_k))^{b_k b'_k}_{a_\alpha a'_\alpha} \equiv \varphi(\lambda_\alpha,\nu_k)\,(R_{\alpha,k}(\lambda_\alpha,\nu_k))^{a_\alpha a'_\alpha}_{b_k b'_k}\,, \tag{35.1}$$

where φ is a scalar function defining the normalization of the L-operator.

Graphically, this L-operator can be represented as shown in Fig. 35.2. The variables a_α, a'_α, b_k, b'_k take values $1, 2, \dots N$, N being the dimension of the "basic" matrix space (R is an $N^2 \times N^2$ dimensional matrix). These variables can be regarded as "spins" sitting on the links of the lattice. So, the N-dimensional vector space V is related to each link (different spaces for different links).

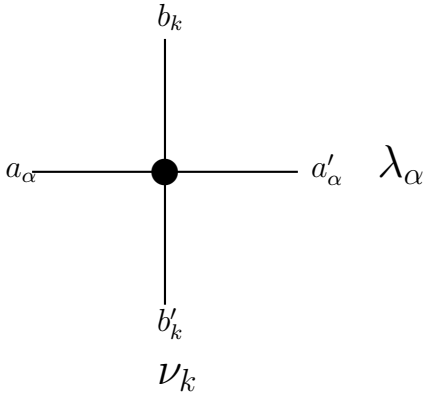

Fig. 35.2: The L-operator $L_{\alpha k}(\lambda_\alpha,\nu_k)$.

The partition function of the vertex model is defined as

$$Z = \sum_{\text{spins}} \prod_{\text{vertices}} (L_{\alpha,k}(\lambda_\alpha,\nu_k))^{b_k^{(\alpha+1)} b_k^{(\alpha)}}_{a_\alpha^{(k+1)} a_\alpha^{(k)}} \tag{35.2}$$

The product here is taken over all the vertices (α, k), and the summation is taken over all the possible values of the spins on the internal links, independently. The partition function thus defined depends on the boundary conditions, i.e., on the values of spins sitting on the external links (edges) of the lattice. These values can be prescribed arbitrarily. These are "right", "left", "down", and "up" spins which we denote $r_\alpha \equiv a_\alpha^{(1)}$, $l_\alpha \equiv a_\alpha^{(M+1)}$, $d_k \equiv b_k^{(1)}$, and $u_k \equiv b_k^{(L+1)}$, so that, in this case, the partition function can be considered as a matrix in the $N^M \times N^L$ dimensional space,

$$Z^{u,d}_{l,r} \equiv Z^{\{u\},\{d\}}_{\{l\},\{r\}}, \tag{35.3}$$

where multiindices are introduced,

$$r \equiv \{r_\alpha\} = \{r_L, r_{L-1}, \dots r_1\}, \qquad l \equiv \{l_\alpha\} = \{l_L, l_{L-1}, \dots l_1\},$$

$$d \equiv \{d_k\} = \{d_M, d_{M-1}, \dots d_1\}, \qquad u \equiv \{u_k\} = \{u_M, u_{M-1}, \dots u_1\}.$$

Usually, one imposes the periodic boundary conditions putting $l_\alpha = r_\alpha$ and $u_k = d_k$ ($\alpha = 1, \ldots, L$, $k = 1, \ldots, M$) and performing then the summation over all possible values of l_α and u_k:

$$Z_{\rm per} = \sum_{u,l} Z^{u,u}_{l,l}. \tag{35.4}$$

This is a partition function for a "closed" $M \times L$ dimensional lattice on a torus. But other types of boundary conditions are also of interest.

The usual definition of the partition function of a physical system is $Z = \sum \exp\{-E/T\}$ where the summation is taken over all possible configurations of the system, E is the energy of the configuration, and T is the temperature (the Boltzman constant $k_{_B}$ is set equal to 1). In our case, the configuration is defined by prescribing spin values to the links, both internal and external. If the statistical weights (i.e., the matrix elements of matrices $L_{\alpha k}$) are positive, the vertex model described above can be given a physical meaning as a model of classical statistical physics. The energy of a given configuration (divided by T) can be then identified with minus the logarithm of the product of statistical weights for the configuration. Usually, the R-matrix (in terms of which the statistical weights are constructed) possesses a free parameter (a "coupling constant"), which allows one also to introduce the temperature.

In our definition of the model, the partition function depends on the set of parameters λ_α, ν_k, the statistical weights prescribed to the vertices are different. This is the "inhomogeneous model" introduced by R.Baxter. Taking all λ's and all ν's equal, $\lambda_\alpha = \lambda$, $\nu_k = \nu$, one comes to the homogeneous model which is usually considered in applications.

Let us discuss now the relation of the fundamental vertex model to the corresponding quantum chain and to the quantum inverse scattering method. Let us consider indices a, a' of the fundamental L-operator (35.2) as "matrix" indices in a matrix space, and indices b, b' as "quantum" indices in a different, local (related to the link) quantum space. The Yang-Baxter equation for the R-matrix can be rewritten as the bilinear relation for the L-operators,

$$R_{\alpha\beta}(\lambda_\alpha, \lambda_\beta) L_{\alpha k}(\lambda_\alpha, \nu_k) L_{\beta k}(\lambda_\beta, \nu_k) = L_{\beta k}(\lambda_\beta, \nu_k) L_{\alpha k}(\lambda_\alpha, \nu_k) R_{\alpha\beta}(\lambda_\alpha, \lambda_\beta), \tag{35.5}$$

for any λ_α, λ_β and ν_k. The subscripts α, β label two different matrix spaces, and the subscript k is related to the third, quantum, space. Note that now the notation $R_{\alpha\beta}$ is used instead of $\overset{(\alpha\beta)}{R}$, i.e., $R_{\alpha\beta} \equiv \overset{(\alpha\beta)}{R}$.

Define the monodromy matrix T_α as the matrix product of the L-operators along the line α (the quantum indices of different local quantum spaces k remain free):

$$T_\alpha \equiv T_\alpha(\lambda_\alpha, \nu) = L_{\alpha M}(\lambda_\alpha, \nu_M) \ldots L_{\alpha 2}(\lambda_\alpha, \nu_2) L_{\alpha 1}(\lambda_\alpha, \nu_1), \tag{35.6}$$

or, in indices,

$$(T_\alpha)^{b^{(\alpha)}b'^{(\alpha)}}_{l_\alpha r_\alpha} = \sum_{a_\alpha} (L_{\alpha M})^{b^{(\alpha)}_M b'^{(\alpha)}_M}_{l_\alpha a^{(M)}_\alpha} \dots (L_{\alpha 2})^{b^{(\alpha)}_2 b'^{(\alpha)}_2}_{a^{(3)}_\alpha a^{(2)}_\alpha} (L_{\alpha 1})^{b^{(\alpha)}_1 b'^{(\alpha)}_1}_{a^{(2)}_\alpha r_\alpha} \tag{35.6a}$$

(α is fixed in both sides of this equation). The matrix element $(T_\alpha)_{l_\alpha, r_\alpha}$ of the $N \times N$ matrix T_α is a quantum operator in a "big" global quantum space, i.e., a matrix acting in the $N^M \times N^M$ dimensional space. The corresponding indices $b_\alpha \equiv b^{(\alpha)}_M, \dots, b^{(\alpha)}_1$ and $b_\alpha \equiv b^{(\alpha)}_M, \dots, b^{(\alpha)}_1$ are free in the both sides of equation (35.6a). Graphically, T_α can be represented as shown in Fig. 35.3.

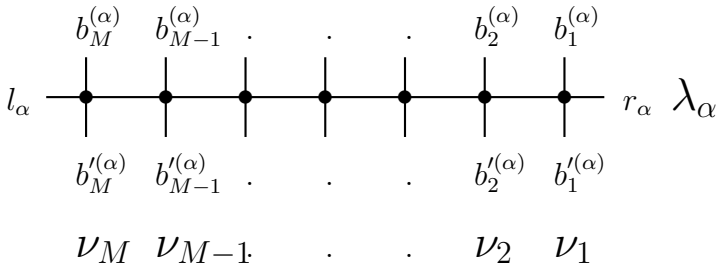

Fig. 35.3: The monodromy matrix T_α.

The R-matrix in equation (34.5) being independent of ν_k, the monodromy matrix satisfies the same relations as the L-operators,

$$R_{\alpha\beta}(\lambda_\alpha, \lambda_\beta) T_\alpha T_\beta = T_\beta T_\alpha R_{\alpha\beta}(\lambda_\alpha, \lambda_\beta). \tag{35.7}$$

Let us emphasize that the quantum space is the same for the monodromy matrices T_α and T_β, but the matrix spaces are different. In the previous equation we have a tensor product of these $N \times N$ matrices, but a the product in the global quantum space is present too. In indices one writes, e.g.,

$$(T_\alpha T_\beta)^{\substack{b^{(\alpha)}b^{(\beta)} \\ l_\alpha r_\alpha \\ l_\beta r_\beta}} = \sum_{b'} (T_\alpha)^{b^{(\alpha)}b'}_{l_\alpha r_\alpha} (T_\beta)^{b'b^{(\beta)}}_{l_\beta r_\beta} \tag{35.8}$$

where b is the multiindex, $b \equiv b_M, \dots, b_1$. The R-matrix $R_{\alpha\beta}$ acts only on the matrix indices l, r. Graphically equation (35.8) can be represented as in Fig. 35.4 (where summation over all internal lines is implied).

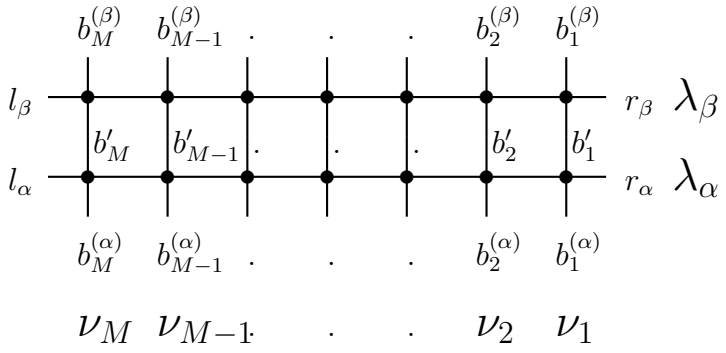

Fig. 35.4: The product $T_\alpha T_\beta$ of two monodromy matrices.

The partition function with fixed boundary conditions (i.e., with fixed r_α, l_α, d_k, u_k), after summing up over the spins on the internal horizontal links, can be represented as

$$Z_{lr}^{ud} = (T_L)_{l_L r_L}^{u b_L} \dots (T_2)_{l_2 r_2}^{b_3 b_2} (T_1)_{l_1 r_1}^{b_2 d} \tag{35.9}$$

where, as usual, the sum over repeated indices $b^{(\alpha)}$ is implied. Hence, the partition function is an "L-fold" tensor product of the monodromy matrices T_α in the matrix spaces and a product of them as quantum operators.

