Squeezing hadronic matter

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Barcellona, September-October 2003

Summary

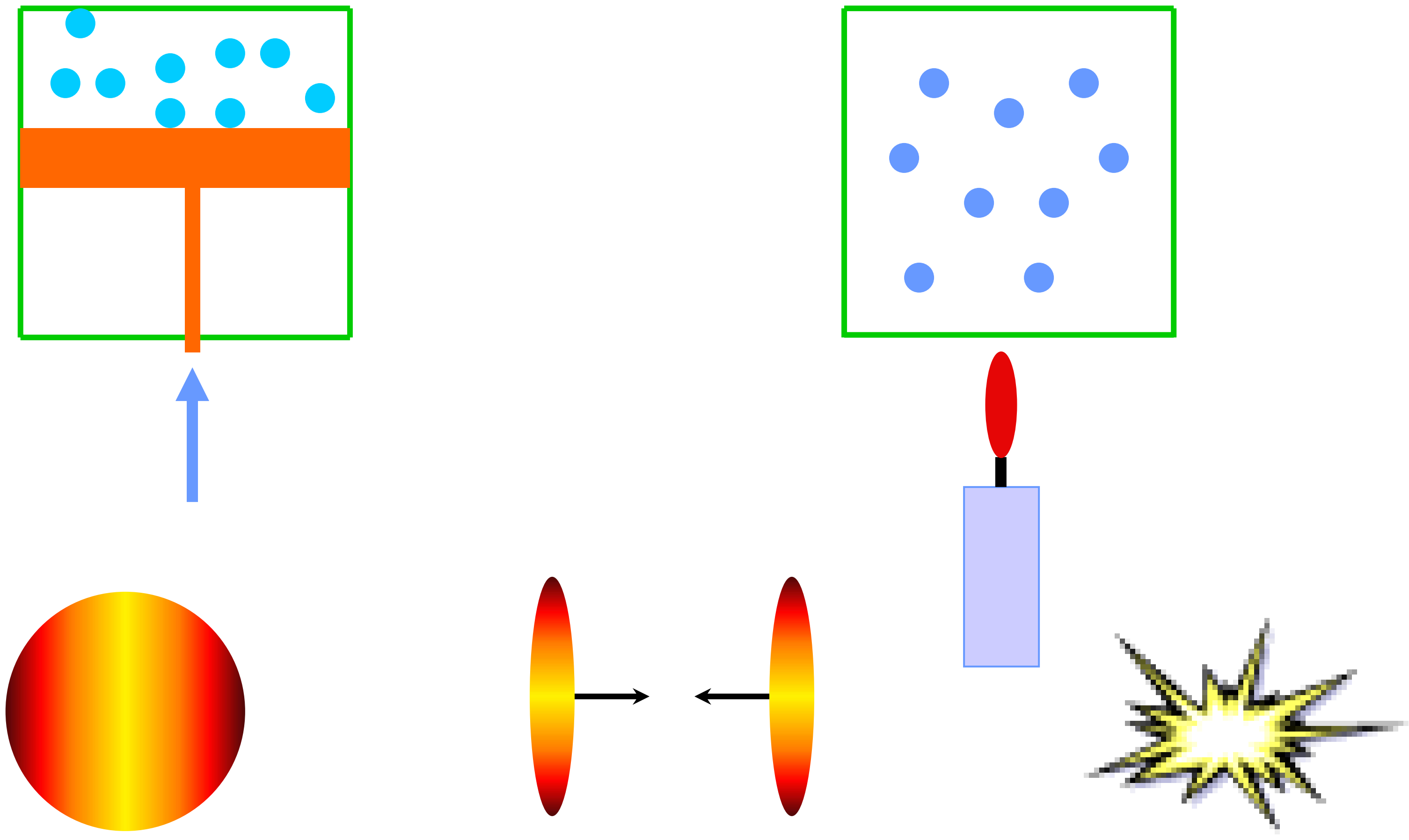
- Introduction and basics in Superconductivity
- Effective theory
- BCS theory
- Color Superconductivity: CFL and 2SC phases
- Effective theories and perturbative calculations
- LOFF phase
- Phenomenology?

Introduction

- Motivations
- Basics facts in superconductivity
- Cooper pairs

Motivations

Studying the QCD vacuum under different and extreme conditions may help our understanding



Neutron star Heavy ion collision Big Bang

- Extreme Conditions in the Universe:
Neutron Stars, Big Bang

- Important to explore the entire phase diagram: Understanding of

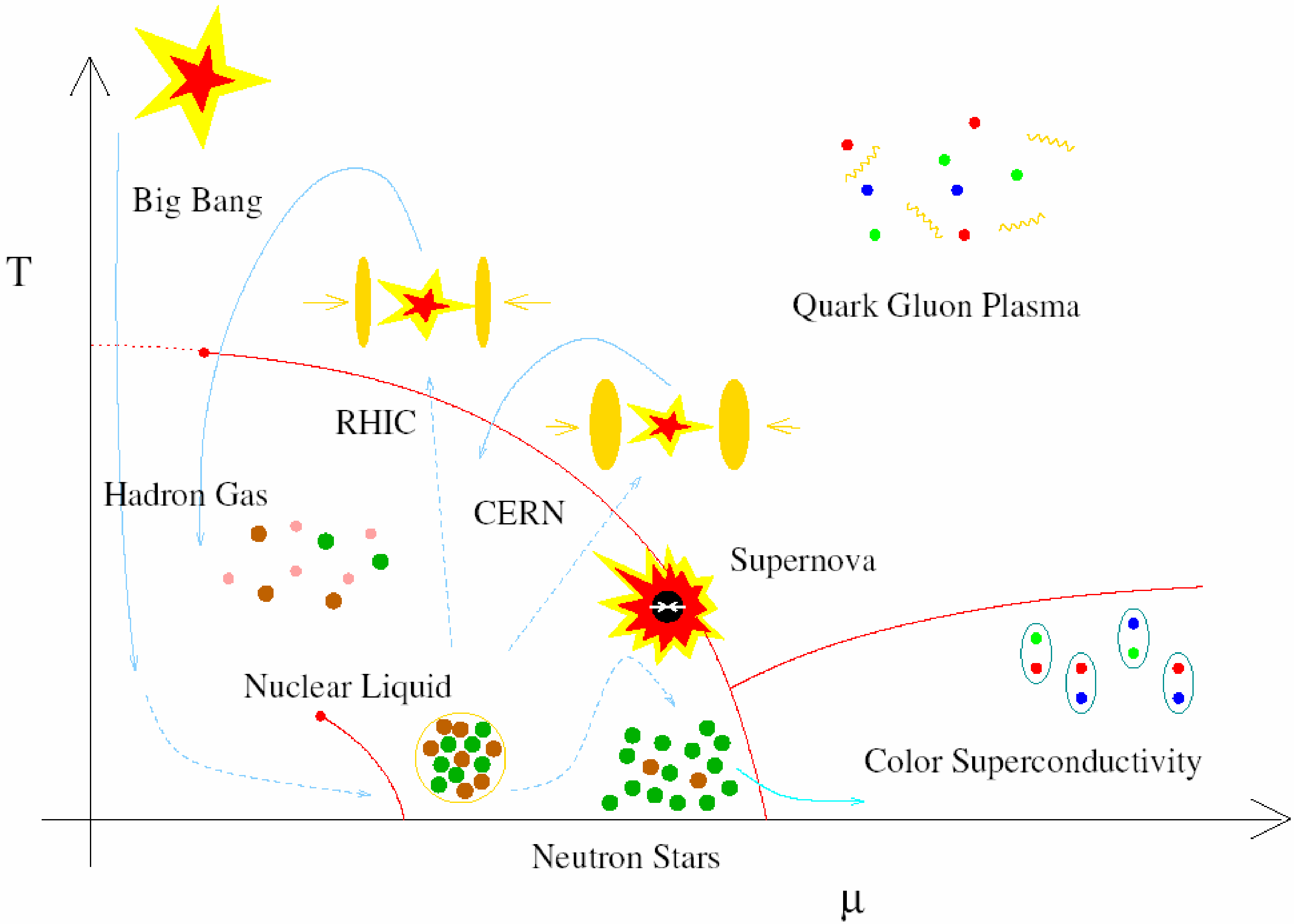
Hadrons → QCD-vacuum

↓

Understanding of its modifications

- QCD simplifies in extreme conditions:

Study QCD when quarks and gluons are the relevant degrees of freedom

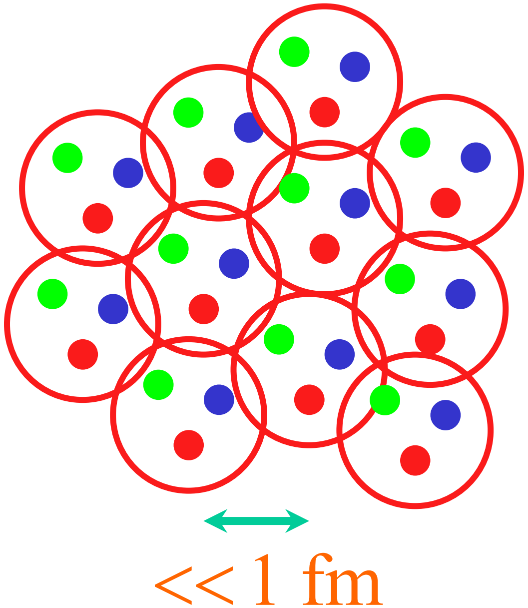

Big Bang
T
Quark Gluon Plasma
RHIC
Hadron Gas
CERN
Supernova
Nuclear Liquid
Color Superconductivity
Neutron Stars
μ

What happens squeezing hadronic matter ?

