Path-Integrals 98

From peV to TeV

A theory of algebraic integration

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Motivations

Classical phase space

$$\Big\downarrow{\scriptstyle QM}$$

Noncommutative algebra

Classical phase space $\longrightarrow$ path-integral
SUSY $\longrightarrow$ nonclassical phase space

$$S = -\int \sqrt{dx^2} \rightarrow S = -\int \sqrt{\omega^2}$$

$$\omega_\mu = dx_\mu - i\bar{\theta}\sigma_\mu d\theta + i d\bar{\theta}\sigma_\mu \theta$$

where θ is a Grassmann spinor. This is quantized via the path integral by using the Berezin integration rules

$$\int d\theta\ \theta = 1, \qquad \int d\theta\ 1 = 0$$

What is the motivation for this rule ?

The standard argument is translational invariance. Because this implies the Schwinger's quantum action principle:

$$0 = \int_{q_i,t_i}^{q_f,t_f} d\mu[q] \frac{\delta}{\delta q(t)} \left(F[q] e^{iS[q]} \right)$$

$$\downarrow$$

$$\langle q_f, t_f | T \left(F[q] \frac{\delta S}{\delta q(t)} \right) | q_i, t_i \rangle = i \langle q_f, t_f | T \left(\frac{\delta F[q]}{\delta q(t)} \right) | q_i, t_i \rangle$$

But this is not very well justified, when looking at possible generalizations as, for instance, quantum groups, M-theory, etc. A more fundamental property seems to us the *combination law for the probability amplitudes*. In the path-integral approach this follows from the factorization property of the measure, implying ($t_i \leq t \leq t_f$)

$$\langle q_f, t_f | q_i, t_i \rangle = \int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle$$

equivalent to the *completeness relation*

$$\int dq | q, t \rangle \langle q, t | = 1$$

We will define the integration over an algebraic structure by requiring the validity of the completeness relation

The completeness can be lifted up to properties of the algebra of the functions of q (see noncommutative geometry).
Start with

$$\int |q\rangle\langle q| dq = 1$$

and with an orthonormal set $\{|\psi_n\rangle\}$ of states

$$\int \langle\psi_m|q\rangle\langle q|\psi_n\rangle dq = \int \psi_m^*(q)\psi_n(q) dq = \delta_{mn}$$

Given this relation, and the completeness for the states, one can reconstruct the completeness in the q-space. The set $\{\psi_n(q)\}$ has the following properties:

- The set $\{\psi_n(q)\}$ spans a vector space.

- The product $\psi_m(q)\psi_n(q)$ can be expressed as a linear combination of $\psi_p(q)$ since the set is complete.

The set $\{\psi_n(q)\}$ is an algebra

From completeness

$$\psi_n^*(q) = \sum_m C_{nm}\psi_m(q)$$

The orthogonality becomes

$$\int \sum_m C_{nm}\psi_m(q)\psi_p(q)dq = \delta_{np}$$

We will generalize this to an algebra with generators $\{x_i\}$

$$\int_{(x)} \sum_j C_{ij}x_jx_k = \delta_{ik}$$

Algebras

ALGEBRA $\mathcal{A}$: A vector space equipped with a bilinear mapping:

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

Basis of $\mathcal{A}$: $\{x_i,\ i = 0, 1, \cdots, n\}$ (possibly $n \to \infty$, or a continuous index).

Structure constants:

$$\boxed{x_i x_j = f_{ijk} x_k}$$

characterize completely the algebra. It is convenient to introduce vectors $\langle x| = (x_0, x_1, \cdots, x_n)$ and $|x\rangle$.

A very important tool is the *algebra of the left and right multiplications*. The associated matrices are:

$$\boxed{R_i|x\rangle = |x\rangle x_i, \qquad \langle x|L_i = x_i\langle x|}$$

We will use also:

$$L_i^T|x\rangle = x_i|x\rangle$$

Left and right multiplications encode the properties of the structure constants. From their definition we have:

$$(R_i)_{jk} = f_{jik}, \quad (L_i)_{jk} = f_{ikj}$$

Since left and right multiplications are linear operators

$$R_a = \sum_{i=1}^{n} a_i R_i \quad \text{if} \quad a = \sum_{i=1}^{n} a_i x_i \in \mathcal{A}$$

Examples:

Abelian algebras:

$$R_i|x\rangle = \underbrace{|x\rangle x_i = x_i|x\rangle}_{x_i x_j = x_j x_i} = L_i^T|x\rangle \Rightarrow \boxed{R_i = L_i^T}$$

Associative algebras:

$$(|x\rangle x_i)x_j = R_i|x\rangle x_j = R_i R_j|x\rangle$$

$$|x\rangle(x_i x_j) = f_{ijk}|x\rangle x_k = f_{ijk}R_k|x\rangle$$

$$\boxed{R_i R_j = f_{ijk}R_k}$$

in analogous way

$$\boxed{L_i L_j = f_{ijk}L_k, \qquad [R_i, L_j^T] = 0}$$

R_i and L_i give a representation of $\mathcal{A}$ (*regular*).

