QCD@work

Effective lagrangians for QCD at high density

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Summary

- Introduction
- Effective theory for CFL phase
- Weak coupling calculations
- Effective theory for LOFF phase
- Conclusions

Introduction

Ideas about **Color Superconductivity** (CS) back to B. Barrois, NP **B129** (1977),390; S. Frautschi, Erice 1978; D. Bailin and A. Love, Phys. Report **107** (1984) 325. Only recently it has been realized that CS methods are very powerful to analyze in a rigorous fashion the high density and zero temperature region of QCD phase space (for a complete review see; K. Rajagopal and F. Wilczek, hep-ph/0011333)

- Naive expectation at very high density: asymptotic freedom $\Rightarrow$ Fermi sphere of almost free quarks

- **BCS proved the instability of the Fermi surface in prexence of an attractive arbitrarily weak interaction.** The previous picture changes to a **coherent state of particle-hole pairs**, the **Cooper pairs**

- The dominant interaction in QCD (gluon exchange) is attractive. **A diquark condensation is expected**

- Two cases of interest ($m_u = m_d = 0$, α, β, color; i,j, flavor)

$m_s = 0$, **CFL phase**

$$\Delta^{ij}_{\alpha\beta L(R)} = \langle q^i_{\alpha L(R)} C\gamma_5 q^j_{\beta L(R)} \rangle$$

$$\propto \left[\epsilon^{ijX}\epsilon_{\alpha\beta X} + \kappa(\delta^i_\alpha \delta^j_\beta + \delta^i_\beta \delta^j_\alpha)\right]$$

$$\propto \left[(\kappa + 1)\delta^i_\alpha \delta^j_\beta + (\kappa - 1)\delta^i_\beta \delta^j_\alpha\right]$$

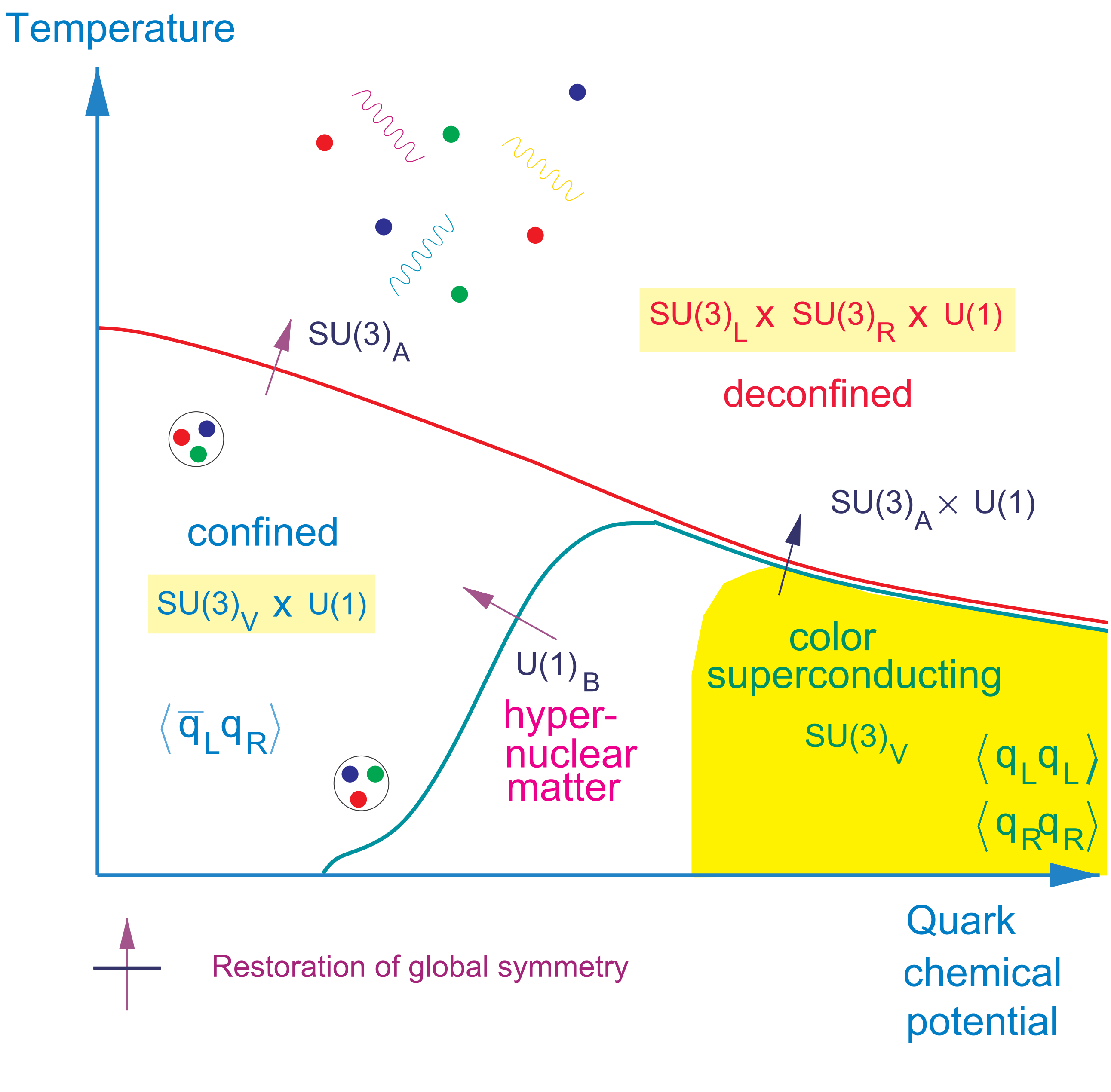
For $\kappa = 0$ the pairing occurs in the $SU(3)_c \otimes SU(3)_{L,R}$ channel $(\bar{\mathbf{3}}, \bar{\mathbf{3}})$. The κ term $\mapsto (\mathbf{6}, \mathbf{6})$. For weak coupling: $\kappa = \mathcal{O}(g)$ (Schäfer, 2000), generally small. **Left, right and color symmetries are locked**

