Color Superconductivity: CFL and 2SC phases

- Introduction
- Hierarchies of effective lagrangians
- Effective theory at the Fermi surface (HDET)
- Symmetries of the superconductive phases

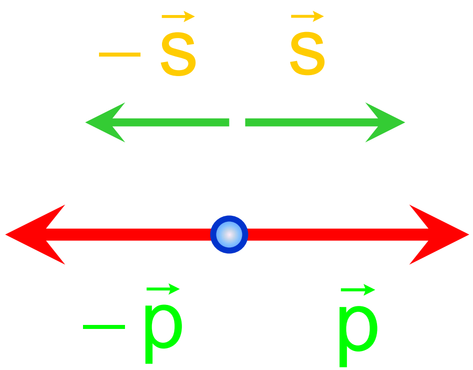
Introduction

- Ideas about CS back in 1975 (Collins & Perry-1975, Barrois-1977, Frautschi-1978).
- Only in 1998 (Alford, Rajagopal & Wilczek; Rapp, Schafer, Schuryak & Velkovsky) a real progress.
- The phase structure of QCD at high-density depends on the number of flavors with mass $m < \mu$.
- Two most interesting cases: $N_f = 2, 3$.
- Due to asymptotic freedom quarks are almost free at high density and we expect difermion condensation in the color channel 3^*.

Consider the possible pairings at very high density

$\langle 0 | \psi_{ia}^{\alpha} \psi_{jb}^{\beta} | 0 \rangle$ α, β color; i, j flavor; a,b spin

- Antisymmetry in spin (a,b) for better use of the Fermi surface
- Antisymmetry in color (α, β) for attraction
- Antisymmetry in flavor (i,j) for Pauli principle



Only possible pairings LL and RR

For $\mu >> m_u, m_d, m_s$

Favorite state for $N_f = 3$, CFL (color-flavor locking) (Alford, Rajagopal & Wilczek 1999)

$$\langle 0 | \psi^{\alpha}_{iL} \psi^{\beta}_{jL} | 0 \rangle = -\langle 0 | \psi^{\alpha}_{iR} \psi^{\beta}_{jR} | 0 \rangle \propto \Delta \varepsilon^{\alpha\beta C} \varepsilon_{ijC}$$

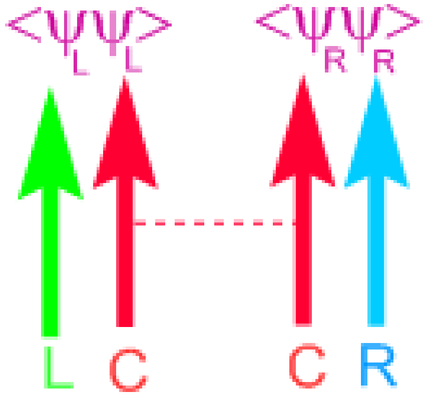
Symmetry breaking pattern

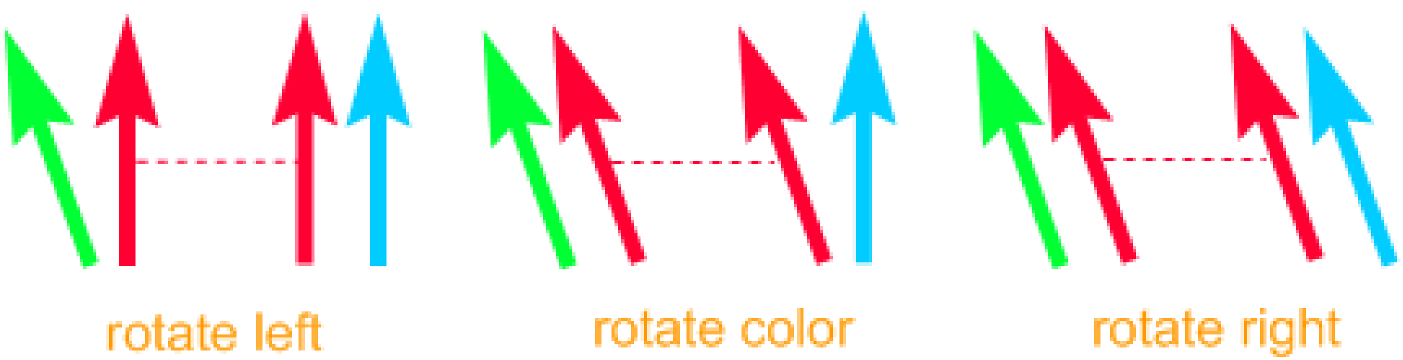
$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \Rightarrow SU(3)_{c+L+R}$$

Why CFL?

$$\langle 0 | \psi^{\alpha}_{iL} \psi^{\beta}_{jL} | 0 \rangle \propto \Delta \varepsilon^{\alpha\beta C} \varepsilon_{ijC}$$

$$\langle 0 | \psi^{\alpha}_{iR} \psi^{\beta}_{jR} | 0 \rangle \propto -\Delta \varepsilon^{\alpha\beta C} \varepsilon_{ijC}$$





What happens going down with μ? If $\mu << m_s$, we get

3 colors and 2 flavors (2SC)

$$\left\langle 0 \middle| \psi^{\alpha}_{iL} \psi^{\beta}_{jL} \middle| 0 \right\rangle = \Delta \varepsilon^{\alpha\beta 3} \varepsilon_{ij}$$

$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \Rightarrow SU(2)_c \otimes SU(2)_L \otimes SU(2)_R$$

However, if μ is in the intermediate region we face a situation with fermions having different Fermi surfaces (see later). Then other phases could be important (LOFF, etc.)

Difficulties with lattice calculations

- Define eucldean variables: $x_0 \to -ix_E^4, \quad x^i \to x_E^i$

  $\gamma_0 \to \gamma_E^4, \quad \gamma^i \to -i\gamma_E^i$

- Dirac operator with chemical potential

$$D(\mu) = \gamma_E^\mu D_E^\mu + \mu\gamma_E^4 \qquad \gamma_5 D(0)\gamma_5 = \text{-}D(0)$$

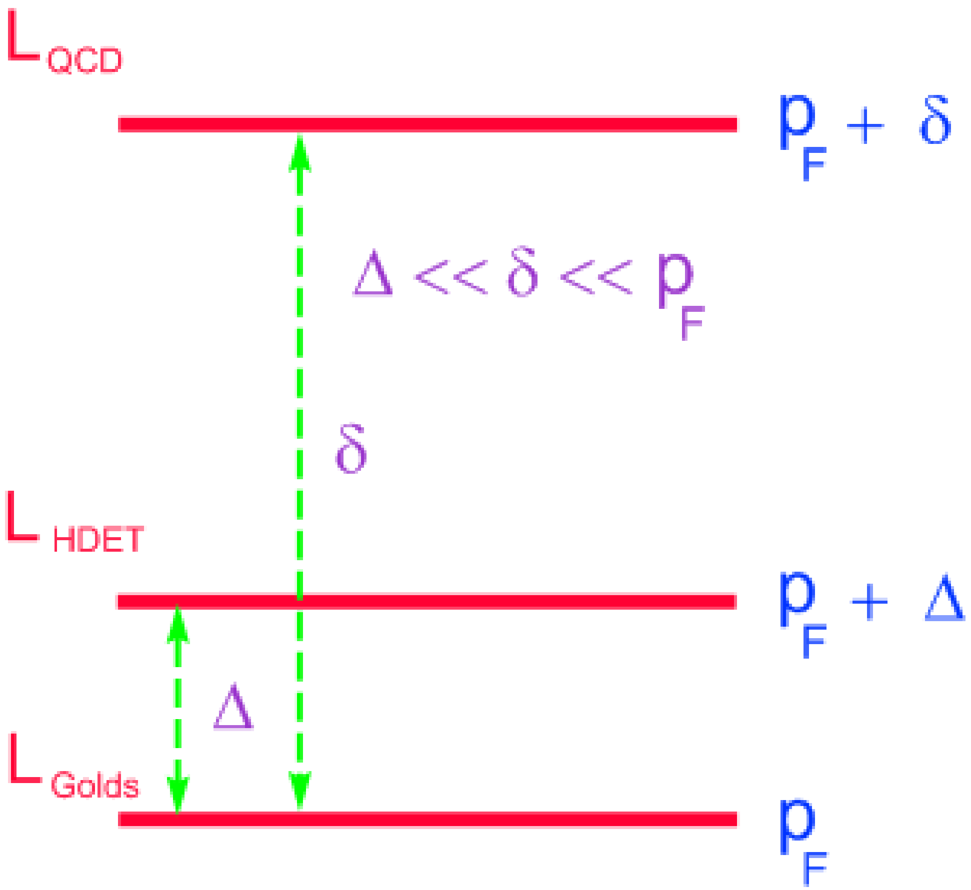
- At $\mu = 0$ $\quad D(0)^\dagger = \text{-}D(0)$
- Eigenvalues of $D(0)$ pure immaginary
- If $|\lambda\rangle$ eigenvector of D(0), $\gamma_5|\lambda\rangle$ eigenvector with eigenvalue $-\lambda$