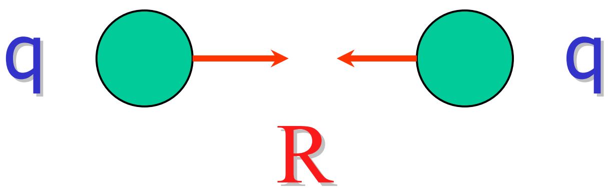
Simple question but COMPLEX ANSWER

When $n_B \gg 1\ fm^{-3}$

free quarks expected



Limiting case $\rho \to \infty$ $(R \to 0)$



Free quarks

Asymptotic freedom:

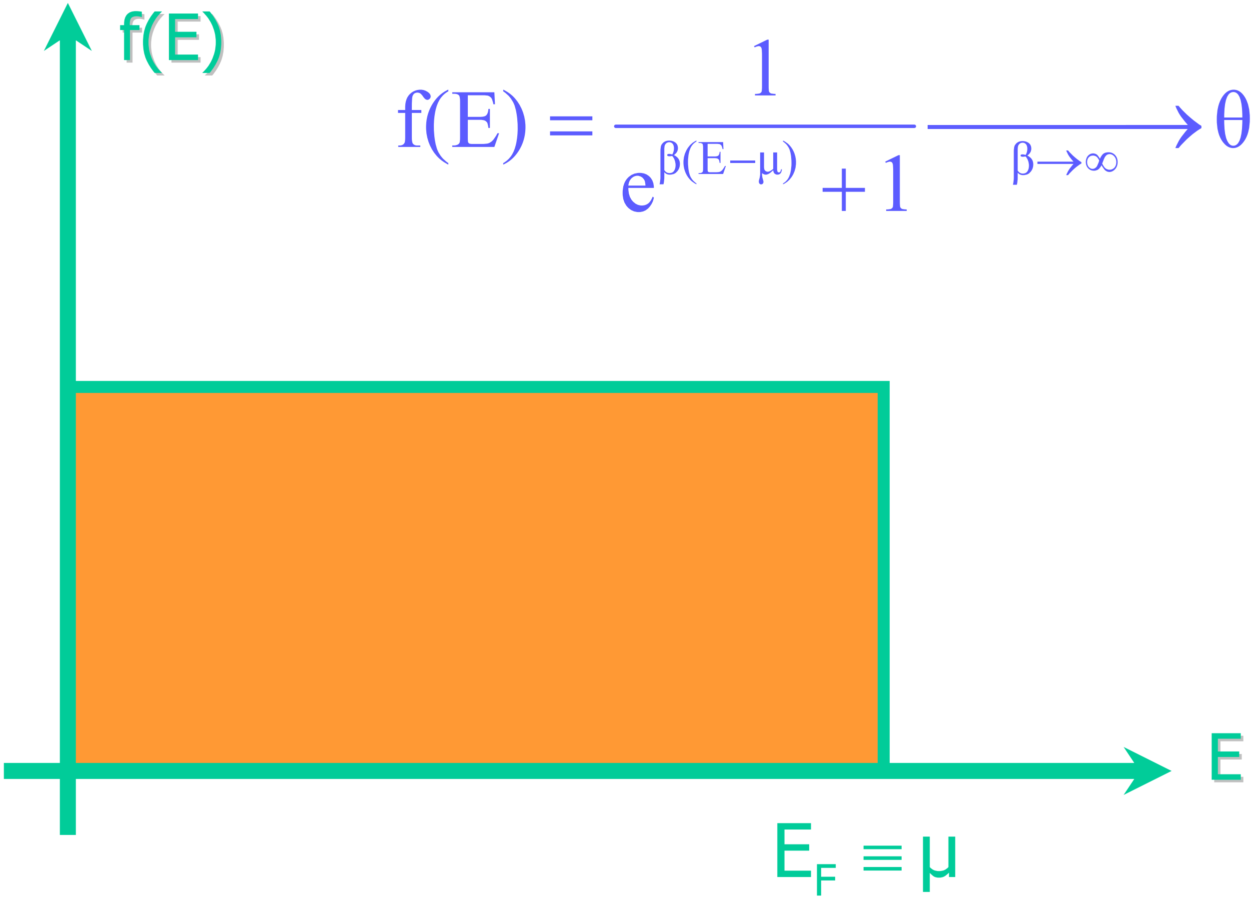
BUT !!!!!!

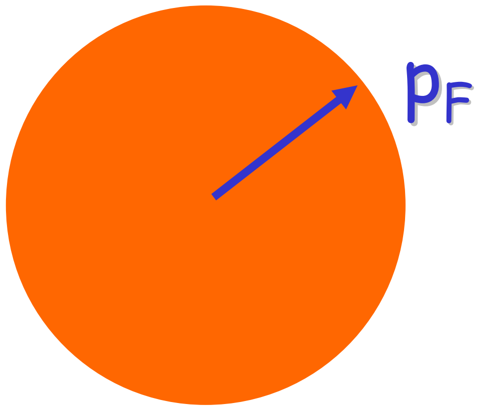
Free Fermi gas and BCS

(high-density QCD)

For $T \to 0 \ (\beta = 1/kT \to \infty)$

$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1} \xrightarrow[\beta \to \infty]{} \theta(\mu - E)$$





- Only quarks at the Fermi surface ($p \sim p_F$) scatter
- High density means high p_F. For

$$p_F >> \Lambda_{QCD}$$

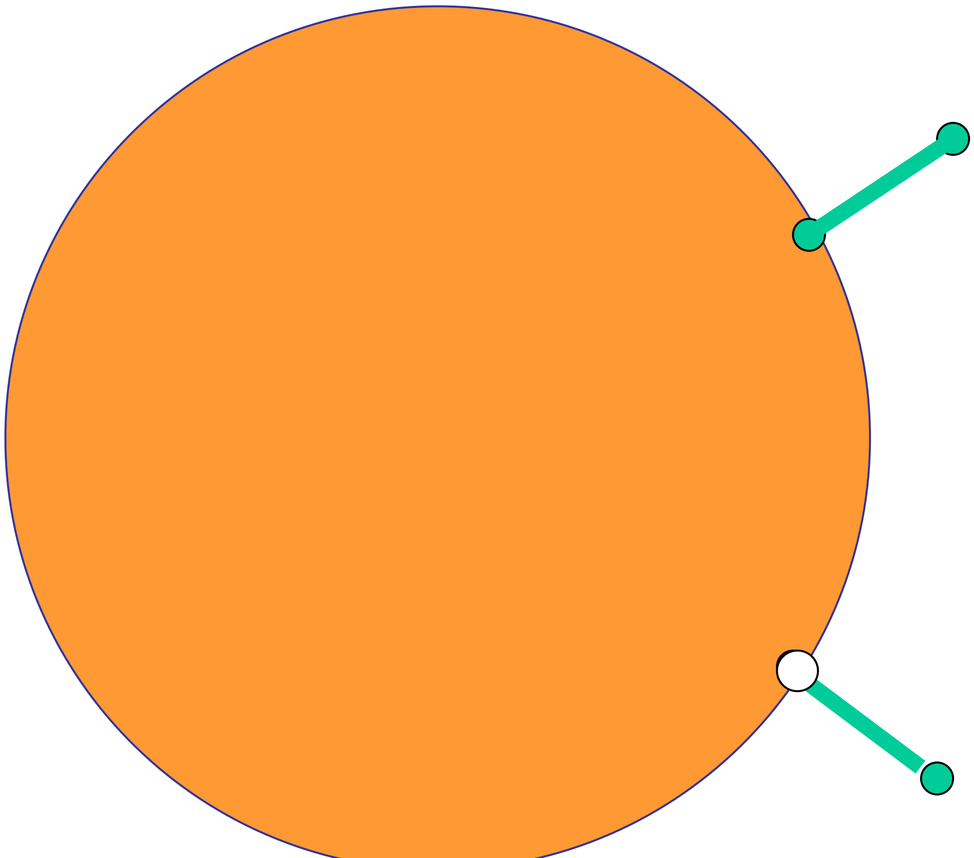
- No chiral breaking
- No confinement
- No generation of masses

Trivial theory ?

Grand potential unchanged: $(F = E - \mu N)$

- Adding a particle to the Fermi surface
- Taking out a particle (creating a hole)

$$F \to (E \pm E_F) - \mu(N \pm 1) = F$$



For an arbitrary attractive interaction it is convenient to form pairs particle-particle or hole-hole (Cooper pairs)

$$E + (\pm 2E_F - E_B) - \mu(N \pm 2) = F - E_B$$

In matter SC only under particular conditions (phonon interaction should overcome the Coulomb force)

$$\frac{T_c(\text{electr.})}{E(\text{electr.})} \approx \frac{1 \div 10\ {}^0\text{K}}{10^4 \div 10^5\ {}^0\text{K}} \approx 10^{-3} \div 10^{-4}$$

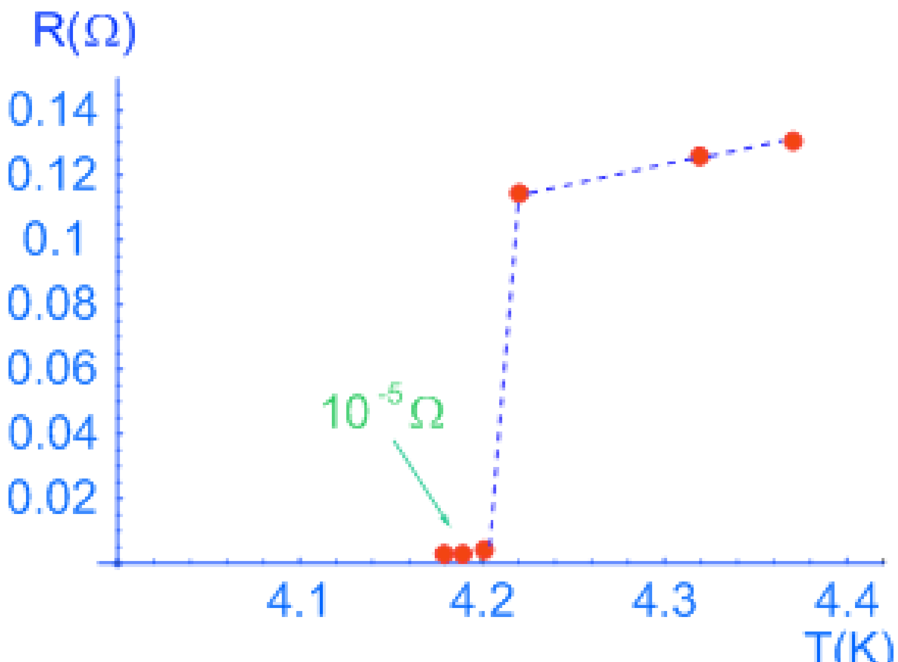
In QCD attractive interaction (antitriplet channel)

$$\frac{T_c(\text{quarks})}{E(\text{quarks})} \approx \frac{50\ \text{MeV}}{100\ \text{MeV}} \approx 1$$

SC much more efficient in QCD

Basics facts in superconductivity

- 1911 – Resistance experiments in mercury lead and thin by Kamerlingh Onnes in Leiden: existence of a critical temperature $T_c \sim$ 4-10 ^{0}K

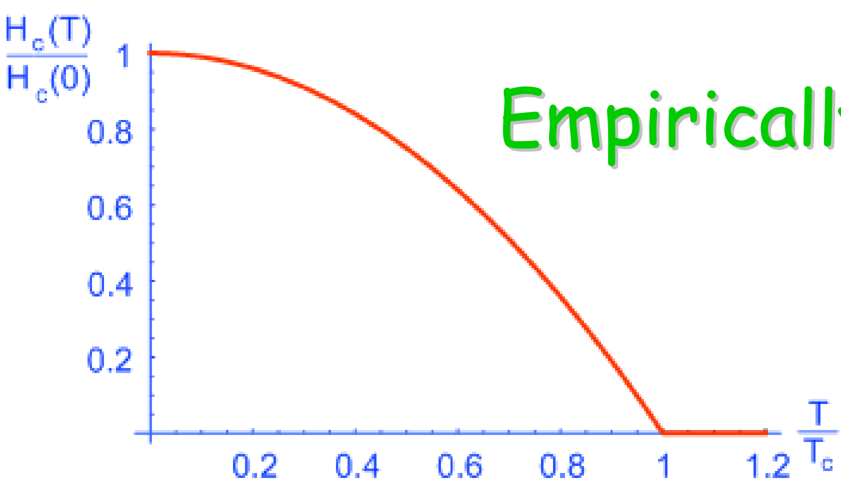


In a superconductor resistivity < 10^{-23} ohm cm

- 1933 - Meissner and Ochsenfeld discover perfect diamagnetism. Exclusion of B except for a penetration depth of ~ 500 Angstrom.

