Effective quantum gravity as a locally covariant QFT

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\textsuperscript{1}Based on the joint work with Klaus Fredenhagen and Romeo Brunetti
Outline of the talk

1. Introduction
   - Effective quantum gravity
   - Local covariance

2. Classical theory
   - Kinematical structure
   - Dynamics and symmetries
   - BV complex

3. Quantization
   - Deformation quantization
   - Quantum BV formalism
   - Background independence
Difficulties in quantum gravity

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- "Points" lose their meaning. The theory is invariant under diffeomorphism transformations.
- As a QFT, quantum gravity is power counting non-renormalizable.
Ways around some of the problems

Non-renormalizability: use Epstein-Glaser renormalization to obtain finite results for any fixed energy scale. Think of the theory as an effective theory.

Outlook: use the renormalization group flow equations to look for a UV fixed point (asymptotic safety program).

Dynamical nature of spacetime: make a tentative split of the metric into background and perturbation, quantize the perturbation as a quantum field on a curved background, show background independence at the end.

Diffeomorphism invariance: use the BV formalism to do the gauge fixing. Possible difficulties: base manifold is Lorentzian and non-compact, symmetry group is infinite dimensional, so is the configuration space.
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- To quantize, consider deformations of such structures.
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- We can think of the measured observable as a function of a perturbation of the fixed background metric: a tentative split into: $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$.
- Diffeomorphism transformation: move our experimental setup to a different region $\mathcal{O}'$. 
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The physical notion of subsystems is realized by the condition of **isotony**, i.e.: $\mathcal{O}_2 \supset \mathcal{O}_1 \Rightarrow \mathcal{A}(\mathcal{O}_2) \supset \mathcal{A}(\mathcal{O}_1)$.

We obtain a net of algebras.
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**Locally covariant quantum field theory (LCQFT)**
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A model in LCQFT is defined by giving a functor \( \mathcal{A} \) from the category of spacetimes to the category **Obs** of observables (for example involutive topological algebras).
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An important advantage of the deformation quantization is the fact that we work all the time on the same set of functionals, but we equip it with different algebraic structures (i.e. Poisson bracket, non-commutative $\star$ product).
Functional approach

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- The first step is to restrict oneself to functionals that are smooth. This requires some tools from calculus on infinite dimensional vector spaces.

- Among all the smooth functionals we can distinguish ones that are particularly relevant for physics. For example, we can consider local functionals, i.e. ones that can be written in the form: \( F(h) = \int_M f(j_x(h))(x) \), where \( h \) is a field configuration, \( f \) is a density-valued function on the jet bundle over \( M \) and \( j_x(h) \) is the jet of \( h \) at \( x \).
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Spacetime localization of a functional

- Another important property of a functional is its spacetime localization.
- For a point $x \in \mathcal{M}$ we want to know if our given functional $F$ is sensitive to fluctuations of field configurations at this point.

Mathematically, this can be expressed as:

$$\text{supp}\ F = \{ x \in \mathcal{M} | \forall \text{ neighbourhoods } U \text{ of } x \exists \text{ configurations } h_1, h_2 \text{ such that } F(h_1 + h_2) \neq F(h_1) \}.$$
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More precisely:
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\text{supp } F = \{ x \in M | \forall \text{ neighbourhoods } U \text{ of } x \exists h_1, h_2 \text{ configurations, supp } h_2 \subset U \text{ such that } F(h_1 + h_2) \neq F(h_1) \} .
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In the classical theory we will consider functionals that are compactly supported and multilocal (i.e. sums of finite products of local functionals).
For the effective theory of gravity the configuration space is \( \mathcal{E}(\mathcal{M}) = \Gamma((T^*M) \otimes 2) \). The space of compactly supported configurations is denoted by \( \mathcal{E}_c(\mathcal{M}) \). The assignment of both \( \mathcal{E}(\mathcal{M}) \) and \( \mathcal{E}_c(\mathcal{M}) \) is functorial.
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The space of multilical functionals will be denoted by \( \mathcal{F}(\mathcal{M}) \). \( \mathcal{F} \) is a covariant functor from \textbf{Loc} to \textbf{Vec} (the category of locally convex topological vector spaces).
Fields as natural transformations