$$\Rightarrow \quad \det[D(0)] = \prod(\lambda)(-\lambda) > 0$$

For μ **not zero** the argument does not apply and one cannot use the sampling method for evaluating the determinant. However for isospin chemical potential and two degenerate flavors one can still prove the positivity.

For finite baryon density no lattice calculation available except for small μ and close to the critical line (Fodor and Katz)

Hierarchies of effective lagrangians



Integrating out heavy degrees of freedom we have two scales. The gap Δ and a cutoff, δ above which we integrate out. Therefore: two different effective theories, L_{HDET} and L_{Golds}

- L_{HDET} is the effective theory of the fermions close to the Fermi surface. It corresponds to the Polchinski description. The condensation is taken into account by the introduction of a mean field corresponding to a Majorana mass. The d.o.f. are quasi-particles, holes and gauge fields. This holds for energies up to the cutoff.

- L_{Golds} describes the low energy modes ($E \ll \Delta$), as Goldstone bosons, ungapped fermions and holes and massless gauge fields, depending on the breaking scheme.

Effective theory at the Fermi surface (HDET)

Starting point: L_{QCD}

$$L_{QCD} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mu \bar{\psi} \gamma_0 \psi, \quad a = 1, \cdots, 8$$

$$D_\mu = \partial_\mu + i g_s A^a_\mu T^a, \quad \not{D} = \gamma_\mu D^\mu, \quad T^a = \frac{\lambda_a}{2}$$

at asymptotic $\mu >> \Lambda_{QCD}$,

$$(\not{p} + \mu\gamma_0)\psi(p) = 0 \;\Rightarrow\; (p_0 + \mu)\psi(p) = \vec{\alpha} \cdot \vec{p}\psi(p)$$

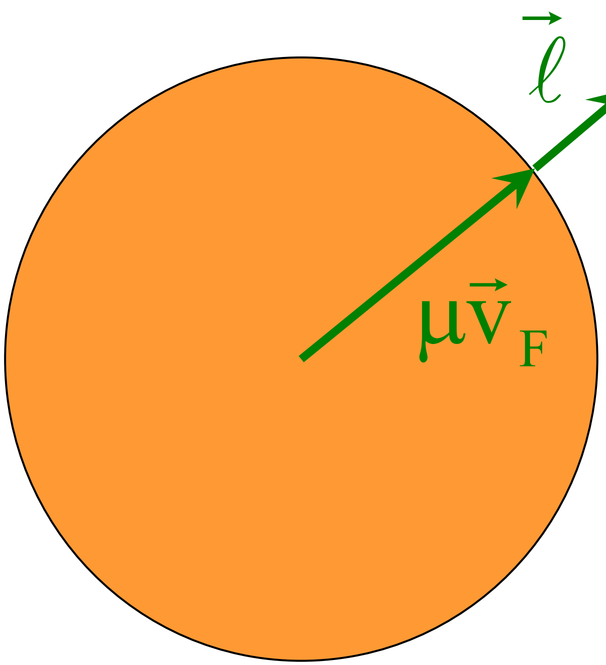
$$\boxed{(p^0 + \mu)^2 = |\vec{p}|^2 \Rightarrow p^0 = E_\pm = -\mu \pm |\vec{p}|}$$

Introduce the projectors:

$$P_{\pm} = \frac{1 \pm \vec{\alpha} \cdot \vec{v}_F}{2}, \quad \vec{v}_F \equiv \vec{v} = \left.\frac{\partial E(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_F} = \left.\frac{\partial |\vec{p}|}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_F} = \hat{\vec{p}}$$

and decomposing: $\vec{p} = \mu \vec{v}_F + \vec{\ell}$

$$\boxed{\begin{aligned} H\psi_+ &= \vec{\alpha} \cdot \vec{\ell}\,\psi_+ \\ H\psi_- &= (-2\mu + \vec{\alpha} \cdot \vec{\ell})\psi_- \end{aligned}}$$



- States ψ_+ close to the FS
- States ψ_- decouple for large μ

Field-theoretical version: $\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\psi(p)$

$$p^\mu = \mu v^\mu + \ell^\mu$$

$$v^\mu = (0, \vec{v}), \left|\vec{v}\right| = 1, \quad \ell^\mu = (\ell^0, \vec{\ell})$$

$$\vec{\ell} = \vec{v}\ell_{\parallel} + \vec{\ell}_{\perp}, \quad \vec{\ell}_{\perp} = \vec{\ell} - (\vec{\ell}\cdot\vec{v})\vec{v}$$