Integration rules

In the following we will consider associative algebras with identity. If the representations of $\mathcal{A}$ generated by R_i and L_i happen to be equivalent, there exists a matrix C such that

$$\boxed{R_i = C^{-1} L_i C}$$

In this case one has

$$L_i |Cx\rangle = |Cx\rangle x_i, \qquad |Cx\rangle = C|x\rangle$$

as it follows from

$$R_i |x\rangle = C^{-1} L_i C |x\rangle = |x\rangle x_i$$

We define the integration over the algebra in such a way that the following identity is satisfied:

$$\boxed{\int_{(x)} |Cx\rangle\langle x| = 1}$$

In components

$$\int_{(x)} C_{ij} x_j x_k = C_{ij} f_{jkp} \int_{(x)} x_p = \delta_{ik}$$

Trouble !!!! $(n+1)^2$ equations for $n+1$ unknown quantities

$$\int_{(x)} x_p$$

But taking $x_k = x_0 = I$, we get

$$\int_{(x)} C_{ij} x_j = \delta_{i0} \Rightarrow \int_{(x)} x_i = (C^{-1})_{i0}$$

This is the **general solution**.

Examples:

Paragrassmann algebras

$\mathcal{G}_1^p$: $(1, \theta) \longrightarrow \theta^{p+1} = 0$, $x_i = \theta^i$

$$\theta^i \theta^j = \theta^{i+j}, \quad i, j, i+j = 0, 1, \cdots, p$$

$$= 0 \qquad \text{otherwise}$$

C matrix: $\theta^i \longrightarrow \theta^{p-i} \longrightarrow C_{ij} = \delta_{i+j,p}$

$$C^T = C, \quad C^2 = 1$$

$$\boxed{\int_{(\theta)} \theta^i = (C^{-1})_{0i} = \delta_{i,p}}$$

$$\int_{(\theta)} 1 = \int_{(\theta)} \theta = \cdots \int_{(\theta)} \theta^{p-1} = 0$$

$$\int_{(\theta)} \theta^p = 1$$

Matrix algebra

Algebra $\mathcal{A}_N$ of the $N \times N$ matrices.

$$A = \sum_{n,m=1}^{N} e^{(nm)} a_{nm}$$

$$e_{ij}^{(nm)} = \delta_i^n \delta_j^m, \qquad e^{(nm)} e^{(pq)} = \delta_{mp} e^{(nq)}$$

C matrix: $e^{(nm)} \rightarrow e^{(mn)} \rightarrow C_{(mn)(rs)} = \delta_{ms}\delta nr$

$$C^T = C, \quad C^2 = 1$$

$$I = \sum_{n=1}^{N} e^{(nn)}$$

$$\boxed{\int_{(e)} e^{(rs)} = \sum_n (C^{-1})_{(nn)(rs)} = \sum_n \delta_{ns}\delta_{nr} = \delta_{rs}}$$

$$\boxed{\int_{(e)} A = \sum_{n,m=1}^{N} a_{nm} \int_{(e)} e^{(nm)} = Tr(A)}$$

Projective group algebras

Given a group G and an arbitrary projective linear representation $\mathcal{A}(G)$:

$$a \to x(a), \quad a \in G, \quad x(a) \in \mathcal{A}(G)$$

$\mathcal{A}(G)$ has a natural algebra structure:

$$x(a)x(b) = e^{i\overbrace{\alpha(a,b)}^{\text{cocycle}}} x(ab)$$

C matrix: $x(a) \to x(a^{-1}) \to C_{ab} = \delta_{ab,e}$
$C^T = C, \quad C^2 = 1$

$$\boxed{\int_{(x)} x(a) = C^{-1}_{e,a} = \delta_{e,a}}$$

Take $G = R^D$ and $\mathcal{A}(G)$ a *vector* representation, then

$$x(\vec{a}) = e^{i\vec{q}\cdot\vec{a}}$$

and

$$\int_{(x)} e^{i\vec{q}\cdot\vec{a}} = \delta^D(\vec{a}) \Rightarrow \int_{(x)} = \int \frac{d^D\vec{q}}{(2\pi)^D}$$

Applications to **group harmonic analysis**.