Turn now to the periodic boundary conditions. In this case, one puts $u = d$, $l = r$ and sums up:

$$Z_{\rm per} = {\rm Sp}\,(\tau_L \tau_{L-1} \dots \tau_2 \tau_1) \tag{35.10}$$

where τ_α is the corresponding transfer matrix, i.e., the trace in the matrix space of T_α,

$$\tau_\alpha = {\rm tr}\, T_\alpha(\lambda_\alpha, \nu), \qquad\qquad \nu \equiv \nu_M, \dots, \nu_1, \tag{35.11}$$

and Sp denotes the trace in the quantum space,

$${\rm Sp}\,(\tau_L \tau_{L-1} \dots \tau_2 \tau_1) = \sum_u (\tau_L \tau_{L-1} \dots \tau_2 \tau_1)^{uu}\,. \tag{35.12}$$

The transfer matrices $\tau_\alpha(\lambda_\alpha, \nu)$ are quantum operators ($N^M \times N^M$ dimensional matrices) with matrix elements $(\tau_\alpha)^{bb'}$. Due to equation (35.7), they are commuting at different values of spectral parameters,

$$\tau_\alpha \tau_\beta = \tau_\beta \tau_\alpha. \tag{35.13}$$

So one can diagonalize them simultaneously in the quantum space. Suppose that the corresponding eigenvalues are $\Theta_p^{(\alpha)} \equiv \Theta_p(\lambda_\alpha; \nu_1, \dots, \nu_M)$ (the subscript p enumerates eigenvalues, $p = 1, \dots, N^M$). Then

$$Z_{\rm per} = \sum_p \prod_\alpha \Theta_p^{(\alpha)}. \tag{35.14}$$

If the model is solvable by means of the Bethe Ansatz, then the eigenvalues can be found by means of it.

Especially elegant are the formulae in the homogeneous case ($\lambda_\alpha \equiv \lambda$, $\nu_k \equiv \nu$). Then all the monodromy matrices are the same, and one has

$$Z_{\text{per}} = \operatorname{Sp} \tau^L = \sum_p (\Theta_p)^L \,. \tag{35.15}$$

Of particular importance is the model in the thermodynamic limit ($M \to \infty$, $L \to \infty$). In this limit, the free energy for one site of the lattice

$$f = -T \lim \left(\frac{1}{ML} \ln Z \right) \tag{35.16}$$

remains finite. Only the maximum eigenvalue $\Theta_{\max}$ of $\tau(\lambda)$ contributes in the limit, so that

$$f = -T \lim \left(\frac{1}{M} \ln \Theta_{\max} \right). \tag{35.17}$$

One should justify, of course, these formulae for concrete models (the limit should exist, being independent of the order of taking the limits in L and M, etc.).

36 The six vertex model.

This is the vertex model generated by the XXZ R-matrix. It is, however, instructive to introduce it first without appealing to the R-matrix.

Consider a two-dimensional square $M \times M$ lattice. The partition function is defined as

$$Z = \sum_{\text{conf}} \exp \left\{ -\frac{E^{(c)}}{T} \right\} \tag{36.1}$$

where the sum is taken over all the possible configurations of the system, and $E^{(c)}$ is the energy of the configuration. The configuration is defined by putting arrows on the links of the lattice, see Fig. 36.1. To define the energy of the configuration, one prescribes energies $\varepsilon_j^{(\alpha k)}$ to the vertex on the intersection of the horizontal line α and of the vertical line k depending on the directions of the four arrows prescribed in the given configuration to the links adjacent to the vertex. If all possible configurations of the arrows are allowed, there can be 16 different weights at the vertex (enumerated, e.g., by the number $j = 1, 2, \ldots, 16$). This is the 16 vertex model (which is not integrable). If energies $\varepsilon_j^{(\alpha k)}$ are different (for fixed j) at different vertices, the model is "inhomogeneous". If, on the other hand, $\varepsilon_j^{(\alpha k)} = \varepsilon_j$, i.e., the vertex energies do not depend on the location of the vertex but only on the configuration j of the arrows on the adjacent links, the model is "homogeneous".

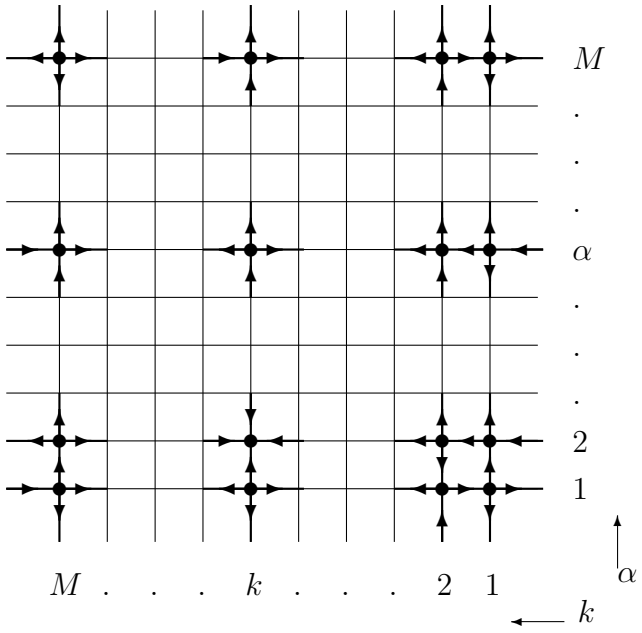

Fig. 36.1: The "sixteen vertex model" on the $M \times M$ lattice.

The energy of a configuration is defined as

$$E^{(c)} = \sum_{\alpha,k} \varepsilon^{(\alpha k)}_{j(\alpha,k)} \tag{36.2}$$

where $j(\alpha, k)$ is defined by the configuration of the arrows adjacent to the vertex (α, k). The partition function can be written as

$$Z = \sum_{\text{conf}} \prod_{\alpha,k} w^{(\alpha k)}_{j(\alpha,k)} \tag{36.3}$$

where quantities

$$w^{(\alpha k)}_j \equiv \exp\left\{-\frac{\varepsilon^{(\alpha k)}_j}{T}\right\}$$

are called statistical weights. For the homogeneous model, $w^{(\alpha k)}_j = w_j$ are the same for all vertices, and

$$Z_{\text{hom}} = \sum_{\text{conf}} \prod_{j=1}^{16} (w_j)^{N^c_j} \tag{36.4}$$

where N^c_j is the number of the vertices of type j at the configuration c.

Let us assume that only the configurations with an even numbers of arrows (i.e., with $0, 2$ or 4 arrows) entering each vertex are allowable (in other words, the energies of other configurations are equal to $+\infty$, or their weights equal 0). Then there are only eight possible weights at the vertex ($j = 1, 2, \ldots, 8$). The usual assumption is also that the energies of the vertices with the inversed directions of all arrows are

the same, so that there are, in fact, four different vertex energies. This is the famous eight-vertex model solved by R.Baxter.

We, however, consider below the six vertex model obtained assuming that the number of arrows going into each vertex should be equal to the number of the outgoing arrows. Thus one has 6 allowed configurations of the arrows at the vertex, shown in Fig. 36.2. Again, the usual assumption is that the vertex energies corresponding to the inversion of all the arrows are the same, so that three different weights can be assigned to each vertex, which we denote a, b and c (see Fig. 36.2),

$$a \equiv w_1 = w_2, \qquad b \equiv w_3 = w_4, \qquad c \equiv w_5 = w_6. \tag{36.5}$$

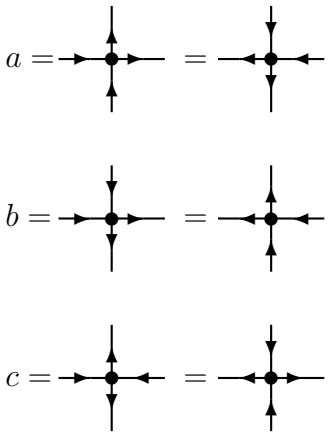

Fig. 36.2: The statistical weights of the six vertex model.

In the inhomogeneous model, the statistical weights are vertex dependent. Following R.Baxter, wee shall consider a special inhomogeneous model. Introduce again, as explained in the previous Section, spectral parameters λ_α associated with the horizontal lines and spectral parameteres ν_k related to the vertical lines. Now $\alpha = 1, \ldots, M$ and $k = 1, \ldots, M$ (we consider a square lattice). A convenient parametrization of the weights is

$$\begin{aligned} a^{(\alpha k)} &\equiv a(\lambda_\alpha, \nu_k) = \sinh(\lambda_\alpha - \nu_k + i\eta), \\ b^{(\alpha k)} &\equiv b(\lambda_\alpha, \nu_k) = \sinh(\lambda_\alpha - \nu_k - i\eta), \\ c^{(\alpha k)} &\equiv c(\lambda_\alpha, \nu_k) = -\sinh(2i\eta) \end{aligned} \tag{36.6}$$

(for the model to have a physical meaning, the parameters should be chosen so that the weights are nonnegative, but from a formal point of view in all the derivations below λ_α, ν_k and η are just arbitrary complex valued parameters).

To get the formulation of the model in terms of the quantum inverse scattering method it is convenient to use the "spin language" instead of the "arrow language" (this was done, in fact, in the previous Section). The spins (in our case, spin up $\uparrow \equiv 1$ and spin down $\downarrow \equiv 2$) are sitting on the links of the lattice. The correspondence with the arrow language is clear from Fig. 36.3 and Fig. 36.4.

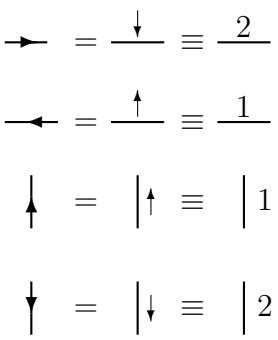

Fig. 36.3: Arrows and spins in the six vertex model.

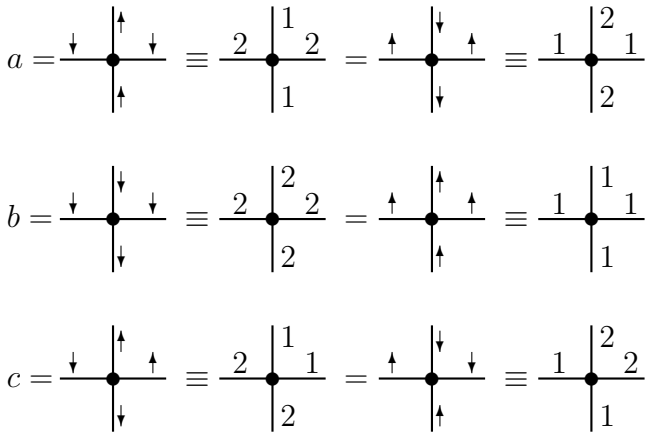

Fig. 36.4: The weights of the six vertex model (the "spin language").

The statistical weights at the vertex αk can be naturally written as a 4×4 dimensional matrix with 6 nonzero elements,

$$(L_{\alpha k}(\lambda_\alpha, \nu_k))^{b_r b'_k}_{a_\alpha a'_\alpha} = \begin{pmatrix} b(\lambda_\alpha, \nu_k) & 0 & 0 & 0 \\ 0 & a(\lambda_\alpha, \nu_k) & c(\lambda_\alpha, \nu_k) & 0 \\ 0 & c(\lambda_\alpha, \nu_k) & a(\lambda_\alpha, \nu_k) & 0 \\ 0 & 0 & 0 & b(\lambda_\alpha, \nu_k) \end{pmatrix} \tag{36.7}$$

(in the left-hand side, the letters a, b are used to label the matrix elements, while in the right-hand side they denote the statistical weights). The matrix elements are enumerated as in equations (13.12), (21.2).