$$[SU(3)_c] \otimes \underbrace{SU(3)_L \otimes SU(3)_R}_{\supset [U(1)_Q]} \otimes U(1)_B$$

$$\downarrow$$

$$\underbrace{SU(3)_{c+L+R}}_{\supset [U(1)_{\tilde{Q}}]} \otimes Z_2$$

(M. Alford, hep-ph/0102047)



- $m_s = \infty$, **2SC phase**:

$$\Delta^{\alpha\beta}_{ijL(R)} = \langle q^{\alpha}_{iL(R)} C \gamma_5 q^{\beta}_{jL(R)} \rangle \propto \epsilon_{ij} \epsilon^{\alpha\beta 3}$$

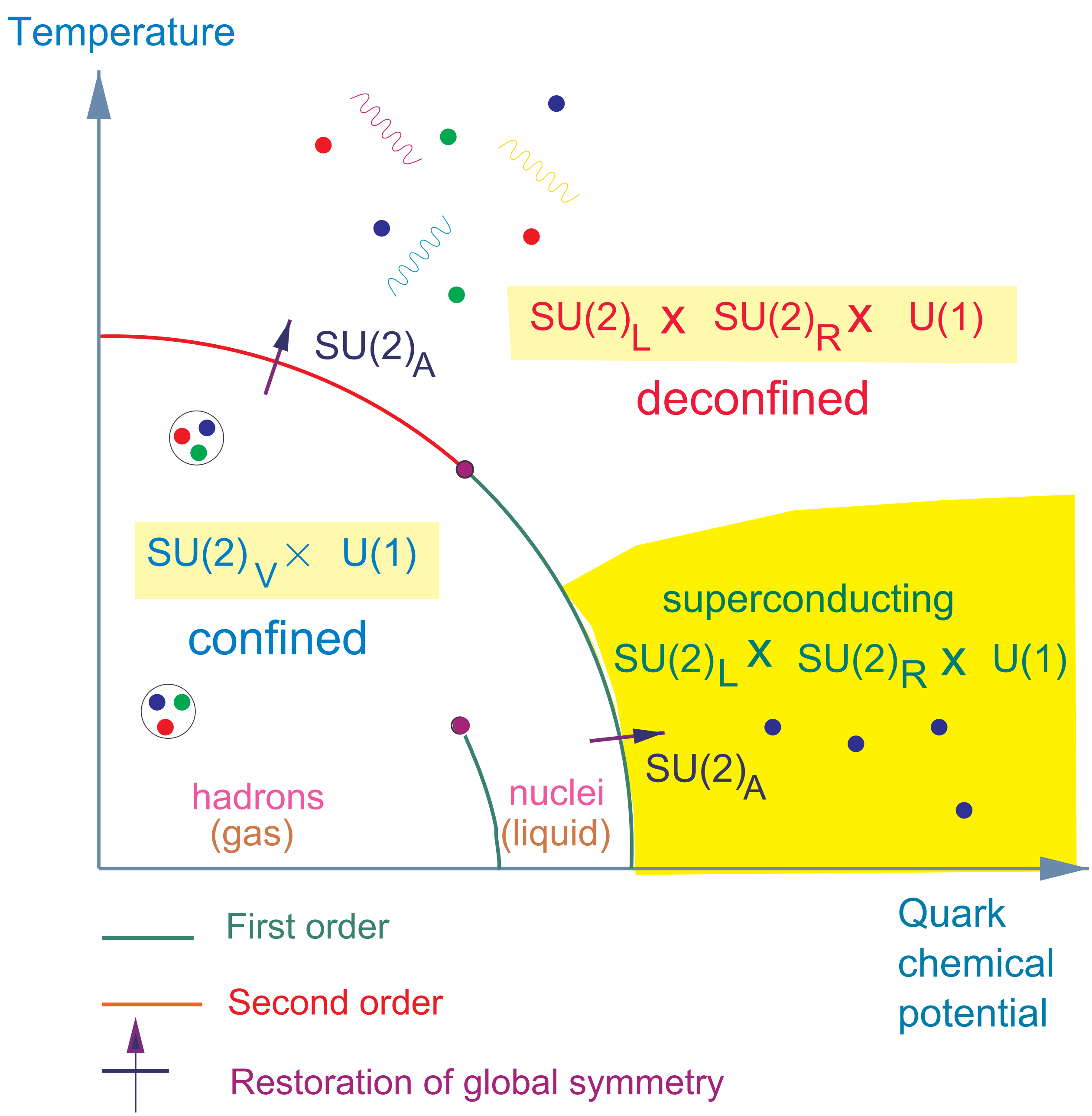
The condensate breaks color but does not break flavor