Derivations

A derivation is a linear mapping on the algebra $D : \mathcal{A} \to \mathcal{A}$, such that

$$D(ab) = (Da)b + a(Db), \quad a, b \in \mathcal{A}$$

We define the matrix d associated to D as

$$Dx_i = d_{ij}x_j$$

If D is a derivation,

$$S = e^{\alpha D}$$

is an algebra automorphism:

$$e^{\alpha D}(ab) = (e^{\alpha D}a)(e^{\alpha D}b)$$

On the contrary if $S(\alpha)$ is a continuous automorphism, then

$$D = \lim_{\alpha \to 0} \frac{S(\alpha) - 1}{\alpha}$$

is a derivation.

The following theorem generalizes the usual integration by part formula

THEOREM:

If D is such that $\int_{(x)} Df(x) = 0$, then the integral is invariant under the related automorphism $\exp(\alpha D)$ *and viceversa.*

Under the hypothesis of the theorem one has the following identities:

$$\boxed{d + C^{-1} d^T C = 0}$$

And by exponentiation

$$\boxed{C^{-1} \exp(\alpha d^T) C = \exp(-\alpha d)}$$

here $s(\alpha) = \exp(\alpha d)$ is the matrix of the automorphism $S(\alpha) = \exp(\alpha D)$.

One can prove also the **THEOREM:**

An associative self-conjugated algebra with identity, and with $C^T = C$, has a set of derivations leading to an invariant integration measure, the inner derivations.

In this case the inner derivations are given by

$$D_a x_i = [x_i, a], \quad a \in \mathcal{A}$$

or

$$d_a = R_a - L_a^T$$

$$(d_a|x\rangle = (R_a - L_a^T)|x\rangle = |x\rangle a - a|x\rangle)$$

It follows

$$C^{-1} d_a^T C = C^{-1}(R_a^T - L_a)C =$$

$$(C^T R_a C^{T-1})^T - R_a = L_a^T - R_a = -d_a$$

For a matrix algebra

$$\int_{(e)} D_B A = \int_{(e)} [A, B] = Tr\,[[A, B]] = 0$$

Integration over a subalgebra

Given self-conjugated algebra $\mathcal{A}$ and subalgebra $\mathcal{B}$, we would like to recover the integration over $\mathcal{B}$ in terms of the integration over $\mathcal{A}$. Given the decomposition

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$$

look for $P \in \mathcal{A}$ such that

$$\int_{\mathcal{A}} \mathcal{C} P = 0, \quad \int_{\mathcal{A}} \mathcal{B} P = \int_{\mathcal{B}} \mathcal{B}$$

or

$$\int_{\mathcal{A}} \mathcal{A} P = \int_{\mathcal{A}} \mathcal{B} P = \int_{\mathcal{B}} \mathcal{B}$$

These are as many conditions as the basis elements of $\mathcal{A}$. P is uniquely determined.

EX: *Grassmann algebra*:

$\mathcal{G}_1$ can be embedded in $\mathcal{A}_2$ (2×2 matrices)

$$\theta \to \sigma_+, \qquad 1 \to 1_2$$

Then

$$P = \sigma_-$$

In fact, given

$$A = a + b\sigma_3 + c\sigma_+ + d\sigma_-$$

one has uniquely: $A = f(\theta) + C$

$$f(\theta) = a - b + c\sigma_+, \quad C = b(1 + \sigma_3) + d\sigma_-$$

and

$$\int_{\mathcal{A}_2} CP = Tr[C\sigma_-] = 0$$

$$\boxed{\int_\theta f(\theta) = Tr[f(\sigma_+)\sigma_-]}$$

that is

$$\int_\theta 1 = Tr[\sigma_-] = 0, \qquad \int_\theta \theta = Tr[\sigma_+\sigma_-] = 1$$

Conclusions and outlook

- Definition of integration for a very large class of algebraic structures (associative self-conjugated algebras with identity).

- I have shown how to recover many known cases and in particular

  - Integration by part theorem can be generalized.

  - Integration over subalgebras is naturally defined.

- The method can be extended to other cases as the algebra of bosonic and q-oscillators, and also to the nonassociative algebra of octonions.

- Next step would be the definition of the path-integral through tensor products and the study of its properties.