- In the framework of locally covariant field theory [Brunetti-Fredenhagen-Verch 2003] fields are natural transformation between certain functors. For the sake of this talk let \( \Phi \in \mathcal{F} \doteq \text{Nat}(\mathcal{D}, \mathcal{F}) \), where \( \mathcal{D} \) is the functor of test function spaces \( \mathcal{D}(\mathcal{M}) = \mathcal{C}_c^\infty(\mathcal{M}) \) (one could substitute \( \mathcal{F} \) with a functor to the category of Poisson or \( C^* \) algebras).
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- In classical gravity we understand physical quantities not as pointwise objects but rather as something defined on all the spacetimes in a coherent way.
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The Euler-Lagrange derivative of $S$ is defined as

$$\langle S'_M(h_0), h \rangle = \langle L_M(f)^{(1)}(h_0), h \rangle,$$

where $f \equiv 1$ on $\text{supp} h$. The field equation is: $S'_M(h_0) = 0$. The space of solutions is denoted by $\mathcal{E}_S(M)$ and multilocal functionals on this space by $\mathcal{F}_S(M)$. 
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A symmetry of $S$ is a direction in $\mathcal{E}(M)$ in which the action is constant, i.e. it is a vector field $X \in \Gamma_c(T\mathcal{E}(M))$ such that $\forall h_0 \in \mathcal{E}(M): 0 = \langle S'_M(h_0), X(h_0) \rangle$.
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For each $M$ we can choose some diffeomorphism $\alpha_M$ and transform $\Phi$ to a new field by relabeling maps $\Phi_M$:

$$((\widetilde{\alpha} \Phi)(M,g))[\tilde{g}] \doteq \Phi(M, \alpha_M \ast g)[\tilde{g}],$$

where $\widetilde{\alpha}$ denotes the family $(\alpha_M)_{M \in \text{Obj}(\text{Loc})}$. 
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From the naturality condition follows that

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The diffeomorphism invariance is a condition:

\[(\bar{\alpha}\Phi)(M,g)(f)[\tilde{g}] = \Phi(M,g)(f)[\tilde{g}].\]
Let us now look at the infinitesimal version, i.e. consider
\[ \alpha_M = \exp(\xi_M), \xi_M \in \mathfrak{X}(M) \cong \Gamma(TM). \]
The family \( \xi \) of “gauge” parameters acts on a field \( \Phi \) by
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(\xi\Phi)_{(M,g)}(f)[\g] = \left\langle (\Phi_{(M,g)}(f))^{(1)}[\g], \mathcal{L}_{\xi_M}\g \right\rangle + \Phi_{(M,g)}(\mathcal{L}_{\xi_M}f)[\g]
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Example: \( \int R[\tilde{g}] f \; d \text{vol}_{(M,\tilde{g})} \) is diffeomorphism invariant, but

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is not.
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**Objective**: characterize $\mathcal{F}_S$, the space of gauge invariant fields on the space of solutions of EOM’s. More precisely, we consider elements of $\mathcal{F} := \text{Nat}(\mathcal{D}, \mathcal{F})$, which are invariant under diffeomorphisms ($\xi \Phi = 0$) and take quotient by the ideal consisting of fields satisfying $\Phi_M(f)(h) = 0$, for all $M$, $f \in \mathcal{D}(M)$, $h \in \mathcal{E}_S(M)$.
Idea: note that $\mathcal{E}_S(\mathcal{M})$ locally can be seen as the critical manifold of the Lagrangian $S_\mathcal{M}(f) : \mathcal{E}(\mathcal{M}) \to \mathbb{R}$ (zero locus of $S'_\mathcal{M}$).
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We can apply standard methods and characterize the space of on-shell fields by its Koszul-Tate resolution. However, one has to be a little bit careful about the topologies and completions, since we work with infinite dimensional spaces!
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Combining the Koszul-Tate and the Chevalley-Eilenberg complex we obtain the BV complex. Its 0th cohomology characterizes then the space of gauge invariant on-shell fields.
We will denote the underlying algebra of the BV complex by $\mathcal{BV}$. Recall that we are working with fields, so elements of $\mathcal{BV}$ are in particular functions from $\mathcal{D}(\mathcal{M})$ to a graded algebra $\mathcal{BV}(\mathcal{M})$, constructed in a “standard” way.
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\( BV(M) \) can be seen as the algebra of functions on \( T^*\overline{E}(M) \), where \( \overline{E}(M) \) is a certain graded manifold (in the simplest case: \( \overline{E}(M) = E(M) \oplus X(M)[1] \)).
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The classical BV differential can be written as