Surprising since from Maxwell, for $E = 0$, B frozen ⟵ $\vec{\nabla} \wedge \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

Destruction of superconductivity for $H = H_c$ $\quad f_s(T) + \frac{H_c^2(T)}{8\pi} = f_n(T)$



Empirically:

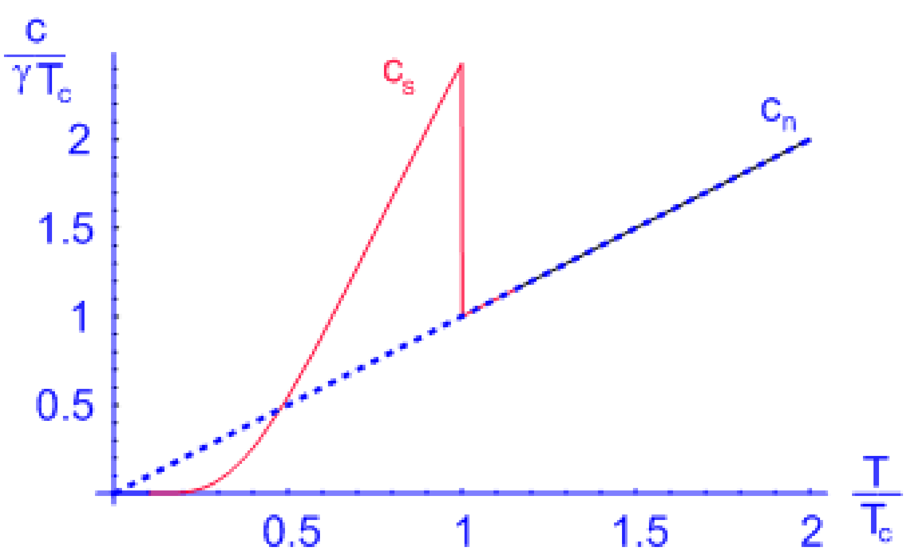
$$H_c(T) \approx H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

- 1950 - Role of the phonons (Frolich).

Isotope effect (Maxwell & Reynolds)

$$T_c \approx H_c(0) \approx \frac{1}{M^{\alpha}}, \quad \alpha \approx 0.45 - 0.5$$

- 1954 - Discontinuity in the specific heat (Corak)

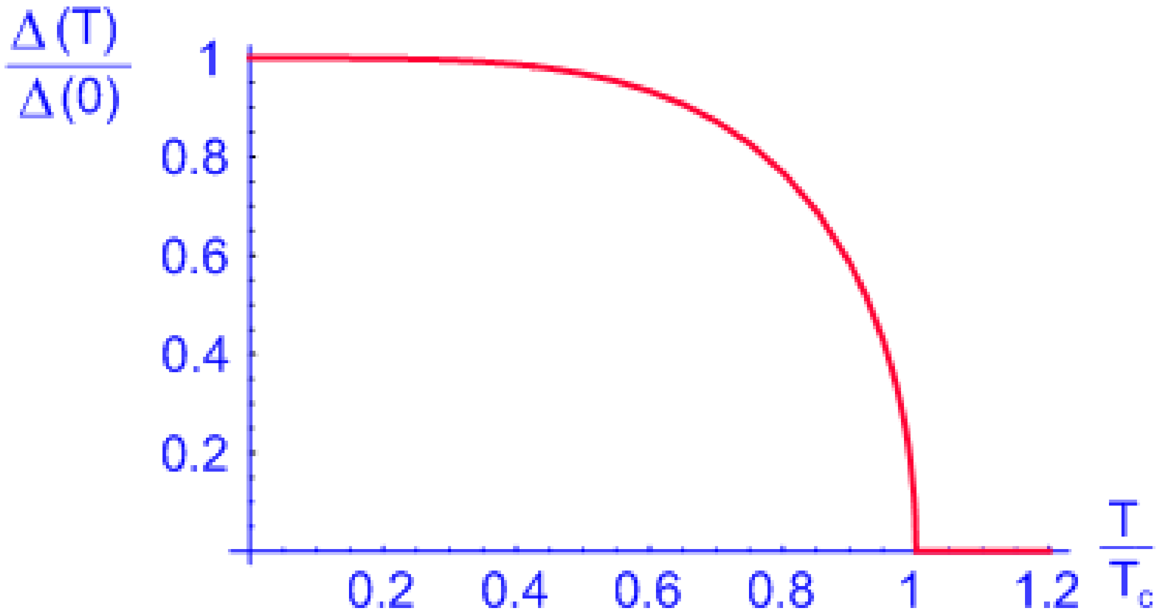


$$c_s \approx a\gamma T_c e^{-bT_c/T}$$

$$c_n \approx \gamma T$$

Excitation energy ~ $1.5\ T_c$

Implication is that there is a gap in the spectrum. This was measured by Glover and Tinkham in 1956



- **Two fluid models**: phenomenological expressions for the free energy in the normal and in the sueprconducting state (Gorter and Casimir 1934)

- **London & London theory, 1935**: still a two-fluid models based on

$$\frac{\partial \vec{J}_s}{\partial t} = \frac{n_s e^2}{m}\vec{E}, \qquad (\vec{J}_s = -e n_s \vec{v}_s)$$

← Newton equation

$$\vec{J}_n = \sigma_n \vec{E}, \qquad (\vec{J}_n = -e n_n \vec{v}_n)$$

$$\vec{\nabla} \wedge \vec{J}_s = -\frac{n_s e^2}{mc}\vec{B}$$

+ Maxwell

$$\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c}\vec{J}_s$$

$$\nabla^2 \vec{B} = \frac{4\pi n_s e^2}{mc^2}\vec{B} = \frac{1}{\lambda_L^2}\vec{B}$$

→

$$B(x) = B(0)e^{-x/\lambda_L}$$

- 1950 - Ginzburg-Landau theory. In the context of Landau theory of second order transitions, valid only around T_c, not appreciated at that time. Recognized of paramount importance after BCS. Based on the construction of an effective theory (modern terms)

$$n_s = |\psi(\vec{r})|^2$$

$$F_s(T) - F_n(T) =$$
$$= \int d^3\vec{r} \left(-\frac{1}{2m} \psi^*(\vec{r}) |\vec{\nabla} + ie^*\vec{A}|^2 \psi(\vec{r}) + \alpha(T) |\psi(\vec{r})|^2 + \frac{1}{2}\beta(T) |\psi(\vec{r})|^4 \right)$$

Cooper pairs

1956 - Cooper proved that two fermions may form a bound state for an arbitrary attractive interaction in a simple model

Only two particle interactions considered. Interactions with the sea neglected but from Fermi statistics

Assume for the ground state:

$$\psi_0(\vec{r}_1-\vec{r}_2)=\underbrace{(\alpha_1\beta_2-\alpha_2\beta_1)}_{\text{spin}}\sum_k g_k \cos\underbrace{(\vec{k}\cdot(\vec{r}_1-\vec{r}_2))}_{\text{zero total momentum}}$$

$$\left[-\frac{1}{2m}(\nabla_1^2+\nabla_1^2)+V(\vec{r}_1-\vec{r}_2)\right]\psi_0(\vec{r}_1-\vec{r}_2)=E\psi_0(\vec{r}_1-\vec{r}_2)$$

$$(E-2\varepsilon_k)g_k=\sum_{k'>k_F}V_{k,k'}g_{k'},\quad \varepsilon_k=\frac{|\vec{k}|^2}{2m}$$

$$V_{k,k'} = \frac{1}{L^3}\int V(\vec{r})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}d^3\vec{r}$$

Cooper assumed that only interactions close to Fermi surface are relevant (see later)

$$V_{k,k'} = \begin{cases} -G, & k_F \le k \le k_c \\ 0, & \text{otherwise} \end{cases}$$

cutoff: $\varepsilon_{k_c} = E_F + \delta$

$$(E - 2\varepsilon_k)g_k = -G\sum_{k'>k_F} g_{k'}$$

Summing over k:

$$\frac{1}{G} = \sum_{k>k_F}\frac{1}{2\varepsilon_k - E}$$

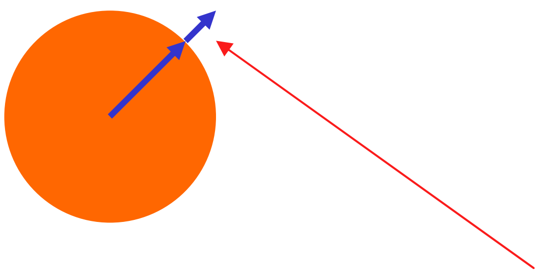
$$\frac{1}{G} = \int_{k_F}^{k_c} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\varepsilon_k - E} = \int_{E_F}^{E_F+\delta} \frac{d\Omega}{(2\pi)^3} k^2 \frac{dk}{d\varepsilon_k} \frac{d\varepsilon}{2\varepsilon - E}$$

Defining the density of the states at the Fermi surface:

$$\rho = 2\int \frac{d\Omega}{(2\pi)^3} k^2 \frac{dk}{d\varepsilon_k}$$

$$\frac{1}{G} = \frac{1}{4}\rho \log \frac{2E_F - E + 2\delta}{2E_F - E}$$

Close to the Fermi surface



$$\varepsilon_k = \mu + (\varepsilon_k - \mu) \approx \mu + \left.\frac{\partial \varepsilon_k}{\partial \vec{k}}\right|_{k=k_F} \cdot (\vec{k} - \vec{k}_F) = \mu + \vec{v}_F(\vec{k}) \cdot \vec{\ell}$$

$$\rho = 2\int \frac{d\Omega}{(2\pi)^3} k^2 \frac{dk}{d\varepsilon_k}$$

For a sphere:

$$\rho = \frac{k_F^2}{\pi^2 v_F}$$

$$E = 2E_F - 2\delta \frac{e^{-4/\rho G}}{1 - e^{-4/\rho G}}$$

For most superconductors $\rho G < 0.3$

Weak coupling approximation:

$$E = 2E_F - 2\delta e^{-4/\rho G} \longrightarrow E_B$$

Very important: result not analytic in G

$$N = \sum_{k>k_F} g_k \qquad \psi_0(\vec{r}) = N \sum_{k>k_F} \frac{\cos(\vec{k}\cdot\vec{r})}{2\varepsilon_k - E}$$

$$\xi_k = \varepsilon_k - E_F \qquad \boxed{\psi_0(\vec{r}) = N \sum_{k>k_F} \frac{\cos(\vec{k}\cdot\vec{r})}{2\xi_k + E_B}}$$