4 d.of. $\boxed{\ell_0, \ell_{\parallel}, \vec{v}}$

Choosing $\vec{v} \parallel \vec{p}$

$$\Downarrow$$

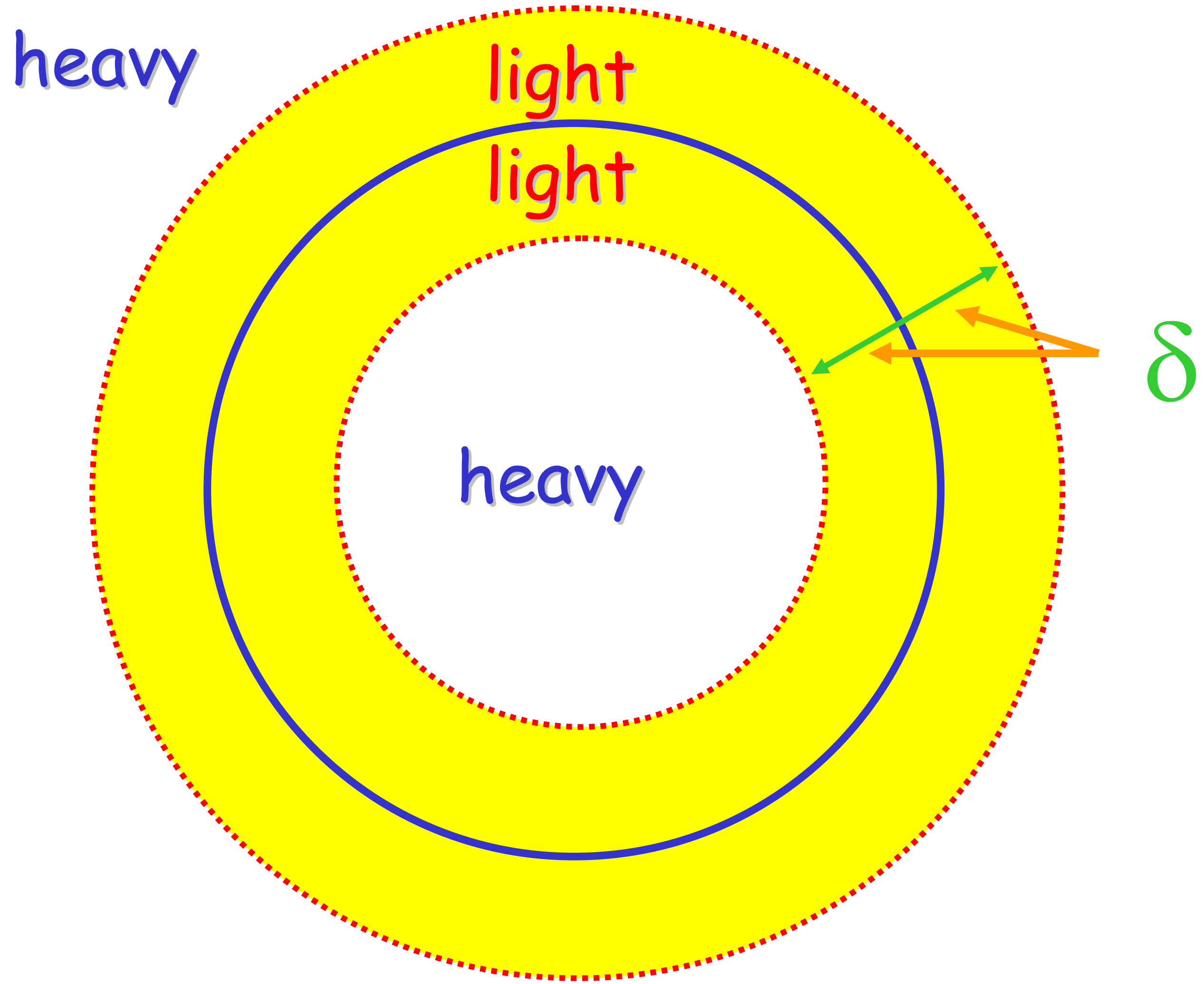
$$\vec{\ell}_{\perp} = 0$$

Separation of light and heavy d.o.f.

light d.o.f. $\mu - \delta \le |\vec{p}\,| \le \mu + \delta, \quad -\delta \le \ell_{\parallel} \le +\delta$

heavy d.o.f. $|\vec{p}\,| \le \mu - \delta, |\vec{p}\,| \ge \mu + \delta, \quad \ell_{\parallel} \le -\delta, \ell_{\parallel} \ge +\delta$

Separation of light and heavy d.o.f.



Momentum integration for the light fields

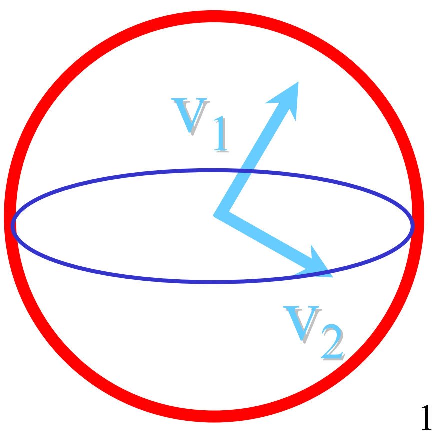
$$\int \frac{d^4p}{(2\pi)^4} \Rightarrow \frac{\mu^2}{(2\pi)^4} \int d\Omega \int_{-\delta}^{+\delta} d\ell_{\|} \int_{-\infty}^{+\infty} d\ell_0 = \int \frac{d\vec{v}}{4\pi} \frac{\mu^2}{\pi} \int \frac{d^2\ell}{(2\pi)^2}$$

The Fourier decomposition becomes

$$\psi(x) = \int \frac{d\vec{v}}{4\pi} e^{-i\mu v\cdot x} \psi_{\vec{v}}(x)$$

$$\psi_{\vec{v}}(x) = \frac{\mu^2}{\pi} \int \frac{d^2\ell}{(2\pi)^2} e^{-i\ell\cdot x} \psi_{\vec{v}}(\ell)$$

$$\left(\psi_{\vec{v}}(\ell) \equiv \psi(p)\right)$$



For any fixed **v, 2-dim theory**

In order to decouple the states corresponding to E_-

Momenta from the Fermi sphere

$$\psi(x) = \int \frac{d\vec{v}}{4\pi} e^{-i\mu v \cdot x} \left[\psi_+(x) + \psi_-(x)\right]$$

$$\psi_\pm(x) \equiv P_\pm \psi_{\vec{v}}(x) = P_\pm \frac{\mu^2}{\pi} \int \frac{d^2\ell}{(2\pi)^2} e^{-i\ell\cdot x} \psi_{\vec{v}}(\ell)$$

substituting inside L_{QCD} and using

$$\int d^4x \psi^\dagger(x)\psi(x) \xrightarrow[\mu\to\infty]{} \frac{\mu^2}{\pi} \int \frac{d\vec{v}}{4\pi} \frac{d^2\ell}{(2\pi)^2} \psi^\dagger_{\vec{v}}(x)\psi_{\vec{v}}(x)$$

$$\int d^4x\,\bar{\psi}(i\not{D}+\mu\gamma_0)\psi \rightarrow \int \frac{d\vec{v}}{4\pi}\Big[\psi_+^\dagger iV\cdot D\psi_+ + \psi_-^\dagger(2\mu + i\tilde{V}\cdot D)\psi_- + (\bar{\psi}_+ i\not{D}_\perp \psi_- + \text{h.c.})\Big]$$

$$V^\mu = (1,\vec{v}), \quad \tilde{V}^\mu = (1,-\vec{v})$$

$$\gamma_{\|}^\mu = (\gamma^0,(\vec{v}\cdot\vec{\gamma})\vec{v}), \qquad \gamma_\perp^\mu = \gamma^\mu - \gamma_{\|}^\mu$$

$$\boxed{\begin{aligned}
\bar{\psi}_+\gamma^\mu\psi_+ &= V^\mu\psi_+^\dagger\psi_+\\
\bar{\psi}_-\gamma^\mu\psi_- &= \tilde{V}^\mu\psi_-^\dagger\psi_-\\
\bar{\psi}_+\gamma^\mu\psi_- &= \bar{\psi}_+\gamma_\perp^\mu\psi_-\\
\bar{\psi}_-\gamma^\mu\psi_+ &= \bar{\psi}_-\gamma_\perp^\mu\psi_+
\end{aligned}}$$

Eqs. of motion:

$$\boxed{\begin{gathered}
iV\cdot D\psi_+ + i\gamma^0\not{D}_\perp\psi_- = 0\\
(2\mu + i\tilde{V}\cdot D)\psi_- + i\gamma^0\not{D}_\perp\psi_+ = 0
\end{gathered}}$$

At the leading order in μ:

$$\boxed{\begin{aligned} iV \cdot D\psi_+ &= 0 \\ \psi_- &= 0 \end{aligned}}$$

At the same order:

$$\boxed{L_D = \int \frac{d\vec{v}}{4\pi} \psi_+^\dagger iV \cdot D\psi_+}$$

Propagator:

$$\boxed{\frac{1}{V \cdot \ell}}$$

$$\frac{1}{\not{p} + \mu\gamma_0} = \frac{(p_0 + \mu)\gamma^0 - \vec{p} \cdot \vec{\gamma}}{(p_0 + \mu)^2 - |\vec{p}|^2} \xrightarrow[\mu \to \infty]{} \frac{\not{V}}{2} \frac{1}{V \cdot \ell} \quad (\langle T(\bar{\psi}\psi) \rangle)$$

$$\frac{\not{V}}{2} = \frac{1}{2}\gamma^0 (1 - \vec{\alpha} \cdot \vec{v}) = P_+ \gamma^0 \quad (\langle T(\psi^\dagger \psi) \rangle)$$

Integrating out the heavy d.o.f.