It is to be noted that the matrix of weights in the eight vertex model has 8 nonzero elements possessing the structure

$$\begin{pmatrix} b & 0 & 0 & d \\ 0 & a & c & 0 \\ 0 & c & a & 0 \\ d & 0 & 0 & b \end{pmatrix},$$

where functions a, b, c, and d are naturally parametrized, following R.Baxter, in terms of the elliptic functions.

Return to the six vertex model. It is easy to see that $L_{\alpha k}$ is just the fundamental L-operator constructed from the R-matrix (21.3), (21.5) of the XXZ model with $\gamma = -2\eta$. Thus it satisfies the bilinear relation (35.5)

$$R_{\alpha\beta}(\lambda_\alpha, \lambda_\beta) L_{\alpha k}(\lambda_\alpha - \nu_k) L_{\beta k}(\lambda_\beta - \nu_k) = L_{\beta k}(\lambda_\beta - \nu_k) L_{\alpha k}(\lambda_\alpha - \nu_k) R_{\alpha\beta}(\lambda_\alpha, \lambda_\beta) \quad (36.8a)$$

(the subscripts α, β label two matrix spaces, while subscript k labels a "quantum" space; all the spaces are of dimension 2). The L-operators $L_{\alpha k}$ and $L_{\beta k}$ are situated on the same vertical line k. It is not difficult to verify that a similar relation is valid for the two L-operators situated on the same horizontal line α,

$$R_{kl}(\nu_l, \nu_k) L_{\alpha k}(\lambda_\alpha - \nu_k) L_{\alpha l}(\lambda_\alpha - \nu_l) = L_{\alpha l}(\lambda_\alpha - \nu_l) L_{\alpha k}(\lambda_\alpha - \nu_k) R_{kl}(\nu_l, \nu_k). \quad (36.8b)$$

Now one can apply the general scheme given in the previous Section. It is to emphasize that in the case of the six vertex model the algebraic Bethe Ansatz can be used to diagonalize the arising transfer matrices τ_α. The reason is that the "vacuum states" exist for the L-operators, and hence, for the monodromy matrices T_α constructed from them. This is evident since the matrix of the weights (36.7) is just the fundamental L-operator for the XXZ Heisenberg chain, which we know to possess the local vacuum state.

A particular case of the "trigonometric" six vertex model considered above is the "rational" six vertex model which is the fundamental vertex model generated by the rational XXX R-matrix. It can be obtained in the limit

$$2\eta = \epsilon c, \qquad \lambda_{XXZ} = \epsilon\lambda, \qquad \epsilon \to 0, \quad (36.9)$$

so that the weights become

$$\begin{aligned} a^{(\alpha k)} &\equiv a(\lambda_\alpha, \nu_k) = (\lambda_\alpha - \nu_k + ic/2)\,, \\ b^{(\alpha k)} &\equiv b(\lambda_\alpha, \nu_k) = (\lambda_\alpha - \nu_k - ic/2)\,, \\ c^{(\alpha k)} &\equiv c(\lambda_\alpha, \nu_k) = -ic. \end{aligned} \quad (36.10)$$

To conclude, let us discuss in more detail the parametrization of the weights in the six vertex model. Consider the trigonometric case (36.6) in the homogeneous model. One can always normalize one of the weights, say the weight c, to be equal to one (see (36.4)); it means that the corresponding vertex energy is set equal to zero. The partition function for the six vertex model on the $M \times M$ dimensional lattice is then given as

$$Z = c^{M^2} \sum_{\text{conf}} \tilde{a}^{N_a} \tilde{b}^{N_b} \quad (36.11)$$

where $\tilde{a} \equiv a/c$, $\tilde{b} \equiv b/c$, and N_a, N_b are the numbers of the corresponding vertices in a configuration. One can define the "physical" partition function demanding that

$\tilde{c} \equiv c/c = 1$, and the weights $\tilde{a}$, $\tilde{b}$ are nonnegative:

$$Z_{\rm phys} = \sum_{\rm conf} \tilde{a}^{N_a} \tilde{b}^{N_b}. \tag{36.12}$$

The region $\tilde{a} > 0, \tilde{b} > 0$ on the plane $(\tilde{a}, \tilde{b})$ corresponds to the physical partition function. It is not difficult to verify that the parameter $\Delta = \cos\gamma$ ($-\infty \le \Delta \le +\infty$, so that either $\mathrm{Im}\,\gamma = 0$, or $\mathrm{Re}\,\gamma = 0$) entering the Hamiltonian (32.2) of the XXZ chain is expressed as

$$\Delta = \cos 2\eta = \frac{a^2 + b^2 - c^2}{2ab}. \tag{36.11}$$

The regions of parameters are presented in Fig. 36.5. The lines $\Delta = \pm 1$ correspond to the rational XXX model. The line $\Delta = 0$ corresponds to the $XX0$ model (it is the free fermion point of the XXZ model).

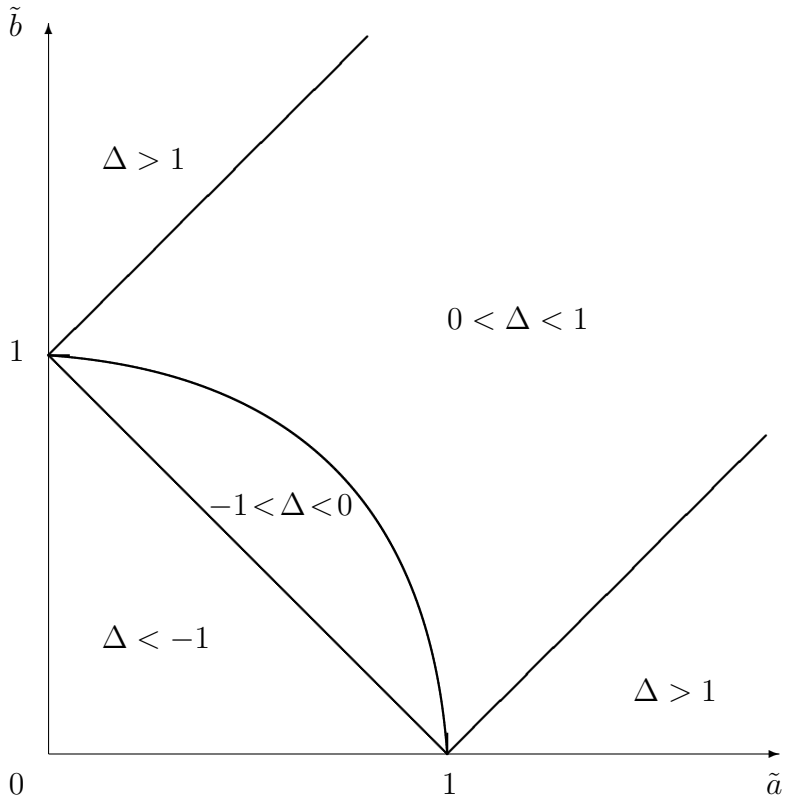

Fig. 36.5: The regions of parameters.

37 The partition function of the six vertex model with the domain wall boundary conditions.

An explicit expression for the partition function of the six vertex model with a special type of boundary conditions is derived in this Section. For simplicity, the rational (XXX) case is considered though an analogous expression for the partition function is also valid in the trigonometric (XXZ) case.

The fundamental L-operator (36.7) with the XXX weights (36.10) can be written as

$$L_{\alpha k} = (\lambda_\alpha - \nu_k) I - \frac{ic}{2}\sigma_3^{(\alpha)}\sigma_3^{(k)} - ic\left(\sigma_+^{(\alpha)}\sigma_-^{(k)} + \sigma_-^{(\alpha)}\sigma_+^{(k)}\right), \tag{37.1}$$

(indices α and k label the spaces). This L-operator is "intertwined" by the R-matrix of the XXX model,

$$R_{\alpha\beta}(\lambda_\alpha,\lambda_\beta)L_{\alpha k}(\lambda_\alpha-\nu_k)L_{\beta k}(\lambda_\beta-\nu_k) = L_{\beta k}(\lambda_\beta-\nu_k)L_{\alpha k}(\lambda_\alpha-\nu_k)R_{\alpha\beta}(\lambda_\alpha,\lambda_\beta), \tag{37.2}$$

$$R_{kl}(\nu_l,\nu_k)L_{\alpha k}(\lambda_\alpha-\nu_k)L_{\alpha l}(\lambda_\alpha-\nu_l) = L_{\alpha l}(\lambda_\alpha-\nu_l)L_{\alpha k}(\lambda_\alpha-\nu_k)R_{kl}(\nu_l,\nu_k), \tag{37.3}$$

where the R-matrix is given by equations (21.3), (21.4) as

$$R(\lambda,\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix},$$

$$f(\mu,\lambda) = 1 + \frac{ic}{\mu-\lambda}, \qquad g(\mu,\lambda) = \frac{ic}{\mu-\lambda},$$

i.e.,

$$R(\lambda,\mu) = E + \frac{ic}{\mu-\lambda}\Pi,$$

where E is the unit matrix, and Π is the permutation matrix.

The L-operator possesses two eigenvectors, $|\uparrow^h_\alpha\rangle\otimes|\uparrow^v_k\rangle$ and $|\downarrow^h_\alpha\rangle\otimes|\downarrow^v_k\rangle$ where $\uparrow$ and $\downarrow$ denote the up and down spins in the corresponding spaces,

$$|\uparrow^h_\alpha\rangle = \begin{pmatrix}1\\0\end{pmatrix}_\alpha, \qquad |\downarrow^h_\alpha\rangle = \begin{pmatrix}0\\1\end{pmatrix}_\alpha, \qquad |\uparrow^v_k\rangle = \begin{pmatrix}1\\0\end{pmatrix}_k, \qquad |\downarrow^v_k\rangle = \begin{pmatrix}0\\1\end{pmatrix}_k. \tag{37.4}$$

We use also the dual vectors,

$$\langle\uparrow^h_\alpha| = |\uparrow^h_\alpha\rangle^\dagger, \qquad \langle\uparrow^h_\alpha|\uparrow^h_\beta\rangle = \delta_{\alpha\beta}, \qquad \text{etc.} \tag{37.4a}$$

(the superscripts h, v are related to the "horizontal" and "vertical" spaces to distinguish, e.g., between the vectors $|\uparrow^h_1\rangle$ and $|\uparrow^v_1\rangle$). It is easily verified that

$$\begin{aligned} L_{\alpha k}(\lambda_\alpha-\nu_k)\,|\uparrow^h_\alpha\rangle\,|\uparrow^v_k\rangle &= \left(\lambda_\alpha-\nu_k-\frac{ic}{2}\right)|\uparrow^v_\alpha\rangle\,|\uparrow^v_k\rangle, \\ L_{\alpha k}(\lambda_\alpha-\nu_k)\,|\downarrow^h_\alpha\rangle\,|\downarrow^v_k\rangle &= \left(\lambda_\alpha-\nu_k-\frac{ic}{2}\right)|\downarrow^h_\alpha\rangle\,|\downarrow^v_k\rangle, \end{aligned} \tag{37.5}$$

the eigenvalues, $(\lambda_\alpha - \nu_k - ic/2)$ being the same.