$$s\Phi_M(f) = \{\Phi_M(f), \tilde{S}_M\} + \Phi_M(\mathcal{L}Cf),$$

where $\tilde{S} \in \mathcal{BV}$ is the so called extended action and $C$ is a ghost.
Equations of motion and Poisson bracket

As an output of the classical theory we have the extended configuration space $\overline{E}$ and the extended action $\tilde{S}$. Now we apply to this data the deformation quantization.
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Using this input, we define the free Poisson bracket on $\mathcal{BV}(\mathcal{M})$:

$$\{F, G\}_0^g = \left\langle F^{(1)}, \Delta_g G^{(1)} \right\rangle, \quad \Delta_g = \Delta^R_A g - \Delta^A_R g,$$
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- This Poisson structure can be naturally extended to a Poisson bracket $\{., .\}_0$ on $\mathcal{BV}$. 

Katharina Rejzner
QG from LCQFT
We start with the deformation quantization of $\mathcal{BV}, \{.,.\}_0$, which is done in the standard way and provides a $\star$-product with the following properties:

$$F \star G \xrightarrow{\hbar \to 0} F \cdot G,$$

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To introduce the interaction one has to define the so called time ordered products. Formally, they are the coefficients in expansion of the S-matrix in powers of the interaction term $V_g$, i.e.:

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Because of the singularity structure of the Feynman propagator, time ordered products of local non-linear functionals are well defined only for arguments with pairwise disjoint supports. In particular the above formula would not make sense for local $V_g$. 
Since most of interaction terms relevant in physics are local, we need to extend maps $\mathcal{T}_n$ to local arguments with arbitrary supports. To this end we use the so called Epstein-Glaser renormalization. Mathematically it reduces to extension of certain distributions.

$$\mathcal{R}_{\mathcal{V}}(\Phi)(M,g)(f) = (e^{\mathcal{V}g\mathcal{T}}) \ast -1 \ast (e^{\mathcal{V}g\mathcal{T}} \mathcal{T}_\Phi(M,g)(f)).$$
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As a result, we obtain a family of maps $\mathcal{T}_n$ from $\mathcal{BV}_{loc}^\otimes n$ to a certain completion of $\mathcal{BV}$. We have shown that these maps can be seen as arising from a binary product $\cdot_{\mathcal{T}}$ defined on a certain domain containing $\mathcal{F}_{loc}$ and $S(V_g) = e^{Vg}_{\mathcal{T}}$ is a time-ordered exponential with respect to this product.
Interaction

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- As a result, we obtain a family of maps $\mathcal{T}_n$ from $\mathcal{BV}_\text{loc}^\otimes n$ to a certain completion of $\mathcal{BV}$. We have shown that these maps can be seen as arising from a binary product $\cdot_{\mathcal{T}}$ defined on a certain domain containing $\mathcal{F}_\text{loc}$ and $S(V_g) = e^{V_g}_{\mathcal{T}}$ is a time-ordered exponential with respect to this product.
- This allows to define the interacting fields by means of the Bogoliubov formula:

\[
(R_V(\Phi))_{(M,g)}(f) \doteq (e^{V_g}_{\mathcal{T}})^{-1} \ast (e^{V_g}_{\mathcal{T}} \cdot_{\mathcal{T}} \Phi_{(M,g)}(f)).
\]
In the framework of [K. Fredenhagen, K.R., CMP 2013], the gauge invariance of the $S$-matrix is guaranteed by the so called quantum master equation (QME):

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where $\{.,.\}$ is the Schouten bracket.