For the heavy d.o.f. we can formally repeat the same steps leading to:

$$\int d^4x\,\bar{\psi}(i\not{D}+\mu\gamma_0)\psi \to \int \frac{d\vec{v}}{4\pi}\left[\psi_+^\dagger iV\cdot D\psi_+ + \psi_-^\dagger(2\mu + i\tilde{V}\cdot D)\psi_- + (\bar{\psi}_+ i\not{D}_\perp\psi_- + \text{h.c.})\right]$$

Eliminating the E_- fields one would get the non-local lagrangian:

$$L_D = \int \frac{d\vec{v}}{4\pi}\left[\psi_+^\dagger iV\cdot D\psi_+ - P^{\mu\nu}\psi_+^\dagger \frac{1}{2\mu + i\tilde{V}\cdot D} D_\mu D_\nu \psi_+\right]$$

$$P^{\mu\nu} = g^{\mu\nu} - \frac{1}{2}\left[V^\mu\tilde{V}^\nu + V^\nu\tilde{V}^\mu\right]$$

Decomposing $\psi_+ = \psi_+^\ell + \psi_+^h$ one gets

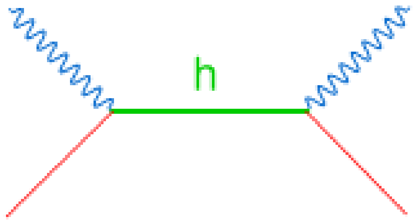
$$L_D = L_D^\ell + L_D^{\ell h} + L_D^h$$

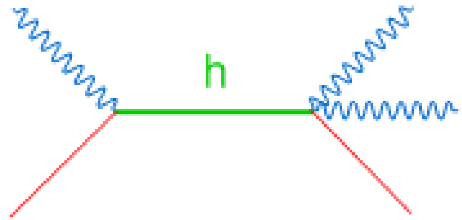
$$L_D^\ell = \int \frac{d\vec{v}}{4\pi} \psi_+^{\ell\dagger} iV \cdot D \psi_+^\ell$$

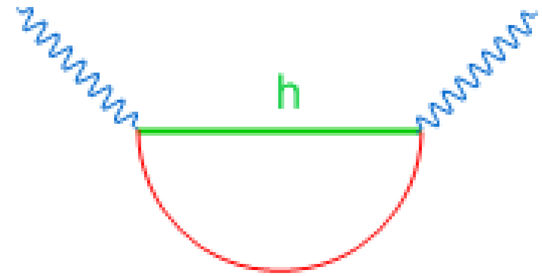
$$L_D^{\ell h} = \int \frac{d\vec{v}}{4\pi} \left(\psi_+^{\ell\dagger} iV \cdot D \psi_+^h - P^{\mu\nu} \psi_+^{\ell\dagger} \frac{1}{2\mu + i\tilde{V} \cdot D} D_\mu D_\nu \psi_+^h + \ell \leftrightarrow h \right)$$

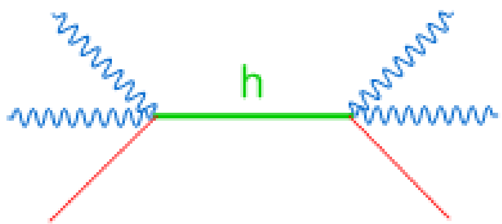
$$L_D^h = \int \frac{d\vec{v}}{4\pi} \left(\psi_+^{h\dagger} iV \cdot D \psi_+^h - P^{\mu\nu} \psi_+^{h\dagger} \frac{1}{2\mu + i\tilde{V} \cdot D} D_\mu D_\nu \psi_+^h \right)$$

When integrating out the heavy fields

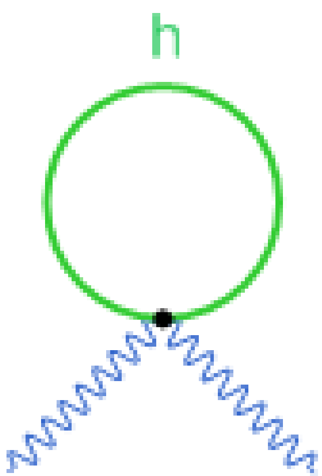








Contribute only if some gluons are hard, but suppressed by asymptotic freedom



This contribution from L^h_D gives the bare Meissner mass

HDET in the condensed phase

Assume $\langle \psi^A C \psi^B \rangle = \Delta_{AB}$ (A, B collective indices)

due to the attractive interaction:

$$L_I = -\frac{G}{4} \varepsilon_{ab} \varepsilon_{\dot{a}\dot{b}} V_{ABCD} \psi_a^A \psi_b^B \psi_{\dot{a}}^{C\dagger} \psi_{\dot{b}}^{D\dagger}$$

$$V_{ABCD} = V_{CDAB}^*, \quad V_{ABCD} = V_{BACD} = V_{ABDC}$$

Decompose →

$$L_I = L_{cond} + L_{int}$$

$$L_{cond} = \frac{G}{4} V_{ABCD} \Gamma^{CD*} \psi^{AT} C \psi^{B} - \frac{G}{4} V_{ABCD} \Gamma^{AB} \psi^{C\dagger} C \psi^{D*}$$

$$L_{int} = -\frac{G}{4} V_{ABCD} \left(\psi^{AT} C \psi^{B} - \Gamma^{AB} \right) \left(\psi^{C\dagger} C \psi^{D*} + \Gamma^{CD*} \right)$$

We define $\Delta_{AB} = \frac{G}{2} V_{CDAB} \Gamma^{CD}, \quad \Delta^{*}_{AB} = \frac{G}{2} V^{*}_{CDAB} \Gamma^{CD*}$

$$\boxed{L_{cond} = \frac{1}{2} \Delta^{*}_{AB} \psi^{AT} C \psi^{B} - \frac{1}{2} \Delta_{AB} \psi^{A\dagger} C \psi^{B*}}$$

and neglect L_{int}. Therefore

$$L_D = \int \frac{d\vec{v}}{4\pi}\frac{1}{2}\sum_{AB}\left[\psi_+^{A\dagger} iV\cdot D\psi_+^B + \psi_-^{A\dagger} iV\cdot D\psi_-^B + \Delta_{AB}^* \psi_-^{AT} C\psi_+^B - \Delta_{AB}\psi_+^{A\dagger} C\psi_-^B\right]$$

$$\psi_\pm(x) \equiv \psi_+(\pm\vec{v}, x)$$

Nambu-Gor'kov basis

$$\chi^A = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi_+^A \\ C\psi_-^{A*}\end{pmatrix}$$

$$\boxed{L_D = \int \frac{d\vec{v}}{4\pi}\chi^{A\dagger}\begin{bmatrix} iV\cdot D_{AB} & -\Delta_{AB} \\ -\Delta_{AB}^* & i\tilde{V}\cdot D_{AB}\end{bmatrix}\chi^B}$$

$$\boxed{S(\ell) = \frac{1}{(V\cdot\ell)(\tilde{V}\cdot\ell) - \Delta\Delta^\dagger}\begin{bmatrix}\tilde{V}\cdot\ell & \Delta \\ \Delta^\dagger & V\cdot\ell\end{bmatrix}} \quad \left(\left[\Delta, \Delta^\dagger\right] = 0\right)$$

From the definition: $$\Delta^*_{AB} = -\frac{G}{2} V_{ABCD} \left\langle \psi^{C\dagger} C \psi^{D*} \right\rangle$$

one derives the gap equation (e.g. via functional formalism)

$$\boxed{\Delta^*_{AB} = iGV_{ABCD} \int \frac{d\vec{v}}{4\pi} \frac{\mu^2}{\pi} \int \frac{d^2\ell}{(2\pi)^2} \Delta^*_{CE} \frac{1}{D_{ED}}}$$

$$\frac{1}{D_{AB}} = \left(\frac{1}{(V \cdot \ell)(\tilde{V} \cdot \ell) - \Delta\Delta^\dagger} \right)_{AB}$$

Four-fermi interaction one-gluon exchange inspired

$$\boxed{L_I = \frac{3}{16} G \bar{\psi}\gamma_\mu \lambda^a \psi \bar{\psi}\gamma^\mu \lambda^a \psi}$$