Wave function maximum in momentum space close to $\boxed{\xi_k = 0}$

Paired electrons within E_B from E_F: $\boxed{E_B << \delta}$

Only d.o.f. close to E_F relevant!! →

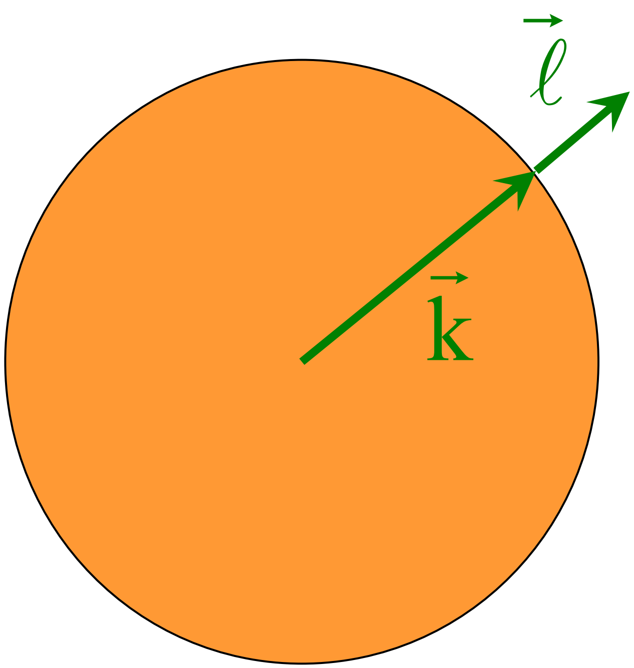
Effective theory

- Field theory at the Fermi surface
- The free fermion gas
- One-loop corrections

Field theory at the Fermi surface

(Polchinski, TASI 1992, hep-th/9210046)

How do fields behave scaling down the energies toward ε_F by a factor $s<1$?



$$\vec{p} = \vec{k} + \vec{\ell}$$

Scaling:

$$E \Rightarrow sE$$
$$\vec{\ell} \Rightarrow s\vec{\ell}$$
$$\vec{k} \Rightarrow \vec{k}$$

Using the invariance under phase transformations, construction of the most general action for the effective degrees of freedom: particles and holes close to the Fermi surface (non-relativistic description)

$$\int dt d^3\vec{p}\left[i\psi_\sigma^\dagger(\vec{p})\partial_t\psi_\sigma(\vec{p}) - \left(\varepsilon(\vec{p}) - \varepsilon_F\right)\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})\right]$$

Expanding around ε_F:

$$\varepsilon(\vec{p}) - \varepsilon_F = \left|\frac{\partial\varepsilon(\vec{p})}{\partial\vec{p}}\right|_{\ell=0} \cdot \vec{\ell} + O\left(\ell^2\right) \equiv v_F\ell + \cdots$$

$$S = \int dt\, d^2\vec{k}\, d\vec{\ell} \left[i\psi_\sigma^\dagger(\vec{p})\, \partial_t \psi_\sigma(\vec{p}) - \ell v_F \psi_\sigma^\dagger(\vec{p})\, \psi_\sigma(\vec{p}) \right]$$

Scaling: $\Rightarrow$ $S \to s^{2d_\psi + 1} S$

$$\ell \to s\ell$$

$$dt \to s^{-1} dt$$

$$d\vec{k} \to d\vec{k}$$

$$d\vec{\ell} \to s\, d\vec{\ell}$$

$$\partial_t \to s\,\partial_t$$

requiring the action S to be invariant

$$\psi \to s^{-1/2} \psi$$

Possible terms in the action:

Quadratic

s^0 $\qquad$ s^{-1}

$$\int dt\, d^2\vec{k}\, d\vec{\ell}\, \mu\left(\vec{k}\right) \psi^\dagger_\sigma\left(\vec{p}\right) \psi_\sigma\left(\vec{p}\right)$$

Scales as s^{-1} → **Relevant**

As a mass term, absorbed into the definition of the Fermi surface. Addition of terms $\sim \ell$, or time derivatives gives marginal (already present) or irrelevant operators

Quartic

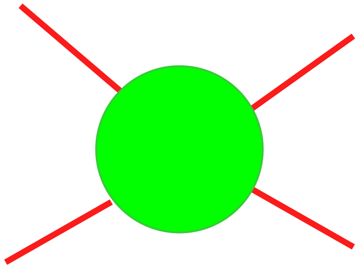
$$\int dt\, d^2\vec{k}_1 d\vec{\ell}_1 d^2\vec{k}_2 d\vec{\ell}_2 d^2\vec{k}_3 d\vec{\ell}_3 d^2\vec{k}_4 d\vec{\ell}_4 \quad \to s^{-1+4}$$

$$V(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\,\psi^\dagger_\sigma(\vec{p}_1)\psi_\sigma(\vec{p}_3)\psi^\dagger_{\sigma'}(\vec{p}_2)\psi_{\sigma'}(\vec{p}_4) \quad \to s^{-4x1/2}$$

$$\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4) \quad \to s^{\delta}\ ??$$

Scales as $s^{1+\delta}$

Scattering: $\vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_3 + \vec{p}_4$



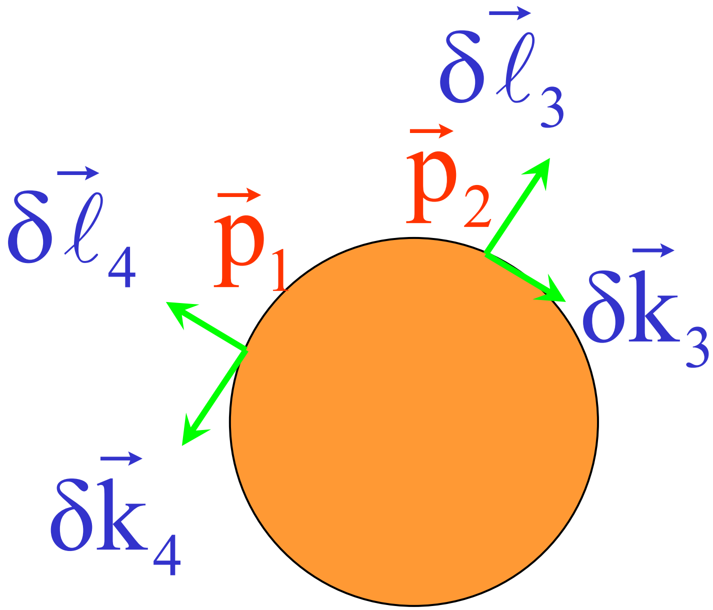
$$\vec{p}_3 = \vec{p}_1 + \delta\vec{k}_3 + \delta\vec{\ell}_3$$

$$\vec{p}_4 = \vec{p}_2 + \delta\vec{k}_4 + \delta\vec{\ell}_4$$

Momentum conservation

$$\delta^3\left(\delta\vec{k}_3 + \delta\vec{k}_4 + \delta\vec{\ell}_3 + \delta\vec{\ell}_4\right)$$

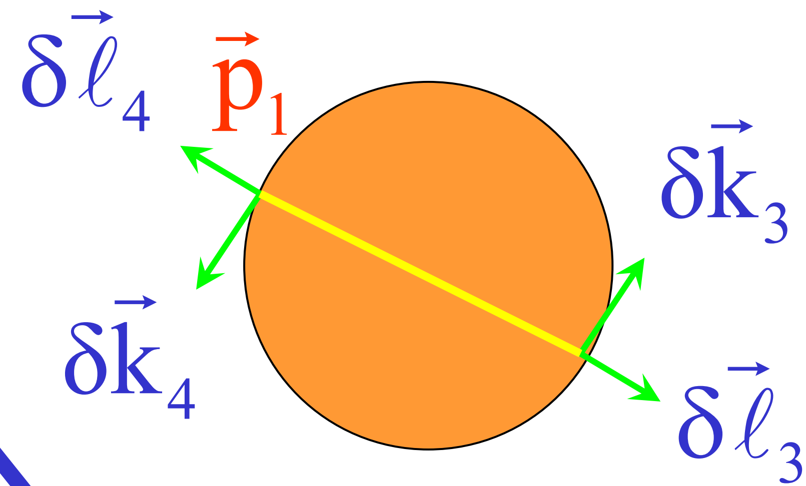
irrelevant



s^0

$$\delta^3\left(\delta\vec{k}_3 + \delta\vec{k}_4 + \delta\vec{\ell}_3 + \delta\vec{\ell}_4\right)$$

marginal



$$\vec{p}_2 = -\vec{p}_1$$

s^{-1}

$$\delta^2(\delta\vec{k}_3 + \delta\vec{k}_4)\delta(\delta\vec{\ell}_3 + \delta\vec{\ell}_4)$$

Higher order interactions irrelevant

Free theory BUT check quantum corrections to the marginal interactions among the Cooper pairs

The free fermion gas

Eq. of motion: $$(i\partial_t - \ell v_F)\psi_\sigma(\vec{p},t) = 0$$

Propagator: $$(i\partial_t - \ell v_F)G_{\sigma,\sigma'}(\vec{p},t) = \delta_{\sigma,\sigma'}\delta(t)$$

$$G_{\sigma,\sigma'}(\vec{p},t) = \delta_{\sigma,\sigma'}G(\vec{p},t) =$$
$$= -i\delta_{\sigma,\sigma'}\left[\theta(t)\theta(\ell) - \theta(-t)\theta(-\ell)\right]e^{-i\ell v_F t}$$

Using: $$\theta(t) = \frac{i}{2\pi}\int d\omega \frac{e^{-i\omega t}}{\omega + i\varepsilon}$$

$$G(\vec{p},t) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\left[\frac{\theta(\ell)}{\omega - \ell v_F + i\varepsilon} + \frac{\theta(-\ell)}{\omega - \ell v_F - i\varepsilon}\right]$$

or:

$$G(\vec{p},t) \equiv \frac{1}{2\pi}\int dp_0\, e^{-ip_0 t} G(p_0,\vec{p})$$

$$G(p) = \frac{1}{(1+i\varepsilon)p_0 - \ell v_F}$$

Fermi field decomposition

$$\psi_\sigma(x) = \sum_{\vec{p}} b_\sigma(\vec{p},t) e^{i\vec{p}\cdot\vec{x}} = \sum_{\vec{p}} b_\sigma(\vec{p}) e^{-ipx}$$

$$x^\mu = (t,\vec{x}), \quad p^\mu = (\ell v_F, \vec{p})$$

with:

$$b_\sigma(\vec{p})|0\rangle = 0 \quad \text{for} \quad |\vec{p}| > p_F$$

$$b_\sigma^\dagger(\vec{p})|0\rangle = 0 \quad \text{for} \quad |\vec{p}| < p_F$$

$$[b_\sigma(\vec{p}), b_\sigma^\dagger(\vec{p})]_+ = \delta_{\vec{p},\vec{p}'}\delta_{\sigma,\sigma'}$$

$$[\psi_\sigma(\vec{x},t), \psi_\sigma^\dagger(\vec{x}',t)]_+ = \delta_{\sigma,\sigma'}\delta^3(\vec{x}-\vec{x}')$$