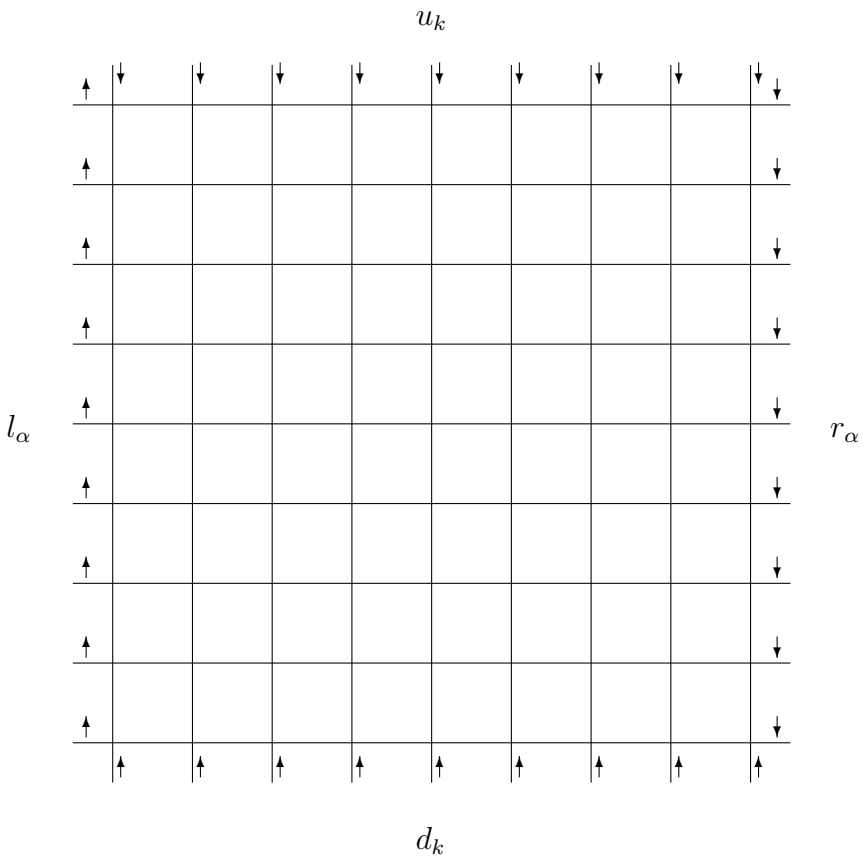

Fig. 37.1: The domain wall boundary conditions for the 9×9-dimensional lattice (the "spin language").

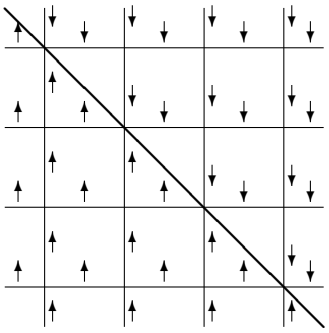

Fig. 37.2: The domain wall in the simplest configuration for the 4×4-dimensional lattice.

Another important property is that the L-operator $L_{\alpha k}$ at the point $\lambda_\alpha - \nu_k = -ic/2$ is proportional to the permutation matrix of spaces α and k,

$$L_{\alpha k}\left(\lambda_\alpha - \nu_k = \frac{ic}{2}\right) = -ic\Pi_{\alpha k}, \tag{37.6}$$

where

$$\Pi_{\alpha k} = \frac{1}{2}\left(I_\alpha \otimes I_k + \vec{\sigma}_\alpha \otimes \vec{\sigma}_k\right).$$

Turn now to the partition function. We consider the model on the $M \times M$ square lattice. Let us impose the "domain wall" boundary conditions putting

$$l_\alpha = \langle\uparrow_\alpha|, \qquad r_\alpha = |\downarrow_\alpha\rangle, \qquad d_k = |\uparrow_k\rangle, \qquad u_k = \langle\downarrow_k| \qquad (\alpha, k = 1, \dots, M), \tag{37.7}$$

see Fig. 37.1 (why the name 'domain wall boundary conditions' is used is clear from Fig. 37.2). The expression for the partition function Z^{ud}_{lr} (see (34.3)) with these boundary conditions reads

$$Z_M = \left(\prod_{\beta=1}^{M} \langle\uparrow^h_\beta| \prod_{l=1}^{M} \langle\downarrow^v_l| \right) \prod_{\alpha=1}^{M} \prod_{k=1}^{M} L_{\alpha k}(\lambda_\alpha - \nu_k) \left(\prod_{\beta=1}^{M} |\downarrow^h_\beta\rangle \prod_{l=1}^{M} |\uparrow^v_l\rangle \right). \tag{37.8}$$

The double product in the middle could be considered as a product of 2×2 weight matrices (37.1) (with noncommuting matrix elements) which should be taken "space" ordered, either first along the rows and then along the columns, or vice versa,

$$\begin{aligned} \prod_{\alpha=1}^{M} \prod_{k=1}^{M} L_{\alpha k}(\lambda_\alpha - \nu_k) &= (L_{MM} \dots L_{M2} L_{M1}) \dots (L_{1M} \dots L_{12} L_{11}) \\ &= (L_{MM} \dots L_{2M} L_{1M}) \dots (L_{M1} \dots L_{21} L_{11}) \end{aligned} \tag{37.9}$$

Let us now introduce the "horizontal" monodromy matrices

$$T_\alpha(\lambda_\alpha) \equiv T_\alpha(\lambda_\alpha, \nu) = \prod_{k=1}^{\overrightarrow{M}} L_{\alpha k}(\lambda_\alpha - \nu_k) \equiv \begin{pmatrix} A(\lambda_\alpha) & B(\lambda_\alpha) \\ C(\lambda_\alpha) & D(\lambda_\alpha) \end{pmatrix} \tag{37.10}$$

and the "vertical" monodromy matrices,

$$\tilde{T}_k(\nu_k) \equiv \tilde{T}_k(\lambda, \nu_k) = \prod_{\alpha=1}^{\overrightarrow{M}} L_{\alpha k}(\lambda_\alpha - \nu_k) \equiv \begin{pmatrix} \tilde{A}(\nu_k) & \tilde{B}(\nu_k) \\ \tilde{C}(\nu_k) & \tilde{D}(\nu_k) \end{pmatrix}. \tag{37.11}$$

Taking into account that

$$\langle\uparrow^h_\alpha|\, T_\alpha(\lambda_\alpha)\, |\downarrow^h_\alpha\rangle = B(\lambda_\alpha), \qquad \langle\downarrow^v_k|\, T_k(\nu_k)\, |\uparrow^v_k\rangle = \tilde{C}(\nu_k),$$

the partition function can be written as

$$\begin{aligned} Z_M(\{\lambda\}, \{\nu\}) &= \left(\prod_{j=1}^{M} \langle\downarrow^v_j| \right) \left(\prod_{\alpha=1}^{M} B(\lambda_\alpha) \right) \left(\prod_{j=1}^{M} |\uparrow^v_j\rangle \right) \\ &= \left(\prod_{\beta=1}^{M} \langle\uparrow^h_\beta| \right) \left(\prod_{j=1}^{M} \tilde{C}(\nu_j) \right) \left(\prod_{\beta=1}^{M} |\downarrow^h_\beta\rangle \right). \end{aligned} \tag{37.12}$$

Let us now establish the characteristic properties of the partition function.

(i) It is clear from the explicit form of the L-operator (37.1) that the matrix elements $A(\lambda_\alpha)$ and $D(\lambda_\alpha)$ of the monodromy matrix $T_\alpha(\lambda_\alpha)$ (37.10) are polynomials

of degree M, and the matrix elements $B(\lambda_\alpha)$ and $C(\lambda_\alpha)$ are polynomials of degree $M-1$ in λ_α. Analogously, $\tilde{A}(\nu_k)$ and $\tilde{D}(\nu_k)$ are polinomyals of degree M in (37.11), and $\tilde{B}(\nu_k)$ and $\tilde{C}(\nu_k)$ are polynomials of degree $M-1$ in ν_k. Taking into account that the dependence on λ_α in representation (37.12) of the partition function enters only $B(\lambda_\alpha)$, one concludes that the partition function is a polynomial of degree $M-1$ in each of λ_α. Also, it is a polynomial of degree $M-1$ in each of the ν_k.

(ii)The monodromy matrices are "intertwined" as the corresponding L-operators ($T_\alpha(\lambda_\alpha)$ as (37.2), and $\tilde{T}_k(\nu_k)$ as (37.3)). Hence, the operators B between themselves and $\tilde{C}$ between themselves commute,

$$B(\lambda_\alpha)B(\lambda_\beta) = B(\lambda_\beta)B(\lambda_\alpha), \qquad \tilde{C}(\nu_j)\tilde{C}(\nu_k) = \tilde{C}(\nu_k)\tilde{C}(\nu_j) \tag{37.13}$$

(equations (37.13) are present explicitly among equations (37.2), (37.3), see (23.4) and (23.5)). So, one concludes immediately that the partition function is symmetric under the permutations of variables λ_α and under the permutations of ν_k,

$$Z_M(\{P\lambda\},\{Q\nu\}) = Z_M(\{\lambda\},\{\nu\}) \tag{37.14}$$

where P and Q denote the permutations of M variables, e.g.,

$$\{P\lambda\} = \{\lambda_{P_1}, \lambda_{P_2}, \ldots, \lambda_{P_M}\}.$$

Let us emphasize that the property (37.14) is not at all obvious from the original definition of the partition function. It is the quantum inverse scattering method formulation that makes it quite evident.

(iii) The other important property of the partition function Z_M on a $M \times M$ dimensional lattice is the following recursion relation expressing the partition function Z_M with the special relations between the spectral parameters via the partition functions on a $(M-1)\times(M-1)$ dimensional lattice:

$$Z_M(\{\lambda_\alpha\},\{\nu_j\})\Big|_{\lambda_\beta-\nu_l-\frac{ic}{2}} =$$

$$-ic\left(\prod_{\substack{k=1\\k\neq l}}^{M}(\lambda_\beta-\nu_k-\tfrac{ic}{2})\ \prod_{\substack{\alpha=1\\\alpha\neq\beta}}^{M}(\lambda_\alpha-\nu_l-\tfrac{ic}{2})\right) Z_{M-1}(\{\lambda_\alpha,\,\alpha\neq\beta\},\{\nu_j,\,j\neq l\}). \tag{37.15}$$

Let us prove the recursion relation. Due to the symmetry property (37.14), it is sufficient to prove relation (37.15) at $\beta=1$, $k=1$, i.e, at $\lambda_1-\nu_1=-ic/2$. Turn to the explicit representation (37.8) of the partition function. The L-operator L_{11} acts nontrivially only onto the product of the two vectors, $|\downarrow_1^h\rangle\ |\uparrow_1^v\rangle$ in the right hand side of (37.8). At $\lambda_1-\nu_1=-ic/2$ the L-operator is proportional to the permutation matrix (see (37.6)), and the result of action is

$$L_{11}\left(\lambda_1-\nu_1=-\frac{ic}{2}\right)|\downarrow_1^h\rangle\ |\uparrow_1^v\rangle = -ic\,|\uparrow_1^h\rangle\ |\downarrow_1^v\rangle$$

(which gives just the factor $-ic$ in the right-hand side of the recursion relation). After this, one has the state vector

$$\left(\prod_{\beta=2}^{M} |\downarrow_\beta^h\rangle \prod_{l=2}^{M} |\uparrow_l^v\rangle\right) \otimes |\uparrow_1^h\rangle\, |\downarrow_1^v\rangle$$

in the right-hand side of (37.8), and using relations (37.5), one can act explicitly with the operators L_{1k} ($k \neq 1$) and $L_{\alpha 1}$ ($\alpha \neq 1$),

$$L_{\alpha 1}(\lambda_\alpha - \nu_1)\, |\downarrow_\alpha^h\rangle\, |\downarrow_1^v\rangle = \left(\lambda_\alpha - \nu_1 - \tfrac{ic}{2}\right) |\downarrow_\alpha^h\rangle\, |\downarrow_1^v\rangle,$$

$$L_{1k}(\lambda_1 - \nu_k)\, |\uparrow_1^h\rangle\, |\uparrow_k^v\rangle = \left(\lambda_1 - \nu_k - \tfrac{ic}{2}\right) |\uparrow_1^h\rangle\, |\uparrow_k^v\rangle.$$

All the other L-operators act in the first horizontal space and in the first vertical spaces trivially (i.e., they are proportional to the unit operators in these spaces). Using the normalization

$$\langle\uparrow_1^h|\uparrow_1^h\rangle = 1, \qquad\qquad \langle\downarrow_1^v|\downarrow_1^v\rangle = 1,$$

one comes just to the recursion relation (37.15) with $\beta = 1$, $l = 1$. The recursion relation is thus proved.