Fierz using:

$$\sum_{a=1}^{8} (\lambda^a)_{\alpha\beta} (\lambda^a)_{\delta\gamma} = \frac{2}{3}(3\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta})$$

$$(\sigma_\mu)_{\dot{a}b} (\tilde{\sigma}^\mu)_{d\dot{c}} = 2\varepsilon_{\dot{a}\dot{c}}\varepsilon_{bd}$$

$$\sigma^\mu = (1, \vec{\sigma}), \quad \tilde{\sigma}^\mu = (1, -\vec{\sigma})$$

$$L_I = -\frac{G}{4} V_{(\alpha i)(\beta j)(\gamma k)(\delta \ell)} \psi_i^\alpha \psi_j^\beta \psi_k^{\gamma\dagger} \psi_\ell^{\delta\dagger}$$

$$V_{(\alpha i)(\beta j)(\gamma k)(\delta \ell)} = -(3\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta})\delta_{ik}\delta_{j\ell}$$

In the 2SC case

$$\Delta_{(\alpha i)(\beta j)} = \varepsilon_{\alpha\beta 3}\varepsilon_{ij}\Delta$$

$$\Delta = 4iG\int\frac{d\vec{v}}{4\pi}\frac{\mu^2}{\pi}\int\frac{d^2\ell}{(2\pi)^2}\frac{\Delta}{\ell_0^2-\ell_\parallel^2-\Delta^2}$$

$$\boxed{\Delta = \frac{G}{2}\rho\int_0^\delta d\xi\frac{\Delta}{\sqrt{\xi^2+\Delta^2}}, \qquad \rho = 4\frac{\mu^2}{\pi^2}}$$

4 pairing fermions

G determined at $T = 0$. M, constituent mass ~ 400 MeV

$$\boxed{1 = 8G\int_0^{\Lambda} \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}|^2 + M^2}}}$$

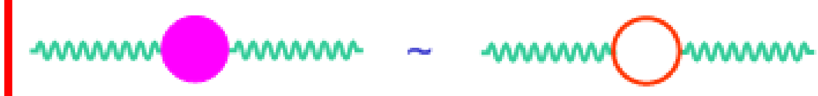
with $\Lambda = \mu + \delta$

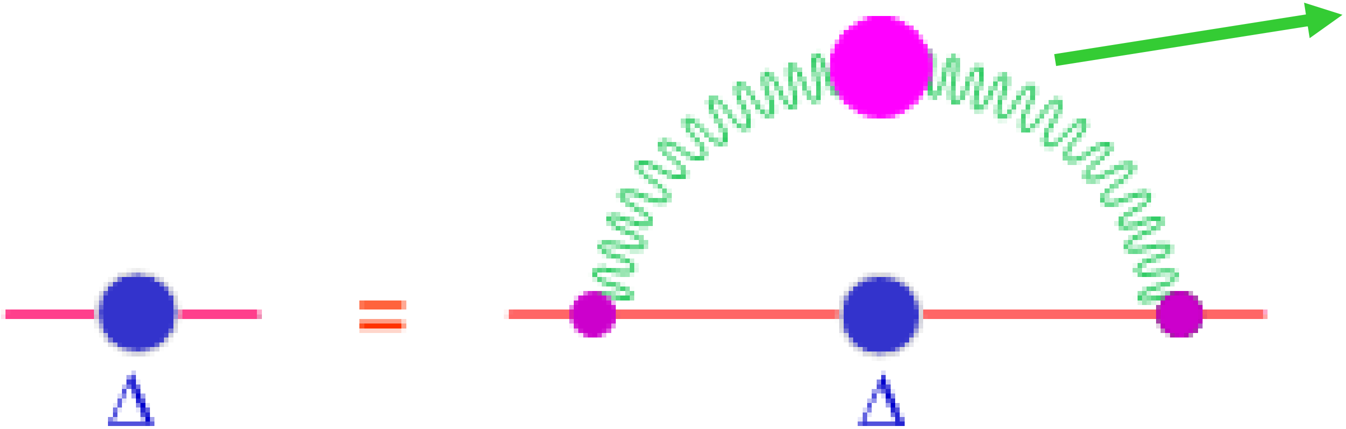
For $\mu = 400 - 500\ \text{MeV}, \quad \Lambda = 800\ \text{MeV}, \quad M = 400\ \text{MeV}$

$$\boxed{\Delta_{2SC} = 33 - 88\ \text{MeV}}$$

Similar values for CFL.

Gap equation in QCD





$$\Delta(p_0) = \frac{g^2}{12\pi^2}\int dq_0 \int d(\cos\theta)\left(\frac{\frac{3}{2}-\frac{1}{2}\cos\theta}{1-\cos\theta+G/(2\mu^2)} + \frac{\frac{1}{2}+\frac{1}{2}\cos\theta}{1-\cos\theta+F/(2\mu^2)}\right)\frac{\Delta(q_0)}{\sqrt{q_0^2+\Delta(q_0)^2}}$$

Hard-loop approximation $q_0 \ll |\vec{q}| \to 0$

$$\boxed{F = m_D^2}$$

electric

$$\boxed{G = \frac{\pi}{4} m_D^2 \frac{q_0}{|\vec{q}|}}$$

magnetic

$$m_D^2 = N_f \frac{g^2\mu^2}{2\pi^2}$$

For small momenta magnetic gluons are unscreened and dominate giving a further logarithmic divergence

$$b = 256\pi^4 \left(\frac{2}{N_f}\right)^{5/2} g^{-5}$$

$$\Delta(p_0) = \frac{g^2}{18\pi^2} \int dq_0 \log\left(\frac{b\mu}{|p_0 - q_0|}\right) \frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}$$

Results:

$$\boxed{\begin{gathered}\Delta(p_0) \approx \Delta_0 \sin\left(\frac{g}{3\sqrt{2}\pi}\log\left(\frac{b}{p_0}\right)\right) \\ \Delta_0 = 2b\mu e^{-\frac{3\pi^2}{\sqrt{2}g}}\end{gathered}}$$

$$\left(1 \approx \frac{g^2}{c}(\log(\delta/\Delta))^2 \to \Delta \approx \delta e^{-c/g}\right)$$

from the double log

To be trusted only for $\mu > 10^5\,\mathrm{GeV}$ but, if extrapolated at 400-500 MeV, gives values for the gap similar to the ones found using a 4-fermi interaction.

However condensation arises at asymptotic values of μ.

Symmetries of superconducting phases

Consider again the 3 flavors, u,d,s and the group theoretical structure of the two difermion condensate:

$$\left\langle \psi^{\alpha}_{ia} \psi^{\beta}_{jb} \right\rangle$$

$$[(3_c, 3_{L(R)}) \otimes (3_c, 3_{L(R)})]_S = (3^*_c, 3^*_{L(R)}) \oplus (6_c, 6_{L(R)})$$

implying in general

$$\left\langle \psi^{\alpha}_{iL}\psi^{\beta}_{jL}\right\rangle = \Delta\varepsilon^{\alpha\beta I}\varepsilon_{ijI} + \Delta_6(\delta^{\alpha}_{i}\delta^{\beta}_{j} + \delta^{\beta}_{i}\delta^{\alpha}_{j}) =$$

$$= \Delta(\delta^{\alpha}_{i}\delta^{\beta}_{j} - \delta^{\beta}_{i}\delta^{\alpha}_{j}) + \Delta_6(\delta^{\alpha}_{i}\delta^{\beta}_{j} + \delta^{\beta}_{i}\delta^{\alpha}_{j}) =$$

$$= (\Delta + \Delta_6)\delta^{\alpha}_{i}\delta^{\beta}_{j} + (\Delta_6 - \Delta)\delta^{\beta}_{i}\delta^{\alpha}_{j}$$

In NJL case with cutoff 800 MeV, constituent mass 400 MeV and $\mu = 400$ MeV

$$\boxed{\Delta = 85.3\,\mathrm{MeV}, \quad \Delta_6 = -1.3\,\mathrm{MeV}}$$

Original symmetry:

$$G_{QCD} = SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \otimes \underbrace{U(1)_A}_{\text{anomalous}}$$

broken to

$$G_{CFL} = SU(3)_{c+L+R} \otimes Z_2 \otimes \underbrace{Z_2}_{\text{anomalous}}$$

\# Goldstones $3 \times 8 + 1 + 1 - 8 = 8 + 8 + 1 + 1 = 17 + \underbrace{1}_{\text{massive}}$