With the use of Master Ward Identity [F.Brennecke, M.Duetsch, RMP 2008], this condition can be rewritten as

$$\frac{1}{2}\{S_{0g} + V_g, S_{0g} + V_g\} = i\hbar \triangle V_g,$$

where $\triangle V_g$ is a certain local linear operator, which we identify with the renormalized BV Laplacian.
If the QME holds, then gauge invariant quantum observables are recovered as the 0th cohomology of the quantum BV operator $\hat{s}$, which acts on quantum fields by

$$(\hat{s}\Phi)_M(f) = e^{-V_g} \cdot \{ e^{V_g} \cdot \Phi_M(f), S_0g \} + \Phi_M(\mathcal{L}c f),$$

where $C$ is the ghost field.
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Again, using the MWI, this can be rewritten as

$$\hat{s}\Phi_M(f) = \{ \Phi_M(f), S_{0g} + V_g \} + \Phi_M(\mathcal{L} cf) - i\hbar \Delta V_g (\Phi_M(f)).$$
Relative Cauchy evolution

Let \( \mathcal{N}_+ \) and \( \mathcal{N}_- \) be two spacetimes that embed into two other spacetimes \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) around Cauchy surfaces, via causal embeddings given by \( \chi_{k,\pm}, k = 1, 2 \).
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Denote $\alpha_{\chi_{i,\pm}} = A \chi_{k,\pm}, k = 1, 2$. 

\[ \beta = \alpha_{\chi_1,\pm} + \alpha_{-1,\chi_2,\pm} + \alpha_{\chi_2,\pm} - \alpha_{-1,\chi_1,\pm} \] is an automorphism of $A(M_1)$. It depends only on the spacetime between the two Cauchy surfaces $M_1 M_2$. 

\[ N_+ \]
\[ \chi_{1,+} \]
\[ M_1 \]
\[ \chi_{1,-} \]
\[ \chi_{2,+} \]
\[ M_2 \]
\[ \chi_{2,-} \]
\[ N_- \]
Relative Cauchy evolution

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- Denote $\alpha_{\chi_{i,\pm}} = \mathcal{A}\chi_{k,\pm}, k = 1, 2$.
- From the time-slice axiom follows that $\beta = \alpha_{\chi_{1,\pm}}^{-1} \alpha_{\chi_{2,\pm}} \alpha_{\chi_{2,-}}^{-1} \alpha_{\chi_{1,-}}$ is an automorphism of $\mathcal{A}(\mathcal{M}_1)$. 

\[ \begin{array}{c}
\begin{array}{c}
\mathcal{N}_+ \\
\chi_{1+} \\
\mathcal{M}_1 \\
\chi_{1-} \\
\mathcal{N}_- \\
\chi_{2+} \\
\mathcal{M}_2 \\
\chi_{2-}
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Background independence

Let $\mathcal{M}_1 = (M, g_1)$ and $\mathcal{M}_2 = (M, g_2)$, where $(g_1)_{\mu\nu}$ and $(g_2)_{\mu\nu}$ differ by a (compactly supported) symmetric tensor $h_{\mu\nu}$ with $\text{supp}(h) \cap J^+(\mathcal{N}_+) \cap J^-(\mathcal{N}_-) = \emptyset$,
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$$\Theta_{\mu\nu}(x) \doteq \frac{\delta \beta_h}{\delta h_{\mu\nu}(x)} \bigg|_{h=0}$$

is a derivation valued distribution which is covariantly conserved.
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The infinitesimal version of the background independence is a condition: $\Theta_{\mu\nu} = 0$. 
Theorem [Brunetti, Fredenhagen, K.R. 2013]

The functional derivative $\Theta_{\mu\nu}$ of the relative Cauchy evolution can be expressed as

$$\Theta_{\mu\nu}(\Phi_M(f)) \overset{o.s.}{=} [ RV_1(\Phi_M(f)), RV_1(T_{\mu\nu}) ]_*, $$

where $T_{\mu\nu}$ is the stress-energy tensor of the extended action and one can define the time-ordered products in such a way that $T_{\mu\nu} = 0$ holds, so the interacting theory is background independent.
Conclusions

- We have constructed a consistent model of perturbative quantum gravity within the framework of locally covariant quantum fields theory.
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- To quantize the theory, we make a tentative split into a free and interacting theory. We quantize the free theory first and then use the Epstein-Glaser renormalization to introduce the interaction.

- We have shown, using the relative Cauchy evolution, that our theory is background independent, i.e. independent of the split into free and interacting part.
Thank you for your attention!