(iv) The partition function at $M = 1$ is

$$Z_1 = -ic \tag{37.16}$$

which is easily calculated from the definition (37.8).

It was already noted that the partition function Z_M is a polynomial of degree $M - 1$ in each individual λ_α, e.g., in λ_1. The recursion relation fixes the value of this polynomial at M points $\lambda_1 = \nu_l$, $l = 1, 2, \ldots, M$. Hence, the recursion relation, together with the "initial" value (37.16) defines the partition function uniquely. So, the function satisfying the properties (i)–(iv) is the partition function.

The remarkable thing is that the solution of the recursion relation for the partition function on the finite dimensional lattice can be found in the explicit form. The answer is the determinant representation

$$Z_M = (-ic)^M \frac{\displaystyle\prod_{j=1}^{M}\prod_{\alpha=1}^{M}\left(\lambda_\alpha - \nu_j + \frac{ic}{2}\right)\left(\lambda_\alpha - \nu_j - \frac{ic}{2}\right)}{\displaystyle\prod_{1\le\beta<\alpha\le M}(\lambda_\alpha - \lambda_\beta)\prod_{1\le k<l\le M}(\nu_k - \nu_l)}\, \det{}_M \mathcal{M} \tag{37.17}$$

where the determinant of the $M \times M$ dimensional matrix $\mathcal{M}$ is in the right-hand side, the matrix elements of the matrix being given as

$$(\mathcal{M})_{\alpha j} = \phi(\lambda_\alpha - \nu_j), \qquad\qquad \alpha, j = 1, 2, \ldots, M. \tag{37.18}$$

The function ϕ here is defined as

$$\phi(x) = \frac{1}{\left(x + \frac{ic}{2}\right)\left(x - \frac{ic}{2}\right)}. \tag{37.19}$$

To prove this representation, it is enough to check that the right hand side of (37.17):

(i) is a polynomial of degree $M-1$ in each individual λ_α (and in each ν_k);

(ii) is symmetric under permutations of the λ_α (and of the ν_k);

(iii) satisfies the recursion relation;

(iv) has the value $(-ic)$ at $M = 1$.

Let us prove it.

(i) The right-hand side of (37.17) is written as a rational function of variables λ_α and ν_k. It is easy to check, however, that the residues at all the poles are equal to zero. Indeed, the poles in the predeterminant factor cancel with the zeroes of the determinant (if $\lambda_\alpha = \lambda_\beta$, then two rows of the matrix $\mathcal{M}$ are equal, and if $\nu_k = \nu_l$, then two columns are equal, so that the determinant equals zero). On the other hand, the poles of the determinant at $\lambda_\alpha - \nu_k = \pm\frac{ic}{2}$ cancel with the corresponding zeroes of the predeterminant factor. So the right-hand side is a polynomial in each λ_α and ν_k. The degree is easily calculated to be $M-1$.

(ii) The change $\lambda_\alpha \leftrightarrow \lambda_\beta$ corresponds to the permutation of two rows of the matrix $\mathcal{M}$, so that it is antisymmetric under this change. Also, it is antisymmetric under the change $\nu_k \leftrightarrow \nu_l$ which permutes the columns. The numerator in the predeterminant factor is symmetric under these changes. The factors $\prod\limits_{1\le\beta<\alpha\le M}(\lambda_\alpha - \lambda_\beta)$ and $\prod\limits_{1\le k<l\le M}(\nu_k - \nu_l)$ in the determinant are easily seen to be antisymmetric under the permutations of λ_α and ν_k, correspondingly. So the right hand-side of (37.17) is symmetric under these permutations.

(iii) Due to the symmetry under the permutations, it is enough to check the recursion relation for the right-hand side of equation (37.17) at $\lambda_1 - \nu_1 = -ic/2$. It is easily done taking into account that

$$\det{}_M\mathcal{M} = -\frac{1}{ic\left(\lambda_1 - \nu_1 + \frac{ic}{2}\right)} \det{}_{M-1}\tilde{\mathcal{M}} + O(1)$$

where $\tilde{\mathcal{M}}$ is the minor of the element 11 of the matrix $\mathcal{M}$, i.e., it is a $(M-1)\times(M-1)$ matrix with entries

$$\left(\tilde{\mathcal{M}}\right)_{\alpha j} = \phi(\lambda_\alpha - \nu_j), \qquad \alpha, j = 2, 3, \ldots, M.$$

(iv) Finally, one easily calculates the value of the right-hand side at $M = 1$ which equals $-ic$ completing the proof of the determinant representation for the partition function.

In the homogeneous case

$$\lambda_\alpha \equiv \lambda, \qquad \nu_k \equiv \nu, \qquad \lambda - \nu \equiv x$$

the partition function can be obtained from the partition function (37.17) by repeating the application of the L'Hôpital rule. Giving the reader an opportunity to produce the corresponding calculation, let us give the result:

$$Z_M^{(\mathrm{hom})} = \frac{(-ic)^M}{\phi^{M^2}(x)\left(\prod_{k=1}^{M-1} k!\right)^2} \det{}_M \mathcal{H} \tag{37.20}$$

where the entries of the $M \times M$ matrix $\mathcal{H}$ are

$$(\mathcal{H})_{\alpha k} = \frac{\mathrm{d}^{\alpha+k-2}}{\mathrm{d}x^{\alpha+k-2}}\phi(x) \tag{37.21}$$

(the function $\phi(x)$ is defined by equation (37.19)). It is to be noted that the determinant $\det{}_M \mathcal{H}$ is, in fact, the Wronskian of the function $\phi(x)$ and its first $M-1$ derivatives.

The determinant representation of this kind can also be obtained for the partition function of the six vertex model with the weights (36.6), (36.7) constructed from the XXZ R-matrix. The result in the inhomogeneous case is

$$Z_M^{XXZ} = (-i\sin 2\eta)^M \frac{\prod_{j=1}^{M}\prod_{\alpha=1}^{M} \sinh(\lambda_\alpha - \nu_j + i\eta)\sinh(\lambda_\alpha - \nu_j - i\eta)}{\prod_{1\le\beta<\alpha\le M}\sinh(\lambda_\alpha - \lambda_\beta)\prod_{1\le k<l\le M}\sinh(\nu_k - \nu_l)} \det{}_M \mathcal{M}^{XXZ}, \tag{37.22}$$

the entries of the $M \times M$ matrix $\mathcal{M}^{XXZ}$ being given as

$$\left(\mathcal{M}^{XXZ}\right)_{\alpha j} = \phi_{XXZ}(\lambda_\alpha - \nu_j), \qquad \alpha, j = 1, 2, \ldots, M. \tag{37.23}$$

The function ϕ_{XXZ} is defined as

$$\phi_{XXZ}(x) = \frac{1}{\sinh(x+i\eta)\sinh(x-i\eta)}. \tag{37.24}$$

In the homogeneous case, the partition fumction is proportional to the Wronskian,

$$Z_M^{XXZ(\mathrm{hom})} = \frac{(-i\sin 2\eta)^M}{\left(\phi_{XXZ}(x)\right)^{M^2}\left(\prod_{k=1}^{M-1} k!\right)^2} \det{}_M \mathcal{H}^{XXZ} \tag{37.25}$$

where the entries of an $M \times M$ matrix $\mathcal{H}^{XXZ}$ are

$$\left(\mathcal{H}^{XXZ}\right)_{\alpha k} = \frac{\mathrm{d}^{\alpha+k-2}}{\mathrm{d}x^{\alpha+k-2}}\phi_{XXZ}(x). \tag{37.26}$$

An interesting and still unsolved problem is how to construct the asymptotical expansion of this partition function in the limit $M \to \infty$.

38 The ground state of the nonrelativistic Bose gas in the periodic box.

Let us return to the nonrelativistic Bose gas with the periodic boundary conditions considered in detail in Sections 2–8 by means of the coordinate Bethe Ansatz, and in Sections 27–29 within the frame of the quantum inverse scattering method. The Bethe eigenstates $\mid \Psi_N(\{\lambda_j\})\rangle$ of the quantum Hamiltonian H (1.2) which are also eigenstates of the momentum operator P and of the operator Q of the number of particles,

$$\begin{aligned} H \mid \Psi_N(\{\lambda_j\})\rangle &= E_N \mid \Psi_N(\{\lambda_j\})\rangle, \\ P \mid \Psi_N(\{\lambda_j\})\rangle &= P_N \mid \Psi_N(\{\lambda_j\})\rangle, \\ Q \mid \Psi_N(\{\lambda_j\})\rangle &= N \mid \Psi_N(\{\lambda_j\})\rangle, \end{aligned} \tag{38.1}$$

are parametrized by a set $\{\lambda_j,\ j = 1, 2, \ldots, N\}$ of different ($\lambda_j \neq \lambda_k,\ j \neq k$) quasimomenta of individual particles. The eigenvalues are given as

$$E_N \equiv E_N(\{\lambda_j\}) = \sum_{j=1}^{N} \epsilon_0(\lambda_j), \qquad \epsilon_0(\lambda) \equiv \lambda^2, \tag{38.2}$$

$$P_N \equiv P_N(\{\lambda_j\}) = \sum_{j=1}^{N} p_0(\lambda_j), \qquad p_0(\lambda) \equiv \lambda. \tag{38.3}$$

The Bethe equations (7.6) for the quasimomenta, after taking the logarithm of both sides, can be put into the form

$$Lp_0(\lambda_j) + \sum_{k=1}^{N} \Theta(\lambda_j - \lambda_k) = 2\pi n_j \qquad (j = 1, 2, \ldots, N). \tag{38.4}$$