8 give mass to the gluons and 8+1 are true massless Goldstone bosons

Notice:

- Breaking of $U(1)_B$ makes the CFL phase superfluid
- The CFL condensate is not gauge invariant, but consider

$$X^k_\gamma = \varepsilon_{\alpha\beta\gamma}\varepsilon^{ijk}(\psi^\alpha_{iL}\psi^\beta_{jL})^\dagger, \quad Y^k_\gamma = \varepsilon_{\alpha\beta\gamma}\varepsilon^{ijk}(\psi^\alpha_{iR}\psi^\beta_{jR})^\dagger$$

$$\Sigma^i_j = (Y^\dagger X)^i_j = \sum_\alpha (Y^j_\alpha)^* X^i_\alpha$$

Σ is gauge invariant and breaks the global part of G_{QCD}. Also

$$\det(X), \quad \det(Y) \quad \text{break} \quad U(1)_B \otimes U(1)_A \Rightarrow Z_2 \otimes Z_2$$

$U(1)_A$ is broken by the anomaly but induced by a 6-fermion operator irrelevant at the Fermi surface. Its contribution is parametrically small and we expect a very light NGB (massless at infinite chemical potential)

Spectrum of the CFL phase

Choose the basis: $$\psi_i^\alpha = \frac{1}{\sqrt{2}} \sum_{A=1}^{9} (\lambda_A)_i^\alpha \psi^A$$

$\lambda_A \quad A = 1, \cdots, 8 \quad \text{Gell} - \text{Mann matrices}$

$$\lambda_9 = \lambda_0 = \sqrt{\frac{2}{3}} \times 1$$

$$\mathrm{Tr}\left[\lambda_A \lambda_B\right] = 2\delta_{AB}$$

Inverting: $$\psi^A = \frac{1}{\sqrt{2}}\sum_{\alpha i}(\lambda_A)^i_\alpha \psi^\alpha_i = \frac{1}{\sqrt{2}}\mathrm{Tr}[\lambda_A \psi]$$

$$\langle \psi^A \psi^B \rangle = \frac{1}{2}\sum (\lambda_A)^i_\alpha (\lambda_B)^j_\beta \Delta \varepsilon^{\alpha\beta I}\varepsilon_{ijI} = \frac{\Delta}{2}\mathrm{Tr}\left[\sum_I \left(\lambda_A \varepsilon_I \lambda_B^T \varepsilon_I\right)\right]$$

$$(\varepsilon_I)_{\alpha\beta} = \varepsilon_{\alpha\beta I}$$

$$\sum_I \varepsilon_I g^T \varepsilon_I = g - \mathrm{Tr}[g]$$ **for any 3x3 matrix g**

We get $$\langle \psi^A \psi^B \rangle = \Delta_A \delta_{AB} \qquad \Delta_A = \begin{cases} A=1,\cdots,8 & \Delta_A = \Delta \\ A = 9 & \Delta_9 = -2\Delta \end{cases}$$

$\boxed{\text{quasi-fermions}}$

$$\boxed{\varepsilon_A(\vec{p}) = \sqrt{(\vec{v}\cdot\vec{\ell})^2 + \Delta_A^2}}$$

gluons

Expected $m_g^2 \approx g^2 F^2$

NG coupling constant

but wave function renormalization effects important (see later)

NG bosons

Acquire mass through quark masses except for the one related to the breaking of $U(1)_B$

NG boson masses quadratic in m_q since the approximate invariance

$$(Z_2)_L \otimes (Z_2)_R : \quad \psi_{L(R)} \to -\psi_{L(R)}$$

and quark mass term: $\bar{\psi}_L M \psi_R + \mathrm{h.c.}$

$$M \to -M$$

Notice: anomaly breaks $(Z_2)_L \otimes (Z_2)_R$ through instantons, producing a chiral condensate (6 fermions -> 2), but of order $(\Lambda_{QCD}/\mu)^8$

In-medium electric charge

$$D_\mu \psi = \partial_\mu \psi - i g_\mu^a T_a \psi - i \psi Q A_\mu$$

The condensate breaks $U(1)_{em}$ but leaves invariant a combination of Q and T_8.

CFL vacuum: $X_\alpha^i = Y_\alpha^i = \delta_\alpha^i$

Define: $Q_{SU(3)_c} \equiv -\frac{2}{\sqrt{3}} T_8 = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, +\frac{2}{3}\right) = Q$

$$\tilde{Q} = Q_{SU(3)_c} \otimes 1 - 1 \otimes Q = Q \otimes 1 - 1 \otimes Q$$

leaves invariant the ground state

$$Q\langle X \rangle - \langle X \rangle Q \rightarrow Q_{\alpha\beta}\delta_{\beta i} - \delta_{\alpha j} Q_{ji} = 0$$

$\mathrm{Eigs}(\tilde{Q}) = 0, \pm 1$ Integers as in the Han-Nambu model

$$A_\mu = \tilde{A}_\mu \cos\theta - \tilde{G}_\mu \sin\theta$$

$$g_\mu^8 = \underbrace{\tilde{A}_\mu \sin\theta + \tilde{G}_\mu}_{\text{rotated fields}} \cos\theta$$

new interaction:

$$g_s g_\mu^8 T_8 \otimes 1 + e A_\mu 1 \otimes Q \to \tilde{e}\tilde{Q}\tilde{A}_\mu + \tilde{g}_s \tilde{G}_\mu \tilde{T}$$

$$\tan\theta = \frac{2}{\sqrt{3}} \frac{e}{g_s}, \tilde{e} = e\cos\theta, \tilde{g}_s = \frac{g_s}{\cos\theta}$$

$$\tilde{T} = -\frac{\sqrt{3}}{2}\left[\left(\cos^2\theta\right) Q \otimes 1 + \left(\sin^2\theta\right) 1 \otimes Q\right]$$

The rotated “photon” remains massless,
whereas the rotated gluons acquires a mass through the Meissner effect

A piece of CFL material for massless quarks would respond to an em field only through NGB:

“bosonic metal”

For quarks with equal masses, no light modes:

“transparent insulator”

For different masses one needs non zero density of electrons or a kaon condensate leading to massless excitations

In the 2SC case, new $\mathbf{Q}$ and $\mathbf{B}$ are conserved

$$\tilde{Q} = Q \otimes 1 - \frac{1}{\sqrt{3}} 1 \otimes T_8 = \left(\frac{2}{3}, -\frac{1}{3} \right) \otimes 1 - 1 \otimes \frac{1}{6} (1, 2, -2)$$

$$\tilde{B} = B - \frac{2}{\sqrt{3}} T_8 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) - \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right) = (0, 0, 1)$$

	$\tilde{Q}$	$\tilde{B}$
$u^{\alpha}, \alpha = 1, 2$	1/2	0
$d^{\alpha}, \alpha = 1, 2$	-1/2	0
u^{3}	1	1
d^{3}	0	1

Spectrum of the 2SC phase

Remember $\Delta_{(\alpha i)(\beta j)} = \varepsilon_{\alpha\beta 3}\varepsilon_{ij}\Delta$

No Goldstone bosons

$\psi_i^\alpha \quad \alpha = 1,2 \quad \text{gapped}$ (equal gap)