The following representation holds:

$$G_{\sigma,\sigma'}(x) = -i\delta_{\sigma,\sigma'}\sum_{\vec{p}}\langle 0|T(b_\sigma(\vec{p},t)b_\sigma^\dagger(\vec{p},0)|0\rangle e^{i\vec{p}\cdot\vec{x}} = \delta_{\sigma,\sigma'}\sum_{\vec{p}} G(\vec{p},t)$$

In fact, using

$$\langle 0|b_\sigma^\dagger(\vec{p})b_\sigma(\vec{p})|0\rangle = \theta(p_F - p) = \theta(-\ell)$$

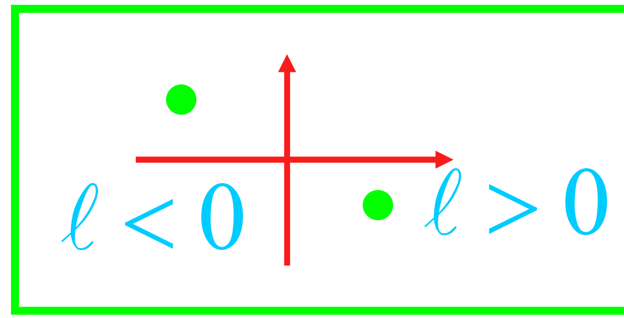
$$\langle 0|b_\sigma(\vec{p})b_\sigma^\dagger(\vec{p})|0\rangle = 1 - \theta(p_F - p) = \theta(p - p_F) = \theta(\ell)$$

$$G(\vec{p},t) = \begin{cases} -i\theta(\ell)e^{-i\ell v_F t}, & t>0 \\ i\theta(-\ell)e^{-i\ell v_F t}, & t<0 \end{cases}$$

The following property is useful:

$$\lim_{\delta\to 0^+} G_{\sigma,\sigma}(\vec{0},-\delta) = -i \lim_{\delta\to 0^+} \langle 0 | T(\psi_\sigma(\vec{0},-\delta)\psi^\dagger_\sigma(0) | 0 \rangle =$$

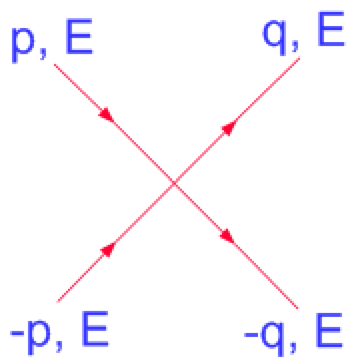
$$= i \langle 0 | \psi^\dagger_\sigma \psi_\sigma | 0 \rangle \equiv i\rho_F$$

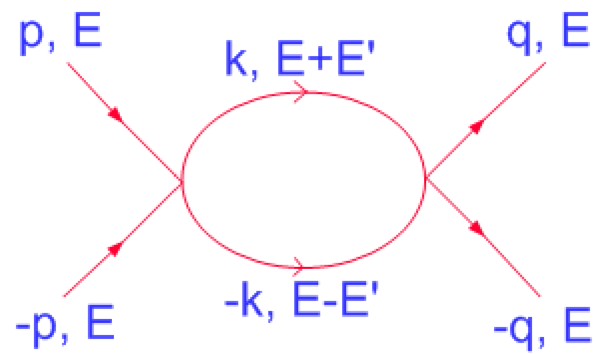


$$\rho_F = -2i \lim_{\delta\to 0^+} \sum_\sigma G_{\sigma,\sigma}(\vec{0},-\delta) = -2i \lim_{\delta\to 0^+} \int \frac{d^4p}{(2\pi)^4} e^{ip_0\delta} \frac{1}{(1+i\varepsilon)p_0 - \ell v_F}$$

$$\rho_F = 2\int \frac{d^3\vec{p}}{(2\pi)^3}\theta(-\ell) = 2\int \frac{d^3\vec{p}}{(2\pi)^3}\theta(p_F - p) = \frac{p_F^3}{3\pi^2}$$

One-loop corrections

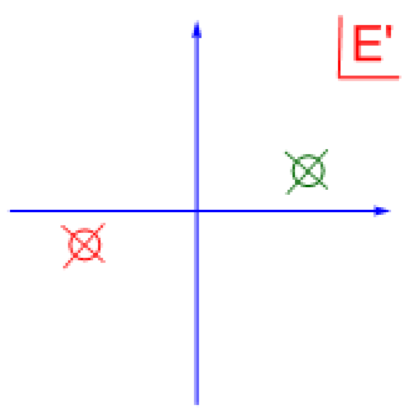




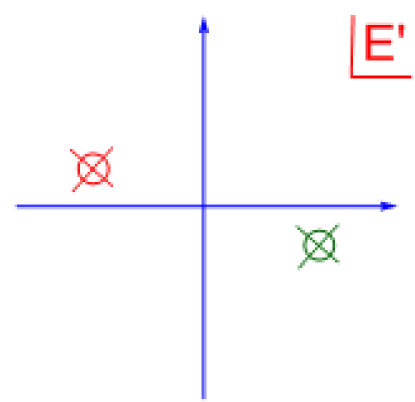
$$\frac{1}{(1+i\varepsilon)p_0 - \ell v_F}$$

$$G(E) = G - G^2 \int \frac{dE' \, d^2\vec{k} \, d\ell}{(2\pi)^4} \frac{1}{((E+E')(1+i\varepsilon) - v_F \ell)((E-E')(1+i\varepsilon) - v_F \ell)}$$

l > 0



l < 0



Closing in the upper plane we get

$$G(E) = G - \frac{1}{2} G^2 \rho \log(\delta/E) + O(G^3)$$

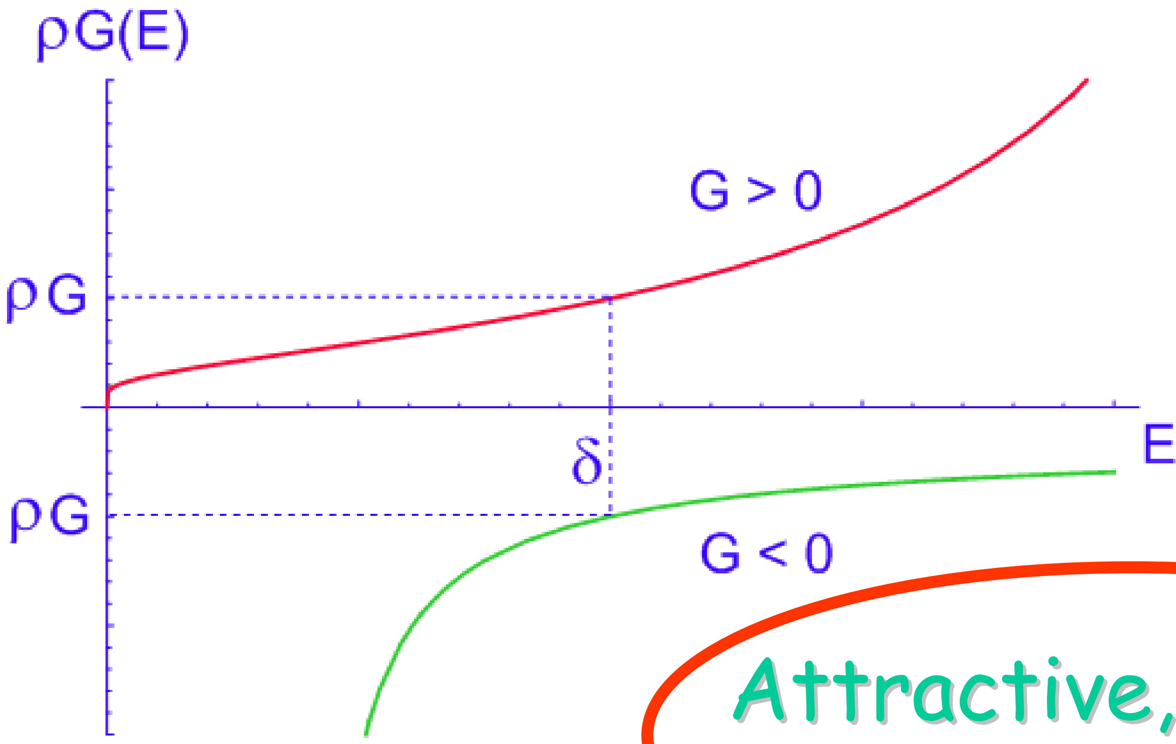
$$\rho = 2\int \frac{d^2\vec{k}}{(2\pi)^3} \frac{1}{v_F(\vec{k})}$$

δ, UV cutoff

From RG equations:

$$\frac{dG(E)}{dE} = \frac{1}{2E} \rho G(E)^2$$

$$\rho G(E) = \frac{\rho G}{1 + \frac{\rho G}{2}\log(\delta/E)}$$



$E \to 0$

BCS instability

Attractive, stronger for $E \to 0$

BCS theory

- A toy model
- BCS theory
- Functional approach
- The critical temperature
- The relevance of gauge invariance

A toy model

Solution to BCS instability

↓

Formation of condensates

Studied with variational methods, Schwinger-Dyson, CJT, etc.