The numbers n_j ($j = 1, 2, \ldots, N$) here are integers (half integers) if the number of particles N is odd (even). The "bare scattering phase" $\Theta(\lambda)$ is given in equation (4.12). Below, having in mind going to the thermodynamic limit, only the repulsive case (the coupling constant $c > 0$) will be considered. In this case, all the solutions λ_j of the Bethe equations are real and

$$\Theta(\lambda) = 2 \arctan\left(\frac{\lambda}{c}\right). \tag{38.5}$$

The solutions λ_j of the Bethe equations (38.4) can be parametrized uniquely by sets of numbers $\{n_j\}$ ($j = 1, 2, \ldots, N$). To prove this, let us introduce the following function (called Yang's action):

$$S(\{\lambda\}) = \frac{L}{2} \sum_{j=1}^{N} \lambda_j^2 - 2\pi \sum_{j=1}^{N} n_j \lambda_j + \frac{1}{2} \sum_{j,k=1}^{N} \Theta_1(\lambda_j - \lambda_k), \tag{38.6}$$

where function $\Theta_1(\lambda)$ is defined as the "antiderivative" of the bare scattering phase,

$$\Theta_1(\lambda) \equiv \int_0^\lambda \Theta(\mu) d\mu. \tag{38.7}$$

Bethe equations (38.4) are equivalent to the stationary point equations for Yang's action,

$$\frac{\partial S}{\partial \lambda_j} = 0 \qquad\qquad (j = 1, 2, \ldots, N). \tag{38.8}$$

The matrix of the second derivatives is easily calculated to be

$$\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_k} = \delta_{jk} \left[L + \sum_{m=1}^{N} K(\lambda_j - \lambda_m) \right] - K(\lambda_j - \lambda_k) \ ,$$

where the function $K(\lambda)$ is given as

$$K(\lambda) = \Theta'(\lambda) = \frac{2c}{\lambda^2 + c^2} > 0 \qquad\qquad (c > 0, \ \mathrm{Im}\, \lambda = 0). \tag{38.9}$$

The matrix of second derivatives is positively definite. This follows from the estimate of the corresponding quadratic form,

$$\sum_{j,k=1}^{N} \frac{\partial^2 S}{\partial \lambda_j \partial \lambda_k} u_j u_k = \sum_{j=1}^{N} L u_j^2 + \frac{1}{2} \sum_{j,l=1}^{N} K(\lambda_j - \lambda_l)(u_j - u_l)^2 > 0, \qquad \text{if } u_j \neq 0.$$

Hence, equation (38.8) defines a unique minimum, the set $\{\lambda_j\}$, for a given set of numbers $\{n_j\}$.

In the Bethe equations all the λ_j's should be taken different ("the Pauli principle" for one-dimensional interacting bosons). It can be shown that all different sets $\{n_j;\ n_j \neq n_k,\ j \neq k\}$, parametrize uniquely the solutions $\{\lambda_j;\ \lambda_j \neq \lambda_k,\ j \neq k\}$. Moreover, $n_j > n_k$ if $\lambda_j > \lambda_k$. So, integers (N odd) or half integers (N even) $\{n_j\}$ can be considered as "quantum numbers" parametrizing the solutions uniquely.

What is the ground state of the gas in the periodic box of length L? It is evident, of course, from the expression of the energy (38.2) that if one consider the states with any number of particles N then the ground state of the Hamiltonian H (1.2) is just the Fock vacuum $|0\rangle$, $H|0\rangle = 0$. Let us construct, however, the ground state in the sector with given N.

What is the ground state of the system of N one-dimensional bosons? It is evident from the "physical" point of view (and can be proved) that it is the state where the corresponding integers (or, half integers) $n_j^{(0)}$ fill symmetrically the interval around zero, without "free vacancies", with $n_{\max}^{(0)} = \frac{N-1}{2}$ and $n_{\min}^{(0)} = -\frac{N-1}{2}$, so that the ground state set is

$$\left\{ n_j^{(0)} : \qquad n_j^{(0)} = j - \frac{N+1}{2}, \qquad j = 1, 2, \ldots, N \right\}, \tag{38.10}$$

and the Bethe equations for the ground state read

$$Lp_0(\lambda_j) + \sum_{k=1}^{N} \Theta(\lambda_j - \lambda_k) = 2\pi n_j^{(0)} \qquad (j = 1, 2, \ldots, N). \tag{38.11}$$

The corresponding picture for $N = 6$ is shown in Fig. 38.1 (the crosses and "holes" denote the occupied and free vacancies, correspondingly).

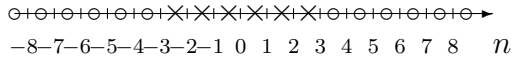

Fig. 38.1: The ground state in the case $N = 6$.

39 The ground state in the thermodynamic limit. The Lieb equation.

Our aim is now to consider the gas in the thermodynamic limit ($N \to \infty$, $L \to \infty$) fixing its density $D = N/L$. It was already discussed (see Section 8) that the gas of finite density can be obtained only in the repulsive case ($c > 0$).

In the thermodynamic limit the permitted values λ_j of momenta of the particles in the ground state are condensed, the difference of the neighbouring momenta being of the order of $1/L$,

$$\Delta\lambda_j \equiv \lambda_{j+1} - \lambda_j = O\left(\frac{1}{L}\right). \tag{39.1}$$

The particles fill the symmetric interval $[-q, q]$ where $q = \lim \lambda_N = -\lim \lambda_1$ (see equation (38.10)). In other words, the ground state particles fill the Fermi zone (or, the Dirac sea), q being the Fermi momentum.

Let us introduce now the density $\rho(\lambda)$ of the ground state ("vacuum") particle distribution in momentum space which is a smooth function of λ, and at points $\lambda = \lambda_j$ it is defined as

$$\rho(\lambda_j) = \lim \frac{1}{L\Delta\lambda_j} \tag{39.2}$$

(lim denotes the thermodynamic limit). The function $\rho(\lambda)$ remains finite in the limit. The quantity

$$L \int_a^b \rho(\lambda) d\lambda, \qquad -q \le a \le \lambda \le b \le q \tag{39.3}$$

gives the number of particles in the ground state possessing the momenta between a and b; this number becomes infinite as $L \to \infty$. The gas density D of the particles

in the coordinate space remains finite, being given as

$$D = \int_{-q}^{q} \rho(\lambda) d\lambda. \tag{39.4}$$

Let us derive the linear integral equation for the function $\rho(\lambda)$. Subtracting the Bethe equation (38.11) with index j from the equation with index $j+1$ and neglecting the terms of order $1/L$, one gets within the needed accuracy

$$L p_0'(\lambda_j)\Delta\lambda_j + \Delta\lambda_j \sum_{k=1}^{N} K(\lambda_j, \lambda_k) = 2\pi \tag{39.5}$$

where $\Delta\lambda_j$ is given by equation (39.1), and function $K(\lambda - \mu)$ (the derivative of the bare scattering phase, see equation (38.9)) is

$$K(\lambda, \mu) = K(\lambda - \mu) = \frac{2c}{(\lambda - \mu)^2 + c^2}. \tag{39.6}$$

In the thermodynamic limit, one can change the sum in equation (39.5) for an integral according to the rule (taking into account also the definition (39.2))

$$\sum_{k=1}^{N} f(\lambda_k) = \sum_{k=1}^{N} \frac{f(\lambda_k)}{\Delta\lambda_k}\Delta\lambda_k \approx L \sum_{k=1}^{N} f(\lambda_k)\rho(\lambda_k)\Delta\lambda_k \longrightarrow L \int_{-q}^{q} f(\mu)\rho(\mu)d\mu, \tag{39.7}$$

coming thus to the linear integral equation for the density of vacuum particles in momentum space

$$\rho(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda, \mu)\rho(\mu)d\mu = \frac{1}{2\pi} p_0'(\lambda) \tag{39.8}$$

(for the Bose gas, of course, $p_0'(\lambda) = \lambda$, see (38.3), but for other integrable models the equation has the form (39.8), with different functions $p_0'(\lambda)$). This equation is called the Lieb equation. The kernel $K(\lambda - \mu)$ is defined by the R-matrix. The momentum $p_0(\lambda)$ is different for different models with the same R-matrix. It should be mentioned that the Lieb equation for the Bose gas ($c > 0$) has a unique positive solution $\rho(\lambda) > 0$ defining this function on the whole λ-axis.

The energy E_0 of the ground state becomes infinitely large in the thermodynamic limit,

$$E_0 = L \int_{-q}^{q} \epsilon_0(\lambda)\rho(\lambda)d\lambda, \qquad \epsilon_0(\lambda) = \lambda^2, \tag{39.9}$$

being proportional to L. The density $\mathcal{E}$ of the energy in the ground state,

$$\mathcal{E}_0 = \frac{E_0}{L} = \int_{-q}^{q} \epsilon_0(\lambda)\rho(\lambda)d\lambda, \tag{39.10}$$

remains finite.

Let us introduce some notations which will be used further. It is convenient to write the Lieb equation (39.8) in the form

$$\left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)\rho = \frac{1}{2\pi}p_0', \qquad \text{or} \qquad \left[\left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)\rho\right](\lambda) = \frac{1}{2\pi}p_0'(\lambda) \quad (39.11)$$

where $\hat{I}$ is the unity operator, and $\hat{K}$ is the operator acting on functions $f(\lambda)$ on the interval $[-q, q]$ according to the rule

$$\left(\hat{K}f\right)(\lambda) \equiv \int_{-q}^{q} K(\lambda, \mu)f(\mu)d\mu \quad (39.12)$$

(the action can be continued on the whole λ-axis by means of this formula). Using the symmetry property of the kernel (39.6),

$$K(\lambda, \mu) = K(\mu, \lambda), \quad (39.13)$$

one can show that the density ρ satisfies also the conjugated (transposed) equation,

$$\left[\rho\left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)\right](\lambda) = \frac{1}{2\pi}p_0'(\lambda), \quad (39.14)$$

or, more explicitly,

$$\rho(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} \rho(\mu)K(\mu, \lambda)d\mu = \frac{1}{2\pi}p_0'(\lambda).$$

As usual, one can introduce the resolvent operator $\hat{W}$:

$$\begin{aligned} &\left(\hat{I} + \hat{W}\right)\left(\hat{I} - \frac{1}{2\pi}\hat{K}\right) = \left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)\left(\hat{I} + \hat{W}\right) = \hat{I}, \\ &\left(\hat{I} + \hat{W}\right)\hat{K} = 2\pi\hat{W}, \qquad \left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)\hat{W} = \frac{1}{2\pi}\hat{K}. \end{aligned} \quad (39.15)$$

The kernel $W(\lambda, \mu)$ of the resolvent operator is also symmetric,

$$W(\lambda, \mu) = W(\mu, \lambda). \quad (39.16)$$

In terms of the resolvent operator, equations (39.8), (39.14) can be rewritten as

$$\rho(\lambda) = \frac{1}{2\pi}\left[\left(\hat{I} + \hat{W}\right)p_0'\right](\lambda), \quad (39.17)$$

$$\rho(\lambda) = \frac{1}{2\pi}\left[p_0'\left(\hat{I} + \hat{W}\right)\right](\lambda), \quad (39.17a)$$

i.e., the solution of the Lieb equation can be represented in terms of the resolvent operator.