$\psi_i^3 \quad \text{ungapped}$

$$SU(3)_c \rightarrow SU(2)_c$$

$$\Downarrow$$

$$8 - 3 = 5 \quad \text{massive gluons}$$

Light modes:

$$\boxed{\psi_i^3 + 3 \text{ gluons } (M = 0)}$$

2+1 flavors

It could happen $\mu \approx m_s, \quad \mu \gg m_u, m_d$

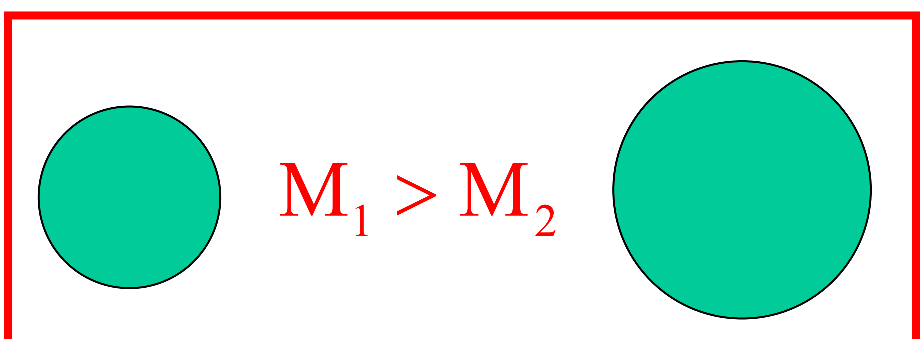
$$\mu \gg m_u, m_d, m_s$$

$$\Downarrow$$

Phase transition expected

$$m_u, m_d < \mu < m_s$$

$$E_F = \mu = \sqrt{p_F^2 + M^2} \rightarrow p_F = \sqrt{\mu^2 - M^2}$$



The radius of the Fermi sphere decrases with the mass

Simple model: $p_{F_1} = \sqrt{\mu^2 - m_s^2}\,, \quad p_{F_2} = \mu$

$$\Omega_{unpair} = 2\int_0^{p_{F_1}} \frac{d^3\vec{p}}{(2\pi)^3}\left(\sqrt{\vec{p}^{\,2} + m_s^2} - \mu\right) + 2\int_0^{p_{F_2}} \frac{d^3\vec{p}}{(2\pi)^3}\left(|\vec{p}| - \mu\right)$$

$$\Omega_{pair} = 2\int_0^{p_{F_{comm}}} \frac{d^3\vec{p}}{(2\pi)^3}\left(\sqrt{\vec{p}^{\,2} + m_s^2} - \mu\right) + 2\int_0^{p_{F_{comm}}} \frac{d^3\vec{p}}{(2\pi)^3}\left(|\vec{p}| - \mu\right) - \underbrace{\frac{\mu^2\Delta^2}{4\pi^2}}_{\text{condensation energy}}$$

$$\frac{\partial \Omega_{pair}}{\partial p_{F_{comm}}} = 0 \quad \rightarrow \quad p_{F_{comm}} = \mu - \frac{m_s^2}{4\mu}$$

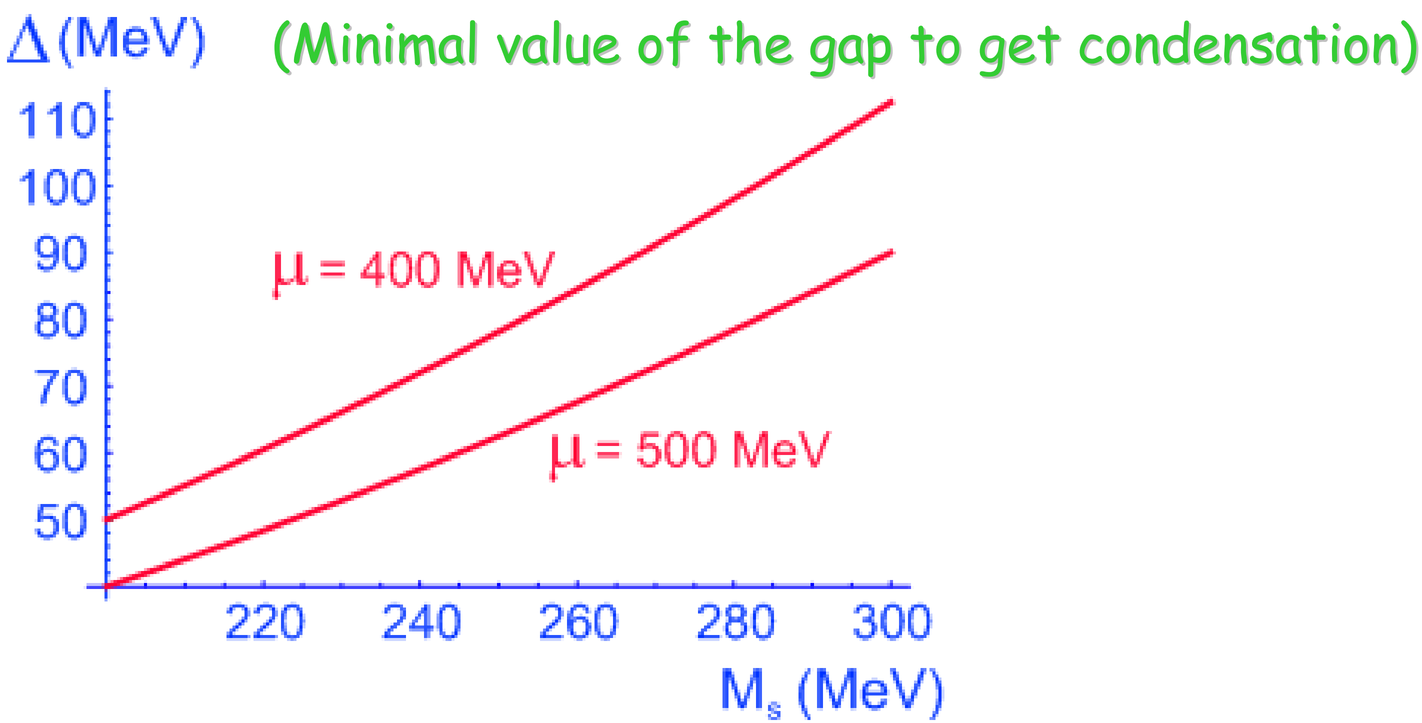
$$\boxed{\Omega_{pair} - \Omega_{unpair} \approx \frac{1}{16\pi^2}\left(m_s^4 - 4\Delta^2\mu^2\right)}$$

Condensation if:

$$\boxed{\mu > \frac{m_s^2}{2\Delta}}$$

Notice that the transition must be first order because for being in the pairing phase

$$\Delta > \frac{m_s^2}{2\mu} \neq 0$$



Effective lagrangians

- Effective lagrangian for the CFL phase
- Effective lagrangian for the 2SC phase

Effective lagrangian for the CFL phase

NG fields as the phases of the condensates in the $(\bar{3},\bar{3})$ representation

$$X_\gamma^k = \varepsilon_{\alpha\beta\gamma}\varepsilon^{ijk}\left\langle \psi_{iL}^\alpha \psi_{jL}^\beta \right\rangle^*, \quad Y_\gamma^k = \varepsilon_{\alpha\beta\gamma}\varepsilon^{ijk}\left\langle \psi_{iR}^\alpha \psi_{jR}^\beta \right\rangle^*$$

Quarks and X, Y transform as

$$\boxed{\begin{gathered}\psi_L \to e^{i(\alpha+\beta)} g_c \psi_L g_L^T, \quad \psi_R \to e^{i(\alpha-\beta)} g_c \psi_R g_R^T \\ g_c \in SU(3)_c,\, g_{L,R} \in SU(3)_{L,R},\, e^{i\alpha} \in U(1)_B,\, e^{i\beta} \in U(1)_A\end{gathered}}$$

$$\boxed{X \to g_c X g_L^T e^{-2i(\alpha+\beta)}, \quad Y \to g_c Y g_R^T e^{-2i(\alpha-\beta)}}$$

Since $X, Y \in U(3)$

The number of NG fields is

$$\#X + \#Y = (1+8) + (1+8) = 18$$

8 of these fields give mass to the gluons. There are only 10 physical NG bosons corresponding to the breaking of the global symmetry (we consider also the NGB associated to $U(1)_A$)

$$SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes U(1)_A$$

$$\Downarrow$$

$$SU(3)_{L+R} \otimes Z_2 \otimes Z_2$$

Better use fields belonging to SU(3). Define

$$X = \hat{X} e^{2i(\varphi+\theta)} \qquad Y = \hat{Y} e^{2i(\varphi-\theta)}$$

and

$$d_X = \det(X) = e^{6i(\varphi+\theta)} \qquad d_Y = \det(Y) = e^{6i(\varphi-\theta)}$$

transforming as

$$\hat{X} \to g_c \hat{X} g_L^T \qquad \hat{Y} \to g_c \hat{Y} g_R^T$$

$$\varphi \to \varphi - \alpha \qquad \theta \to \theta - \beta$$

The breaking of the global symmetry can be described by the gauge invariant order parameters

$$\boxed{\Sigma_j^i = (\hat{Y}_\alpha^j)^* \hat{X}_\alpha^i \to \Sigma = \hat{Y}^+ \hat{X}, \quad d_X, \quad d_Y}$$

Σ, d_X, d_Y are 10 fields describing the physical NG bosons. Also

$$\Sigma \to g_R^* \Sigma g_L^T$$

shows that Σ^T transforms as the usual chiral field.