Idea of quasi-particles through a toy model (Hubbard toy-model)

2 Fermi oscillators:

$$H = \varepsilon\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) + G a_1^\dagger a_2^\dagger a_1 a_2$$

Trial wave function:

$$\left|\Psi\right\rangle_{\text{trial}} = (\cos\theta + \sin\theta\, a_1^\dagger a_2^\dagger)\left|0\right\rangle$$

$$\Gamma = \left\langle\Psi\right|a_1 a_2\left|\Psi\right\rangle = -\sin\theta\cos\theta$$

Decompose:

$$H = H_0 + H_{res}$$

$$H_0 = \varepsilon\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) - G\Gamma\left(a_1 a_2 - a_1^\dagger a_2^\dagger\right) + G\Gamma^2$$

$$H_{res} = G a_1^\dagger a_2^\dagger a_1 a_2 + G\Gamma\left(a_1 a_2 - a_1^\dagger a_2^\dagger\right) - G\Gamma^2 =$$

$$= G\left(a_1^\dagger a_2^\dagger + \Gamma\right)\left(a_1 a_2 - \Gamma\right)$$

Mean field theory assumes $H_{res} = 0$

$$\langle \Psi | H_0 | \Psi \rangle = 2\varepsilon \sin^2\theta - G\Gamma^2$$

Minimize w.r.t. θ

$$2\varepsilon \sin 2\theta + 2G\Gamma \cos 2\theta = 0 \Rightarrow \tan 2\theta = -\frac{G\Gamma}{\varepsilon}$$

From the expression for Γ:

$$\Gamma = -\frac{1}{2}\sin 2\theta = \frac{1}{2}\frac{G\Gamma}{\sqrt{\varepsilon^2 + G^2\Gamma^2}}$$

Gap equation

$$1 = \frac{1}{2} \frac{G}{\sqrt{\varepsilon^2 + \Delta^2}}$$

$$\Delta = G\Gamma$$

$|\Psi\rangle_{\text{trial}}$ Is the fundamental state in the broken phase where the condensate Γ is formed

In fact, via Bogolubov transformation

$$A_1 = a_1 \cos\theta - a_2^\dagger \sin\theta$$

$$A_2 = a_1^\dagger \sin\theta + a_2 \cos\theta$$

one gets: $$A_{1,2}\left|\Psi\right\rangle_{\text{trial}} = 0$$

$$H_0 = \left(\varepsilon - \sqrt{\varepsilon^2 + \Delta^2}\right) + \sqrt{\varepsilon^2 + \Delta^2}\left(A_1^\dagger A_1 + A_2^\dagger A_2\right)$$

Energy of quasi-particles (created by $A^+_{1,2}$)

$$E = \sqrt{\varepsilon^2 + \Delta^2}$$

$$\langle\Psi|H_0|\Psi\rangle = 2\varepsilon\sin^2\theta - G\Gamma^2 = \left(\varepsilon - \frac{\varepsilon^2}{\sqrt{\varepsilon^2+\Delta^2}}\right) - \frac{\Delta^2}{G}$$

at weak coupling:

$$\frac{1}{2}\frac{\Delta^2}{\varepsilon} - \frac{\Delta^2}{G}, \quad \text{from gap eq. } 1 = \frac{G}{2\varepsilon}$$

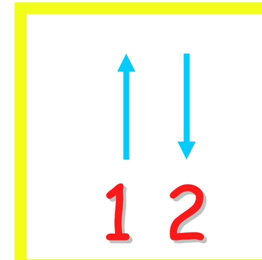
$$\langle\Psi|H_0|\Psi\rangle = 0$$

BCS theory

$$\tilde{H} = H - \mu N = \sum_{k\sigma} \xi_k b_\sigma^\dagger(k) b_\sigma(k) + \sum_{kq} V_{kq} b_1^\dagger(k) b_2^\dagger(-k) b_2(-q) b_1(q)$$

$$\xi_k = \varepsilon_k - E_F = \varepsilon_k - \mu$$

$$\tilde{H} = H_0 + H_{res}$$



$$H_0 = \sum_{k\sigma} \xi_k b_\sigma^\dagger(k) b_\sigma(k) + \sum_{kq} V_{kq} \left[b_1^\dagger(k) b_2^\dagger(-k) \Gamma_q + b_2(-q) b_1(q) \Gamma_k^* - \Gamma_q \Gamma_k^* \right]$$

$$H_{res} = \sum_{kq} V_{kq} \left(b_1^\dagger(k) b_2^\dagger(-k) - \Gamma_k^* \right) \left(b_2(-q) b_1(q) - \Gamma_q \right)$$

$$\Gamma_k = \left\langle b_2(-k) b_1(k) \right\rangle$$

$$H_0 = \sum_{k\sigma} \xi_k b_\sigma^\dagger(k) b_\sigma(k) + \sum_k \left[\Delta_k b_1^\dagger(k) b_2^\dagger(-k) + \Delta_k^* b_2(-k) b_1(k) - \Delta_k \Gamma_k^* \right]$$

$$\Delta_k = -\sum_q V_{kq} \Gamma_q$$

Bogolubov-Valatin transformation:

$$b_1(k) = u_k^* A_1(k) + v_k A_2^\dagger(k),$$

$$b_2^+(-k) = -v_k^* A_1(k) + u_k A_2^\dagger(k)$$

$$|u_k|^2 + |v_k|^2 = 1$$

To bring H_0 in canonical form we choose

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right), \quad |v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$$

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$$

$$H_0 = \sum_{k\sigma} E_k A_\sigma^\dagger(k) A_\sigma(k) + \langle H_0 \rangle$$

$$|0\rangle_{BCS} = \prod_k (u_k + v_k b_1^\dagger(k) b_2^\dagger(-k)) |0\rangle$$

$$A_1(k)|0\rangle_{BCS} = A_2(k)|0\rangle_{BCS} = 0$$

$$\Gamma_k = \left\langle b_2(-k) b_1(k) \right\rangle = u_k^* v_k \left\langle (1 - A_1^\dagger(k) A_1(k) - A_2^\dagger(k) A_2(k) \right\rangle =$$

$$= u_k^* v_k = \frac{1}{2} \frac{\Delta_k}{E_k}$$

$$\Gamma_k = \frac{1}{2} \frac{\Delta_k}{E_k}$$

$$\Delta_k = -\sum_q V_{kq} \Gamma_q$$

$$\Delta_k = -\frac{1}{2} \sum_q V_{kq} \frac{\Delta_q}{E_q}$$

Gap equation

As for the Cooper case choose:

$$V_{k,k'} = \begin{cases} -G, & |\xi_k|, |\xi_{k'}| < \delta \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta_k \approx \Delta$$

$$\langle H_0 \rangle = 2 \sum_{k>k_F} \underbrace{\left(\xi_k - \frac{\xi_k^2}{E_k} \right)}_{\text{Kinetic energy}} - \underbrace{\frac{\Delta^2}{G}}_{\text{Interaction term}}$$

$$\langle H_0 \rangle = \rho \int_0^{\delta} d\xi \left(\xi - \frac{\xi^2}{\sqrt{\xi^2 + \Delta^2}} \right) - \frac{\Delta^2}{G} =$$

$$= \rho \left[\delta^2 - \delta \sqrt{\delta^2 + \Delta^2} + \Delta^2 \log \overbrace{\frac{\delta + \sqrt{\delta^2 + \Delta^2}}{\Delta}}^{\frac{2}{\rho G}} \right] - \frac{\Delta^2}{G}$$

$$\Delta = \frac{1}{2}\rho G \int_0^{\delta} d\xi \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}} = \frac{1}{2}\rho G \Delta \log \frac{\delta + \sqrt{\delta^2 + \Delta^2}}{\Delta}$$

$$\langle H_0 \rangle = \frac{\rho}{2}\left[\delta^2 - \delta\sqrt{\delta^2 + \Delta^2} + \frac{2\Delta^2}{\rho G}\right] - \frac{\Delta^2}{G} =$$

$$= \frac{\rho}{2}\left[\delta^2 - \delta\sqrt{\delta^2 + \Delta^2}\right] \approx -\frac{1}{4}\rho\Delta^2$$

$\rho G << 1$, or $\Delta << \delta$

Pair condensation energy

$$\langle H_0 \rangle \approx -\frac{1}{4}\rho\Delta^2$$

$$\Delta \approx 2\delta e^{-2/\rho G}$$

$T \neq 0$

$$\langle O \rangle_T = \frac{\mathrm{Tr}\left[e^{-H/T} O\right]}{\mathrm{Tr}\left[e^{-H/T}\right]}$$

For a single Fermi oscillator $H = Eb^\dagger b$

$$\mathrm{Tr}[e^{-Eb^\dagger b/T}] = 1 + e^{-E/T}$$

$$\mathrm{Tr}[b^\dagger b e^{-Eb^\dagger b/T}] = e^{-E/T}$$

$$\boxed{\langle b^\dagger b \rangle_T = f(E) = \frac{1}{e^{E/T} + 1}}$$

Fermi distribution

$$\Gamma_k = u_k^* v_k \left\langle (1 - A_1^\dagger(k) A_1(k) - A_2^\dagger(k) A_2(k) \right\rangle_T = u_k^* v_k (1 - 2f(E_k))$$

$$\Delta_k = -\sum_q V_{kq} u_q^* v_q (1 - 2f(E_q)) = -\sum_q V_{kq} \frac{\Delta_q}{2E_q} \tanh \frac{E_q}{2T}$$

$$\boxed{1 = \frac{1}{4} \rho G \int_{-\delta}^{+\delta} \frac{d\xi}{E} \tanh \frac{E}{2T}, \quad E = \sqrt{\xi^2 + \Delta^2}}$$

Functional approach

$$S\left[\psi,\psi^{\dagger}\right]=\int d^4x\left[\psi^{\dagger}(i\partial_t-\varepsilon(|\vec{\nabla}|)+\mu)\psi+\frac{G}{2}(\psi^{\dagger}\psi)^2\right]$$

Fierzing $(C=i\sigma_2)$

$$\psi_a^{\dagger}\psi_a\psi_b^{\dagger}\psi_b=-\psi_a^{\dagger}\psi_b^{\dagger}\psi_a\psi_b=$$

$$=-\frac{1}{4}\varepsilon_{ab}\varepsilon_{ab}\psi_c^{\dagger}\psi^{\dagger c}\psi_d\psi^d=-\frac{1}{2}\psi^{\dagger}C\psi^{*}\psi^{T}C\psi$$

$$S\left[\psi,\psi^{\dagger}\right]=\int d^4x\left[\psi^{\dagger}(i\partial_t-\varepsilon(|\vec{\nabla}|)+\mu)\psi-\frac{G}{4}(\psi^{\dagger}C\psi^{*})(\psi^{T}C\psi)\right]$$

Quantum theory

$$Z=\int D(\psi,\psi^{\dagger})e^{iS\left[\psi,\psi^{\dagger}\right]}$$

$$\text{const.} = \int D(\Delta,\Delta^*) e^{-\frac{i}{G}\int d^4x \left[\Delta - \frac{G}{2}(\psi^T C\psi)\right]\left[\Delta^* + \frac{G}{2}(\psi^\dagger C\psi^*)\right]}$$

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D(\psi,\psi^\dagger) D(\Delta,\Delta^*) e^{iS_0[\psi,\psi^\dagger] + i\int d^4x \left[-\frac{|\Delta|^2}{G} - \frac{1}{2}\Delta(\psi^\dagger C\psi^*) + \frac{1}{2}\Delta^*(\psi^T C\psi)\right]}$$

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi \\ C\psi^* \end{pmatrix}$$

$$S_0 + \cdots = \int d^4x \left(\chi^\dagger S^{-1}\chi - \frac{|\Delta|^2}{G} \right)$$

$$S^{-1}(p) = \begin{bmatrix} p_0 - \xi_p & -\Delta \\ -\Delta^* & p_0 + \xi_p \end{bmatrix}$$

Since ψ^* appears already in χ we are double-counting. Solution: integrate over the fermions with the "replica trick":

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\left[\det(S^{-1})\right]^{1/2} e^{-i\int d^4x \frac{|\Delta|^2}{G}} \equiv e^{iS_{eff}}$$

$$S_{eff}(\Delta,\Delta^*) = -\frac{i}{2}\mathrm{Tr}\left[\log(S_0 S^{-1})\right] - \int d^4x \frac{|\Delta|^2}{G}$$