Let us sum up. The ground state $|\Omega_N\rangle$ of the gas in the sector with fixed number N of particles is given as the state over the "bare" vacuum $|0\rangle$,

$$|\Omega_N\rangle = \int_0^L d^N z\, \chi_N(\{\lambda\}|\{z\})\, \psi^+(z_1)\psi^+(z_2)\dots\psi^+(z_N)|0\rangle, \tag{39.18}$$

where the values $\lambda_1, \lambda_2, \dots, \lambda_N$ of the momenta of the ground state particles (defined by the set $n_1^{(0)}, n_2^{(0)}, \dots, n_N^{(0)}$ of the vacuum quantum numbers) should be defined from the system (38.10) of the Bethe equations. In the thermodynamic limit ($N \to \infty$, $L \to \infty$, $D = N/L$ fixed) the permitted values of the momenta are condensed (see eq. (39.1)), filling the Fermi zone (in the language of statistical physics), or the Dirac sea (in the language of quantum field theory). So, the ground state $|\Omega\rangle$ in the thermodynamic limit is formed by the vacuum particles with momenta distributed over the interval $-q \le \lambda \le q$ (q is the Fermi momentum) with density $\rho(\lambda)$ satisfying the Lieb equation (39.8). The density D of the gas remains finite and is given by equation (39.4). This equation gives the relation between the density and the Fermi momentum, $D = D(q)$. If the density D is fixed, then one should choose the Fermi momentum q entering the Lieb equation (39.8) in such a way that equation (39.4) is valid. In other words, for a given density D one should consider the Fermi momentum q in the Lieb equation as a parameter, $q = q(D)$, which should be defined so that equation (39.4) is satisfied.

40 Excitations at zero temperature.

Consider now the excitations assuming that the total number N of particles is kept fixed. In the ground state, the set of quantum numbers $n_j^{(0)}$ is given by equation (38.10),

$$\{n_j^{(0)} :\ n_j^{(0)} = j - \frac{N+1}{2},\ j = 1, 2, \dots, N\},$$

all the other states are parametrized by other sets

$$\{n_j;\ j = 1, 2, \dots, N;\ n_j \neq n_k,\ j \neq k\}.$$

These other sets can be obtained by removing some of the numbers $n_j^{(0)}$ from the ground state (making "holes" in the ground state) and adding an equal number (N is fixed) of numbers n_j ($|n_j| > \frac{N-1}{2}$) (adding "particles"). In the thermodynamic limit ($N \to \infty$, $L \to \infty$, $D = N/L$ fixed) these excited states are obtained by removing some particles with different momenta λ_h, $|\lambda_h| < q$, from the Fermi zone (holes), and by adding an equal number of particles with momenta λ_p, $|\lambda_p| > q$, outside the Fermi zone. The momenta λ_h and λ_p can be chosen otherwise arbitrary (they are not quantized in the thermodynamic limit).

Consider first the simplest excitation containing one hole and one particle. In the presence of the particle and the hole, the momenta of the ground state particles are changed, $\lambda_j \to \tilde{\lambda}_j$ (the effect of "vacuum polarization"). The energy of the excitation (the energy of the new state minus the ground state energy) is given as

$$\begin{aligned}\Delta E_{ph} = \Delta E(\lambda_p, \lambda_h) &= \epsilon_0(\lambda_p) - \epsilon_0(\lambda_h) + \sum_{j=1}^{N} \Big(\epsilon_0(\tilde{\lambda}_j) - \epsilon_0(\lambda_j)\Big) \\ &\approx \epsilon_0(\lambda_p) - \epsilon_0(\lambda_h) + \sum_{j=1}^{N} \epsilon_0'(\lambda_j) \left(\frac{\tilde{\lambda}_j - \lambda_j}{\Delta\lambda_j}\right) \Delta\lambda_j\end{aligned} \tag{40.1}$$

where $\Delta\lambda_j = \lambda_{j+1} - \lambda_j$, see (39.1). In the thermodynamic limit, $\Delta\lambda_j = O(1/L)$ and also $\tilde{\lambda}_j - \lambda_j = O(1/L)$. Let us introduce the polarization function $F(\lambda|\lambda_p, \lambda_h)$ as (compare with (39.2))

$$F(\lambda|\lambda_p, \lambda_h) \equiv F_{\rm ph}(\lambda) \lim \frac{\tilde{\lambda}_j - \lambda_j}{\Delta\lambda_j}. \tag{40.2}$$

This function remains finite in the thermodynamic limit. It depends on the variable λ, and on the parameters λ_p and λ_h parametrizing the excitation. In the thermodynamic limit, equation (40.1) turns into

$$\Delta E_{ph} = \Delta E(\lambda_p, \lambda_h) = \epsilon_0(\lambda_p) - \epsilon_0(\lambda_h) - \int_{-q}^{q} \epsilon_0'(\lambda) F(\lambda|\lambda_p, \lambda_h)\, d\lambda. \tag{40.3}$$

Let us now derive the equation for the polarization function. The Bethe equations for the polarized vacuum particles in the excited state are

$$Lp_0(\tilde{\lambda}_j) + \sum_{k=1}^{N} \Big(\Theta(\tilde{\lambda}_j - \tilde{\lambda}_k) + \Theta(\tilde{\lambda}_j - \lambda_p) - \Theta(\tilde{\lambda}_j - \lambda_h)\Big) = 2\pi \left(j - \frac{N+1}{2}\right) \qquad (j = 1, 2, \ldots, N,\ j \neq n_h). \tag{40.4}$$

Here, n_h is the number ($|n_h| < \frac{N-1}{2}$) corresponding to the hole. Subtracting equations (40.4) from the vacuum equations (38.11) with the same j and neglecting terms of order $1/L$, one has

$$\begin{aligned}&Lp_0'(\lambda_j)(\lambda_j - \tilde{\lambda}_j) - \Theta(\lambda_j - \lambda_p) + \Theta(\lambda_j - \lambda_h) + \\ &\qquad + (\lambda_j - \tilde{\lambda}_j) \sum_{k=1}^{N} K(\lambda_j - \lambda_k) - \sum_{k=1}^{N} K(\lambda_j - \lambda_k)(\lambda_k - \tilde{\lambda}_k) = 0.\end{aligned} \tag{40.5}$$

Taking into account that

$$Lp_0'(\lambda_j) + \sum_{k=1}^{N} K(\lambda_j - \lambda_k) \approx 2\pi L \rho(\lambda_j)$$

(see equations (39.5) and (39.2)), one rewrites equation (40.5) as

$$2\pi\frac{\lambda_j-\tilde{\lambda}_j}{\Delta\lambda_j}-\sum_{k=1}^{N}K(\lambda_j,\lambda_k)\frac{\lambda_k-\tilde{\lambda}_k}{\Delta\lambda_k}\Delta\lambda_k=\Theta(\lambda_j-\lambda_p)-\Theta(\lambda_j-\lambda_h)$$

obtaining in the thermodynamic limit the linear integral equation for the polarization function $F_{ph}(\lambda)$ (40.2),

$$\begin{aligned}\left[\left(\hat{I}-\frac{1}{2\pi}\hat{K}\right)F_{ph}\right](\lambda)&=\frac{1}{2\pi}\left(\Theta(\lambda-\lambda_p)-\Theta(\lambda-\lambda_h)\right)\\&=-\frac{1}{2\pi}\int_{\lambda_h}^{\lambda_p}K(\lambda-\mu)d\mu,\end{aligned}\tag{40.6}$$

where we used the brief notations introduced at the end of Section 39, and also relation (38.9). Equation (40.6) differs from the Lieb equation only by the right-hand side, the integral operator being the same.

By means of the resolvent operator $\hat{W}$ (39.15), acting with operator $\hat{I}+\hat{W}$ on both sides in equation (40.6) and taking into account the symmetry (39.16) of the resolvent kernel, one obtains the representation for the polarization function in the form

$$F_{ph}(\lambda)=-\int_{\lambda_h}^{\lambda_p}W(\lambda,\mu)d\mu\tag{40.7}$$

which allows to write excitation's energy (40.3) as

$$\Delta E_{ph}=\epsilon_0(\lambda_p)-\epsilon_0(\lambda_h)+\int_{\lambda_h}^{\lambda_p}\left[\hat{W}\epsilon_0'\right](\mu)d\mu\,.\tag{40.8}$$

Let us now prove the following important statement. The energy of the one-particle one-hole excitation can be represented in the form

$$\Delta E_{ph}\equiv\Delta E(\lambda_p,\lambda_h)=\tilde{\varepsilon}(\lambda_p)-\tilde{\varepsilon}(\lambda_h).\tag{40.9}$$

The function $\tilde{\varepsilon}(\lambda)$ here is the solution of the linear integral equation

$$\tilde{\varepsilon}(\lambda)-\frac{1}{2\pi}\int_{-q}^{q}K(\lambda,\mu)\tilde{\varepsilon}(\mu)d\mu=\tilde{\varepsilon}_0(\lambda),\tag{40.10}$$

where

$$\tilde{\varepsilon}_0(\lambda)=\epsilon_0(\lambda)-h=\lambda^2-h.$$

The constant $h=h(q)$ (independent of λ) is defined by the condition

$$\tilde{\varepsilon}(q)=\tilde{\varepsilon}(-q)=0\ ;\tag{40.11}$$

The fact that $\tilde{\varepsilon}(\lambda) = \tilde{\varepsilon}(-\lambda)$ is, of course, evident from equation (40.10). Equation (40.10) supplied by condition (40.11) is a particular case of a more general nonlinear integral equation, the so called Yang-Yang equation, presented in next Section.

Let us give the proof of our previous statement. Differentiating equation (40.10) with respect to λ and taking into account that due to condition (40.11) the boundary terms vanish when integrating by parts, one gets

$$\left[\left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)\tilde{\varepsilon}'\right](\lambda) = \tilde{\varepsilon}_0'(\lambda) = \epsilon_0'(\lambda) \qquad (\epsilon_0'(\lambda) = 2\lambda) \tag{40.12}$$

(since h does not depend on λ). Applying operator $(\hat{I} + \hat{W})$ to both sides of (40.12) one gets (see (39.15))

$$\left[\hat{W}\epsilon'\right](\lambda) = \tilde{\varepsilon}'(\lambda) - \epsilon_0'(\lambda). \tag{40.13}$$

Now it is sufficient to use equation (40.13) to calculate the integral in the right-hand side of equation (40.8), recovering the result of equation (40.11). The statement is thus proved.

Quite similarly, one gets for the momentum of the one-particle one-hole excitation

$$\begin{aligned}\Delta P_{ph} &= p_0(\lambda_p) - p_0(\lambda_h) - \int_{-q}^{q} \left[\Theta(\lambda - \lambda_p) - \Theta(\lambda - \lambda_h)\right] \rho(\lambda)\, d\lambda \\ &= 2\pi \int_{\lambda_h}^{\lambda_p} \rho(\lambda)\, d\lambda.\end{aligned} \tag{40.14}$$

Due to the linearity of the integral equations at zero temperature, all the other eigenstates of the Hamiltonian with fixed number of particles in the thermodynamic limit can be constructed as the states containing equal numbers of particles and holes, the energy and momentum of an excitation being equal to the sums of the energies and momenta of individual particles and holes.

What is the "physical meaning" of Yang-Yang equation? It is not difficult to prove that $\tilde{\epsilon}(\lambda) > 0$ if $|\lambda| > q$, and $\tilde{\epsilon}(\lambda) < 0$ if $|\lambda| < q$. The excitation we considered is a "two-particle" (one-particle one-hole) excitation conserving the total number of particles. Since $|\lambda_p| > q$ and $|\lambda_h| < q$ are otherwise free, and since the integral equation is linear, this excitation can be regarded as consisting of two one-particle excitations: a particle with the bare momentum λ_p and a hole with the bare momentum $-\lambda_h$. The energy of the excitation is positive, being equal to zero only if $|\lambda_p| = |\lambda_h| = q$. It should be so, of course, since the ground state is the state with the minimal energy (and there is no gap in the spectrum).