Consider the currents:

$$J_X^\mu = \hat{X} D^\mu \hat{X}^\dagger = \hat{X}(\partial^\mu \hat{X}^\dagger + \hat{X}^\dagger g^\mu) = \hat{X}\partial^\mu \hat{X}^\dagger + g^\mu$$

$$J_Y^\mu = \hat{Y} D^\mu \hat{Y}^\dagger = \hat{Y}(\partial^\mu \hat{Y}^\dagger + \hat{Y}^\dagger g^\mu) = \hat{Y}\partial^\mu \hat{Y}^\dagger + g^\mu$$

$$g^\mu = i g_s g_a^\mu T^a$$

$$\boxed{J_{X,Y}^\mu \to g_c J_{X,Y}^\mu g_c^\dagger}$$

Most general lagrangian up to two derivatives invariant under G, the space rotation group O(3) and Parity (R.C. & Gatto 1999)

$$P: \quad (X \leftrightarrow Y, \varphi \leftrightarrow \varphi, \theta \leftrightarrow -\theta)$$

$$L = -\frac{F_T^2}{4}\mathrm{Tr}\left[\left(J_X^0 - J_Y^0\right)^2\right] - \alpha_T \frac{F_T^2}{4}\mathrm{Tr}\left[\left(J_X^0 + J_Y^0\right)^2\right] + \frac{1}{2}(\partial_0\varphi)^2 + \frac{1}{2}(\partial_0\theta)^2 +$$

$$+\frac{F_S^2}{4}\mathrm{Tr}\left[\left(\vec{J}_X - \vec{J}_Y\right)^2\right] + \alpha_S \frac{F_S^2}{4}\mathrm{Tr}\left[\left(J_X^0 + J_Y^0\right)^2\right] - \frac{v_\varphi^2}{2}\left|\vec{\nabla}\varphi\right|^2 - \frac{v_\theta^2}{2}\left|\vec{\nabla}\theta\right|^2$$

$$\boxed{\begin{aligned} L = &-\frac{F_T^2}{4}\mathrm{Tr}\left[\left(\hat{X}\partial_0\hat{X}^\dagger - \hat{Y}\partial_0\hat{Y}^\dagger\right)^2\right] - \alpha_T \frac{F_T^2}{4}\mathrm{Tr}\left[\left(\hat{X}\partial_0\hat{X}^\dagger + \hat{Y}\partial_0\hat{Y}^\dagger + 2g_0\right)^2\right] + \\ &+\frac{F_S^2}{4}\mathrm{Tr}\left[\left(\hat{X}\vec{\nabla}\hat{X}^\dagger - \hat{Y}\vec{\nabla}\hat{Y}^\dagger\right)^2\right] + \alpha_S \frac{F_S^2}{4}\mathrm{Tr}\left[\left(\hat{X}\vec{\nabla}\hat{X}^\dagger + \hat{Y}\vec{\nabla}\hat{Y}^\dagger + 2\vec{g}\right)^2\right] + \\ &+\frac{1}{2}(\partial_0\varphi)^2 + \frac{1}{2}(\partial_0\theta)^2 - \frac{v_\varphi^2}{2}\left|\vec{\nabla}\varphi\right|^2 - \frac{v_\theta^2}{2}\left|\vec{\nabla}\theta\right|^2 \end{aligned}}$$

Using $SU(3)_c$ gauge invariance we can choose:

$$\hat{X} = \hat{Y}^{\dagger} = e^{i\Pi^a T^a / F_T}$$

Expanding at the second order in the fields

$$L \approx \frac{1}{2}\left(\partial_0 \Pi^a\right)^2 + \frac{1}{2}\left(\partial_0 \varphi\right)^2 + \frac{1}{2}\left(\partial_0 \theta\right)^2 - \frac{v^2}{2}\left|\vec{\nabla}\Pi^a\right|^2 - \frac{v_\varphi^2}{2}\left|\vec{\nabla}\varphi\right|^2 - \frac{v_\theta^2}{2}\left|\vec{\nabla}\theta\right|^2$$

$$v^2 = \frac{F_S^2}{F_T^2}$$

Gluons acquire Debye and Meissner masses (not the rest masses, see later)

$$m_D^2 = \alpha_T g_s^2 F_T^2, \quad m_s^2 = \alpha_S g_s^2 F_S^2 = \alpha_S v^2 g_s^2 F_T^2$$

Low energy theory supposed to be valid at energies « gap. Since we will see that gluons have masses of order Δ they can be decoupled

$$L = -\frac{F_T^2}{4}\mathrm{Tr}\left[\left(\hat{X}\partial_0\hat{X}^\dagger - \hat{Y}\partial_0\hat{Y}^\dagger\right)^2\right] - \alpha_T\frac{F_T^2}{4}\mathrm{Tr}\left[\left(\hat{X}\partial_0\hat{X}^\dagger + \hat{Y}\partial_0\hat{Y}^\dagger + 2g_0\right)^2\right] +$$

$$+\frac{F_S^2}{4}\mathrm{Tr}\left[\left(\hat{X}\vec{\nabla}\hat{X}^\dagger - \hat{Y}\vec{\nabla}\hat{Y}^\dagger\right)^2\right] + \alpha_S\frac{F_S^2}{4}\mathrm{Tr}\left[\left(\hat{X}\vec{\nabla}\hat{X}^\dagger + \hat{Y}\vec{\nabla}\hat{Y}^\dagger + 2\vec{g}\right)^2\right] +$$

$$+\frac{1}{2}(\partial_0\varphi)^2 + \frac{1}{2}(\partial_0\theta)^2 - \frac{v_\varphi^2}{2}\left|\vec{\nabla}\varphi\right|^2 - \frac{v_\theta^2}{2}\left|\vec{\nabla}\theta\right|^2$$

$$\boxed{g_\mu = -\frac{1}{2}\left(\hat{X}\partial_\mu\hat{X}^\dagger + \hat{Y}\partial_\mu\hat{Y}^\dagger\right)}$$

we get the gauge invariant result:

$$L_{NGB} = \overbrace{\frac{F_T^2}{4}\left(\mathrm{Tr}\left[\partial_t\Sigma\partial_t\Sigma^+\right] - v^2\mathrm{Tr}\left[\vec{\nabla}\Sigma\vec{\nabla}\Sigma^+\right]\right)}^{\sim\ \chi\text{-lagrangian}} -$$

$$-\frac{1}{2}\left((\partial_t\varphi)^2 - v_\varphi^2\,|\vec{\nabla}\varphi|^2\right) - \frac{1}{2}\left((\partial_t\theta)^2 - v_\theta^2\,|\vec{\nabla}\theta|^2\right)$$

$$\Sigma = \hat{Y}^+\hat{X}$$

Effective lagrangian for the 2SC phase

Light modes: 2 ungapped fermions and 3 massless gluons of $SU(2)_c$. Consider gluons:

- Gauge invariance: $E_i^a = F_{oi}^a, \quad B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a$
- O(3) + Parity $$\boxed{L = \frac{1}{g^2}\sum_{a=1}^{3}\left(\frac{\varepsilon}{2}\vec{E}^a\cdot\vec{E}^a - \frac{1}{2\lambda}\vec{B}^a\cdot\vec{B}^a\right)}$$
- Gluon velocity $$\boxed{v = \frac{1}{\sqrt{\varepsilon\lambda}}}$$