Evaluating the saddle point:

$$\Delta = iG\int \frac{d^4p}{(2\pi)^4}\frac{\Delta}{p_0^2 - \xi_p^2 - |\Delta|^2} \quad \Longrightarrow \quad \Delta = \frac{G}{2}\int \frac{d^3p}{(2\pi)^3}\frac{\Delta}{\sqrt{\xi_p^2 + |\Delta|^2}}$$

At T not 0, introducing the Matsubara frequencies

$$\omega_n = (2n+1)\pi T$$

$$\Delta = GT \sum_{n=-\infty}^{+\infty} \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\omega_n^2 + \xi_p^2 + |\Delta|^2}$$

and using

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\omega_n^2 + \xi_p^2 + |\Delta|^2} = \frac{1}{2E_p T}\underbrace{(1-2f(E_p))}$$

$$\Delta = \frac{G}{2} \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\sqrt{\xi_p^2 + |\Delta|^2}} \tanh(E_p / 2T)$$

By saddle point:

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D(\psi,\psi^\dagger)D(\Delta,\Delta^*)e^{iS_o[\psi,\psi^\dagger]+i\int d^4x\left[-\frac{|\Delta|^2}{G}-\frac{1}{2}\Delta(\psi^\dagger C\psi^*)+\frac{1}{2}\Delta^*(\psi^T C\psi)\right]}$$

$$\Delta = \frac{G}{2}\left\langle \psi^T C\psi \right\rangle$$

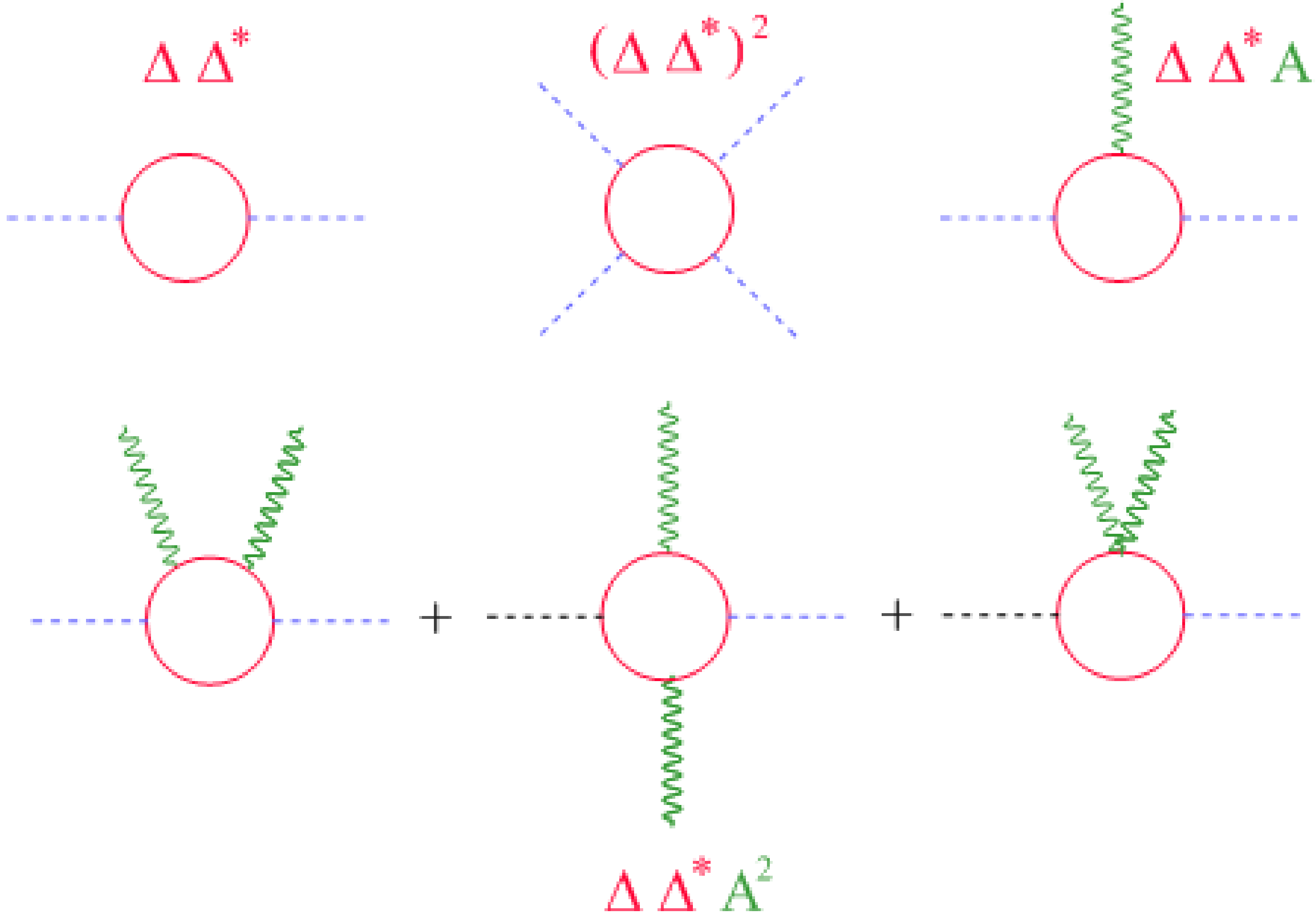
Introducing the em interaction in S_0 we see that Z is gauge invariant under

$$\psi \to \psi e^{i\alpha(x)}, \quad \Delta \to \Delta e^{2i\alpha(x)}$$

Therefore also S_{eff} must be gauge invariant and it will depend on the space-time derivatives of Δ through

$$D_\mu = \partial_\mu + 2ieA_\mu$$

In fact, evaluating the diagrams (Gor'kov 1959):



got the result (with a convenient renormalization of the fields):

$$H = \int d^3\vec{r} \left(-\frac{1}{4m} \psi^*(\vec{r}) \,|\, (\vec{\nabla} + i2e\vec{A}) \,|^2\, \psi(\vec{r}) + \alpha \,|\, \psi(\vec{r}) \,|^2 + \frac{1}{2} \beta \,|\, \psi(\vec{r}) \,|^4 \right)$$

charge of the pair

This result gave full justification to the Landau treatment of superconductivity

The critical temperature

By definition at T_c the gap vanishes. One can perform a GL expansion of the grand potential

$$\Omega = \frac{1}{2}\alpha\Delta^2 + \frac{1}{4}\beta\Delta^4$$

with extrema: $\alpha\Delta^2 + \beta\Delta^4 = 0$

α and β from the expansion of the gap equation up to normalization

$$\Delta = GT\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\frac{\Delta}{\omega_n^2 + \xi_p^2 + |\Delta|^2}$$

To get the normalization remember (in the weak coupling and relatively to the normal state):

$$\langle H_0 \rangle = \Omega = -\frac{1}{4}\rho\Delta^2$$

Starting from the gap equation: $\Delta - \frac{1}{2}\rho G \Delta \log\frac{2\delta}{\Delta} = 0$

Integrating over Δ and using the gap equation one finds:

$$-\frac{G\rho}{8}\Delta^2$$

Rule: Integrate the gap equation and multiply by 2/G

Expanding the gap equation: $(\omega_n = (2n+1)\pi T)$

$$\Delta - 2G\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta} d\xi\left[\frac{\Delta}{(\omega_n^2+\xi^2)} - \frac{\Delta^3}{(\omega_n^2+\xi^2)^2} + \cdots\right] = 0$$

Integrating and applying the rule:

$$\alpha = \frac{2}{G}\left(1 - 2G\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\omega_n^2+\xi^2)}\right)$$

$$\beta = 4\rho T\,\mathrm{Re}\sum_{n=0}^{\infty}\int_0^{\delta}\frac{d\xi}{(\omega_n^2+\xi^2)^2}$$

Integrating over ξ and summing over n up to N

$$\omega_N = \delta \Rightarrow N \approx \frac{\delta}{2\pi T}$$

$$\alpha(T) = \rho \log \frac{\pi T}{\gamma \Delta_0}$$

Requiring $\alpha(T_c) = 0$

$$T_c = \frac{\gamma}{\pi}\Delta_0 \approx 0.56693\,\Delta$$

Also

$$\beta(T) \approx \frac{7\rho}{8\pi^2 T_c^2}\zeta(3)$$

and, from the gap equation $\left(\alpha(T) \approx -\rho\left(1 - \frac{T}{T_c}\right)\right)$

$$\Delta^2(T) = -\frac{\alpha(T)}{\beta(T)} \Rightarrow \Delta(T) \approx \frac{2\sqrt{2}\pi T_c}{\sqrt{7\zeta(3)}}\left(1 - \frac{T}{T_c}\right)^{1/2}$$

Origin of the attractive interaction

- Coulomb force repulsive, need of an attractive interaction

- Electron-phonon interaction (Frolich 1950)

- Simple description: Jellium model (Pines et al. 1958): electrons + ions treated as a fluid.

- Interaction:

$$\frac{4\pi e^2}{q^2+k_s^2}+\underbrace{\frac{4\pi e^2}{q^2+k_s^2}\frac{\omega_q^2}{\omega^2-\omega_q^2}}_{\text{may give attraction}}, \quad \omega_q \approx v_s q$$

$$k_s^2 = \frac{6\pi n e^2}{E_F}$$

may give attraction

Coulomb interaction screned by electrons and ions

The relevance of gauge invariance

(See Weinberg (1990))

In the BCS ground state:

$$\langle O \rangle = \langle \varepsilon_{\alpha\beta} \psi^{\alpha} \psi^{\beta} \rangle \neq 0$$

The $U(1)_{em}$ is broken since $Q_{em}(O) = -2e$.