The constant h has the meaning of a chemical potential. This will become evident in the next Section, where the thermodynamics of the gas at finite temperature T is considered.

41 The thermodynamics of the Bose gas at finite temperature. The thermal equilibrium.

Our aim is to consider the thermodynamics of the gas. To do this, one puts the gas of N particles into a one-dimensional box of length L, going then to the thermodynamic limit,

$$N \to \infty, \qquad L \to \infty, \qquad D = N/L \quad \text{fixed.} \tag{41.1}$$

The "microscopic" description of the gas is done by means of the set of N "quantum numbers" $\{n_j; j = 1, 2, \ldots, N\}$, defining the momenta of particles via the equations (38.4). The state of thermal equilibrium of the gas is not a pure quantum mechanical state; it is formed by a "mixture" of such states. To go to the "macroscopic" description in the limit, let us introduce the densities in momentum space, $\rho_p(\lambda)$, $\rho_h(\lambda)$, and $\rho_t(\lambda) = \rho_p(\lambda) + \rho_h(\lambda)$. Quantities

$$L\rho_p(\lambda)d\lambda, \quad L\rho_h(\lambda)d\lambda, \quad \text{and} \quad L\rho_t(\lambda)d\lambda$$

give the number of particles (occupied vacancies), of holes (unoccupied vacancies), and of vacancies in the interval $d\lambda$, and per unit of length, respectively.

In the thermodynamic limit, Bethe equations (38.4) give the relation between the densities ρ_t and ρ_p

$$\rho_t(\lambda) = \frac{p_0'(\lambda)}{2\pi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\lambda, \mu)\rho_p(\mu)\, d\mu. \tag{41.2}$$

(Note, that for the ground state in the thermodynamic limit, $\rho_t(\lambda) = \rho_p(\lambda)$ ($|\lambda| < q$), and $\rho_t(\lambda) = \rho_h(\lambda)$ ($|\lambda| > q$), so that equation (41.2) in this case is reduced to the Lieb equation (39.8) for the vacuum particle density.) Many sets $\{n_j\}$, i.e., many different eigenstates of the Hamiltonian correspond to a given macroscopic situation described by some fixed function ρ_p, since there are many different possibilities of putting $L\rho_p(\lambda)d\lambda$ particles into $L\rho_t(\lambda)d\lambda$ vacancies.

In a generical situation, taking into account Fermi's exclusion principle, the number of these possibilities, is given as

$$e^{dS} = \frac{[L\rho_t(\lambda)d\lambda]!}{[L\rho_p(\lambda)d\lambda]!\,[L\rho_h(\lambda)d\lambda]!},$$

and using the Stirling formula, $\ln(n!) \cong n(\ln n - 1)$, one comes to the following expression for the entropy differential dS:

$$dS = Ld\lambda\,[\rho_t(\lambda)\ln\rho_t(\lambda) - \rho_p(\lambda)\ln\rho_p(\lambda) - \rho_h(\lambda)\ln\rho_h(\lambda)]\,,$$

so that the entropy of a macroscopic state described in terms of a given density $\rho_p(\lambda)$ is

$$S = L\int_{-\infty}^{\infty} d\lambda\ [\rho_t \ln\rho_t - \rho_p \ln\rho_p - \rho_h \ln\rho_h]\,. \tag{41.3}$$

This is Yang's formula for the entropy of the one-dimensional Bose gas. To a given macroscopic situation (fixed $\rho_p,\ \rho_h,\ \rho_t$) corresponds a number e^S of microscopic states.

We want now to evaluate the partition function of the model, given as

$$Z = \mathrm{tr}\left[e^{-H/T}\right] = e^{-F/T}\ .$$

For finite number of particles N we have:

$$\begin{aligned} Z_N &= \frac{1}{N!}\sum_{n_1,\ldots,n_N} \exp\left[-\frac{E_N}{T}\right] \\ &= \sum_{n_1<\ldots<n_N} \exp\left[-\frac{E_N}{T}\right]\ , \end{aligned} \tag{41.4}$$

where $E_N = \sum_{j=1}^N \lambda_j^2$. We shall consider the thermodynamics of the gas being at rest as a whole; hence the total momentum $P_N = \sum_{j=1}^N \lambda_j$ vanishes; this means that $\sum_{j=1}^N n_j = 0$. We can therefore introduce $N-1$ independent variable $m_j \equiv n_{j+1} - n_j$ and reexpress the partition function as

$$Z_N = \sum_{m_1=1}^{\infty} \ldots \sum_{m_{N-1}=1}^{\infty} \exp\left[-\frac{E_N}{T}\right]. \tag{41.5}$$

In the thermodynamic limit, the energy of these states is given as

$$E_N = L\int_{-\infty}^{\infty} d\lambda\, \rho_p(\lambda)\epsilon_0(\lambda), \tag{41.6}$$

so it depends only on the macroscopic variable $\rho_p(\lambda)$. Actually in the thermodynamic limit, the whole partition function, expressed in (41.5) as a sum over microscopic variables $\{n_j\}$, can be rewritten as a functional integral over the macroscopic densities. To do that, let us make the substitution

$$m_j \longrightarrow \frac{\rho_t(\lambda)}{\rho_p(\lambda)}$$

(which is quite natural, noting that the number of vacancies for the j-th particle, is just m_j; again we are just expressing the same physical quantity in terms of microscopic and macroscopic variables).

Taking also into account that many different sets of microscopic variables, enumerated in (41.3), correspond to given macroscopic variables, partition function (41.5) is rewritten as the following functional integral:

$$Z_N = \mathrm{const}\int \mathcal{D}\left[\frac{\rho_t(\lambda)}{\rho_p(\lambda)}\right]\delta\left(L\int \rho_p(\lambda)d\lambda - DL\right)\exp\left[S - \frac{E_N}{T}\right]\ . \tag{41.7}$$

The fixed number of particles N (we are working in the canonical ensemble) leads to the appearance of the δ-function in (41.7). It can be exponentiated and we finally obtain:

$$Z_N = \text{const} \int dh \int \mathcal{D}\left[\frac{\rho_t(\lambda)}{\rho_p(\lambda)}\right] \exp\left[S - \frac{E_N}{T} + \frac{L}{T}h\left(\int \rho_p(\lambda)d\lambda - D\right)\right] \ . \qquad (41.8)$$

We see that h is simply a Lagrange multiplier associated to the constraint imposed over the total number of particles (fixed to be $N = LD$), *i.e.* the chemical potential. As $L \to \infty$, the steepest descent method can be used to evaluate Z. The variation of the exponent in (41.8) is

$$\int d\lambda \left\{T\left(\delta\rho_t \ln \rho_t - \delta\rho_p \ln \rho_p - \delta\rho_h \ln \rho_h\right) - \lambda^2 \delta\rho_p + h\delta\rho_p\right\} + \delta h \left\{\int d\lambda \rho_p - D\right\} \ . \qquad (41.9)$$

Taking into account that $\rho_t(\lambda) = \rho_p(\lambda) + \rho_h(\lambda)$, remembering Bethe equations (41.2), and introducing the notation

$$\frac{\rho_h(\lambda)}{\rho_p(\lambda)} = \exp\left[\frac{\tilde{\varepsilon}(\lambda)}{T}\right] \ , \qquad (41.10)$$

saddle point equation (41.9) becomes

$$\int d\lambda \delta\rho_p \lambda) \left\{\tilde{\varepsilon}(\lambda) + \frac{T}{2\pi}\int d\mu K(\lambda,\mu) \ln\left[1 + e^{-\tilde{\varepsilon}(\lambda/T}\right] - \lambda^2 + h\right\} + \delta h \left\{\int d\lambda \rho_p - D\right\} \ , \qquad (41.11)$$

leading to the following conditions:

$$\tilde{\varepsilon}(\lambda) = \lambda^2 - h - \frac{T}{2\pi}\int_{-\infty}^{+\infty} d\mu K(\lambda,\mu) \ln\left[1 + e^{-\tilde{\varepsilon}(\lambda)/T}\right] \ , \qquad (41.12a)$$

$$D = \int_{-\infty}^{+\infty} d\lambda \rho_p(\lambda) \ . \qquad (41.12b)$$

It can be proved that the matrix of second derivatives of the exponent in (41.8) has the correct sign; moreover the value of h at the stationary point can be proved to be real; these statements justify the applicability of the steepest descent method.

We can thus calculate the partition function and free energy of the model:

$$Z = e^{-F/T} \ ; \qquad F = Nh - \frac{TL}{2\pi}\int_{-\infty}^{+\infty} d\lambda \ln\left(1 + e^{-\tilde{\varepsilon}(\lambda)/T}\right) \ . \qquad (41.13)$$

By definition, the pressure is the derivative of the free energy with respect to the volume, at fixed temperature:

$$\mathcal{P} = -\left(\frac{\partial F}{\partial L}\right) = \frac{T}{2\pi}\int_{-\infty}^{+\infty} d\lambda \ln\left(1 + e^{-\tilde{\varepsilon}(\lambda)/T}\right) \ . \qquad (41.14)$$

It is just a matter of calculation to verify that the basic thermodynamic identity

$$d\mathcal{P} = \frac{S}{T}dT + Ddh \tag{41.15}$$

is fulfilled, confirming that S as defined in (41.3) and the Lagrange multiplier h are indeed to be identified as the entropy and chemical potential, respectively.

Summarizing the results, we can say that the state of thermodynamic equilibrium is determined by the Bethe equations (41.2), together with the steepest descent equations (41.12). The density $D > 0$, the temperature $T > 0$, and the coupling constant $c > 0$ are free parameters here. The other variables appearing in (41.2) (41.12) are related by the identities

$$\frac{\rho_h(\lambda)}{\rho_p(\lambda)} = e^{\tilde{\varepsilon}(\lambda)/T} \ , \qquad \frac{\rho_p(\lambda)}{\rho_t(\lambda)} = \frac{1}{1+e^{\tilde{\varepsilon}(\lambda)/T}} \equiv \vartheta(\lambda) \ , \tag{41.16}$$

which are obvious consequences of definition (41.10). The equilibrium state is by no means a pure quantum mechanical state: it is not described by a single eigenfunction of the Hamiltonian (as for $T = 0$). It is a mixture of eigenstates, their number being given as $\exp S$. Equation (41.12a) for $\tilde{\varepsilon}(\lambda)$ is the Yang-Yang equation. It is possible to prove that Yang-Yang equation (41.12a) always admits a solution for $\tilde{\varepsilon}(\lambda)$, for physical values of the external parameters T, D and c. The function $\tilde{\varepsilon}(\lambda)$ has a clear physical interpretation as the energy of elementary excitations above the equilibrium state: indeed we recognize in function $\vartheta(\lambda)$ in (41.16) the well known Fermi distribution.

In the zero temperature limit, the logarithm in Yang-Yang equation (41.12a) tends to

$$-\frac{\tilde{\varepsilon}(\lambda)}{T}\ H(-\tilde{\varepsilon}(\lambda))$$

where $H(x)$ is the Heavyside step function, and equation (40.10) is recovered, together with condition (40.11). In a similar way, the thermodynamic limit of Bethe equations at zero temperature, the so called Lieb equation (39.8),can be recovered from the $T \to 0$ limit of the corresponding finite temperature Bethe equations (41.2).