Introduce an order parameter Φ transforming as the operator O:

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\Lambda, \; \psi \rightarrow e^{ie\Lambda}\psi, \; \Phi \rightarrow e^{2ie\Lambda}\Phi$$

As usual the phase of O is the Goldstone field associated to the breaking of the global U(1). Decompose:

$$\Phi(x) = \rho(x) e^{2ie\phi(x)}$$

Order parameter — Goldstone

$\rho(x)$ is gauge invariant, whereas

$$\phi(x) \to \phi(x) + \Lambda(x)$$

- ϕ dependence through $\partial_\mu \phi$
- U(1) broken to Z_2 $\left(\Lambda = 0, \quad \text{and} \quad \Lambda = \frac{\pi}{e} \right)$

- Gauge invariant Fermi field $\tilde{\psi} = e^{-ie\phi}\psi$
- Effective theory in terms of $\tilde{\psi}, A_\mu, \partial_\mu\phi$
- From gauge invariance only combinations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \; A_\mu - \partial_\mu \phi$$

$$\Downarrow$$

$$L = -\frac{1}{4}\int d^3\vec{x}\, F_{\mu\nu}F^{\mu\nu} + L_s(A_\mu - \partial_\mu\phi)$$

Eqs. of motion for ϕ: $0 = \partial_\mu \frac{\delta L_s}{\delta \partial_\mu \phi} = -\partial_\mu \frac{\delta L_s}{\delta A_\mu} = -\partial_\mu J^\mu$

Assume that L_s gives a stable state in absence of A and ϕ. This implies that

$$A_\mu = \partial_\mu \phi$$

is a local minimum and that

$$\left.\frac{\delta^2 L_s}{\delta(A_\mu - \partial_\mu \phi)^2}\right|_{A_\mu = \partial_\mu \phi} \neq 0$$

Well inside the superconductor we will be at the minimum. The em field is a pure gauge and

$$F_{\mu\nu} = 0 \Rightarrow \vec{B} = 0$$

→ Meissner effect

Close the minimum:

$$L_s(A_\mu - \partial_\mu \phi) \approx L_s(0) + \frac{1}{2} \underbrace{\left.\frac{\delta^2 L_s}{\delta(A_\mu - \partial_\mu \phi)^2}\right|_{A_\mu = \partial_\mu \phi}}_{\dim = E \times E^{-2} = L} (A_\mu - \partial_\mu \phi)^2$$

$$L_s \approx \frac{L^3}{\lambda^2} \left| \vec{A} - \vec{\nabla}\phi \right|^2$$

L^3 = volume, λ some typical length where the field is not a pure gauge

$$\left|\vec{A} - \vec{\nabla}\phi\right| \approx BL \quad \Longrightarrow \quad L_s \approx \frac{B^2L^5}{\lambda^2}$$

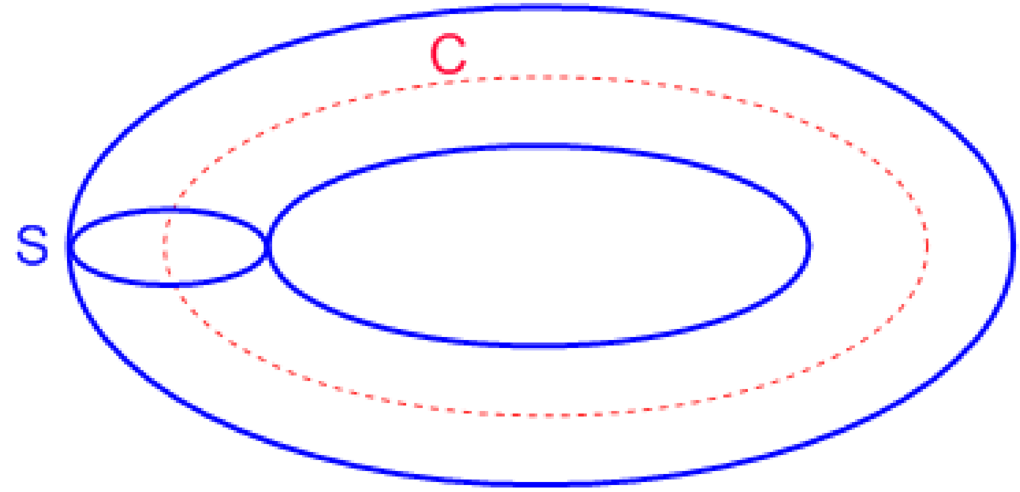
Cost of expelling B $\qquad B^2L^3$

Convenience in expelling B if $\displaystyle \frac{B^2L^5}{\lambda^2} \gg B^2L^3$

$$\boxed{L \gg \lambda}$$

Since $\vec{J} \propto \vec{\nabla} \wedge \vec{B}$ the current flows at the surface in a region of thickness λ

Flux quantization



Inside S:

$$\left|\vec{A} - \vec{\nabla}\phi\right| = 0$$

But not necessarily $\vec{A} = 0, \quad \vec{\nabla}\phi = 0$

$$\int_{\Sigma} \vec{B} \cdot d\vec{S} = \oint_{C} \vec{A} \cdot d\vec{x} = \oint_{C} \vec{\nabla}\phi \cdot d\vec{x} = \frac{n\pi}{e}$$

Since, after 1 cycle

$$\phi \to \phi + \frac{n\pi}{e}$$

Both B and J are at the surface in a layer of thickness λ. Therefore the current cannot decay smoothly but it must jump in such a way that

$$\Delta\Phi(\vec{B}) = \frac{n\pi}{e}$$

Superconductivity

Current density conjugated to ϕ: $$\frac{\delta L_s}{\delta\dot{\phi}} = -\frac{\delta L_s}{\delta A_0} = -J_0$$

Hamilton equation: $$\dot{\phi}(x) = \frac{\delta H_s}{\delta(-J_0(x))} = -V(x)$$

In stationary conditions the voltage $V(x) = 0$, with J not zero

Close to the phase transition the Goldstone field ϕ is not the only long wave-length mode.
Consider again

$$\Phi(x) = \rho(x) e^{2ie\phi(x)}$$

and expand L_s for small Φ

$$L_s \approx \int d^3\vec{x} \left[-\frac{1}{2}\Phi^* \left|(\vec{\nabla} - 2ie\vec{A})\right|^2 \Phi - \frac{1}{2}\alpha|\Phi|^2 - \frac{1}{4}\beta|\Phi|^4 \right]$$

$$L_s \approx \int d^3\vec{x} \left[-2e^2\rho^2(\vec{\nabla}\phi - e\vec{A})^2 - \frac{1}{2}(\vec{\nabla}\rho)^2 - \frac{1}{2}\alpha\rho^2 - \frac{1}{4}\beta\rho^4 \right]$$

$$\lambda = \frac{1}{\sqrt{4e^2\langle\rho^2\rangle}} \qquad \langle\rho^2\rangle = -\frac{\alpha}{\beta}$$

Looking at the fluctuations: $\rho = \rho' + \langle \rho \rangle$

$$\vec{\nabla}^2 \rho' = -2\alpha \rho'$$

Coherence length: $$\xi = \frac{1}{\sqrt{-2\alpha}}$$

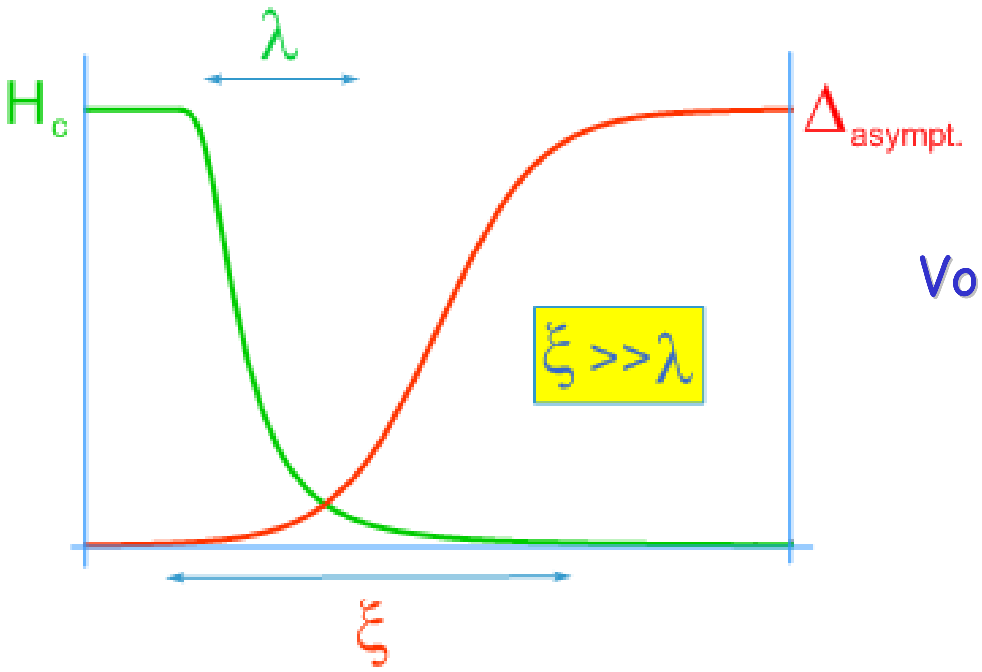
Notice that in the SM: $$\lambda = \frac{1}{M_V^2}, \quad \xi = \frac{1}{M_H^2}$$

Vortex lines can be formed.
Correspondingly to their stability, two types of superconductors

- Type I: $\xi > \lambda$, small penetration of B, vortices are not stable

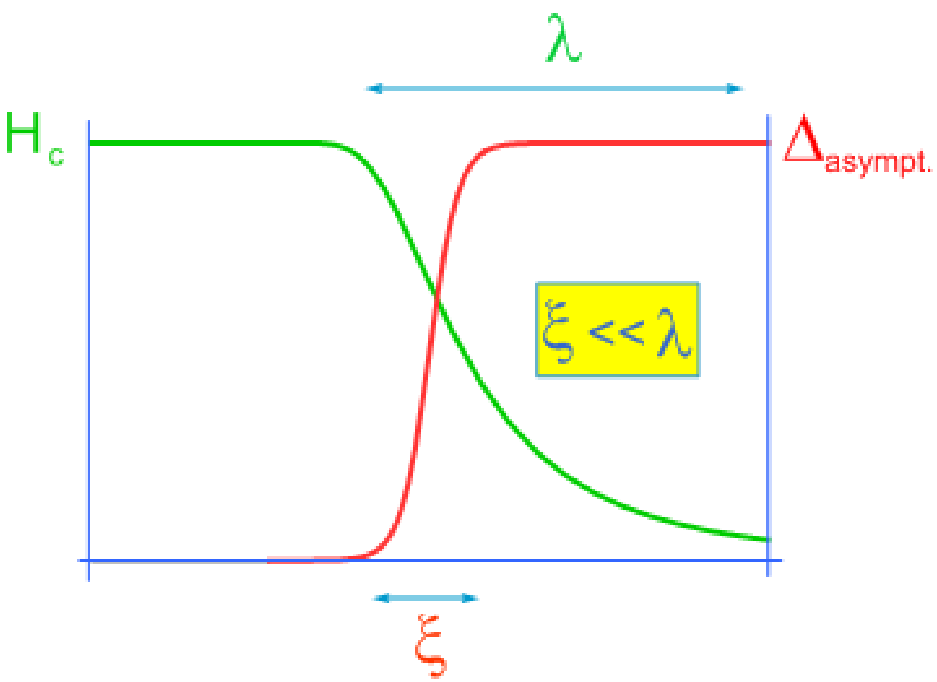
- Type II: $\xi < \lambda$, vortices are stable and penetrate inside. Two critical magnetic fields:

$H < H_{c_1}$	BCS state
$H_{c_1} < H < H_{c_2}$	Vortices
$H > H_{c_2}$	Normal state



Type I

Vortices are not stable due to the small penetration of H



Type II

Vortices are stable and H penetrates inside the vortices because their core smaller than the penetration depth. Two critical magnetic fields. $H>H_1$ (vortices are formed). $H> H_2$ transition to the normal state