$$[SU(3)_c] \otimes [U(1)_Q] \otimes SU(2)_L \otimes SU(2)_R$$

$$\downarrow$$

$$[SU(2)_c] \otimes [U(1)_{\tilde{Q}}] \otimes SU(2)_L \otimes SU(2)_R$$

(M. Alford, hep-ph/0102047)



Effective Theory for the CFL phase

In 3 massless flavors QCD at high density the following condensate is formed ($q^i_{\alpha(L,R)}$ Weyl spinors)

$$\langle q^i_{\alpha L}(\vec{p}) C\, q^j_{\beta L}(-\vec{p})\rangle \approx$$

$$\approx \epsilon_{ijX}\epsilon^{\alpha\beta X} + \kappa(\delta^\alpha_i \delta^\beta_j + \delta^\beta_i \delta^\alpha_j)$$

$$\langle q^i_{\alpha L}(\vec{p}) C\, q^j_{\beta L}(-\vec{p})\rangle = -\langle q^i_{\alpha R}(\vec{p}) C\, q^j_{\beta R}(-\vec{p})\rangle$$

producing the symmetry breaking

$$G = SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_V$$

$$\Downarrow$$

$$H = SU(3)_{c+L+R} \otimes Z_2$$

The $U(1)_A$ symmetry is restored at very high density for $N_f = 3$. The breaking of this symmetry also gives rise to a massless Goldstone boson ($U(1)_A \to Z_2$) (R. Rapp, T. Schäfer, E,V, Shuryak, Velkowsky, hep-ph/9904353; T. Schäfer, hep-ph/9909574; D.T. Son and M.A. Stephanov, Phys. Rev. **D61** (2000) 074012, hep-ph/9910941; *ibidem*, Erratum, **D62** (2000) 059902, hep-ph/0004095)

The condensates belong to the representation $(\bar{3},\bar{3})\oplus(6,6)$ of $SU(3)_c\otimes SU(3)_{L,R}$ To represent the Goldstone fields it is enough to consider one representation, say the $(\bar{3},\bar{3})$. We then introduce **Goldstone fields as the phases of these condensates**

$$X^i_\alpha \approx \epsilon^{ijk}\epsilon_{\alpha\beta\gamma}\langle q^j_{\beta L} q^k_{\gamma L}\rangle^*$$

$$Y^i_\alpha \approx \epsilon^{ijk}\epsilon_{\alpha\beta\gamma}\langle q^j_{\beta R} q^k_{\gamma R}\rangle^*$$

With the notations

$$g_c \in SU(3)_c, \quad g_{L(R)} \in SU(3)_{L(R)}$$

$$\exp(i\alpha) \in U(1)_V, \quad \exp(i\beta) \in U(1)_A$$

we have

$$q_{L(R)} \in (\mathbf{3},\mathbf{3}) \text{ of } SU(3)_c \otimes SU(3)_{L(R)}$$

$$q_L \to \exp i(\alpha+\beta) q_L \text{ under } U(1)_V \otimes U(1)_A$$

$$q_R \to \exp i(\alpha-\beta) q_R \text{ under } U(1)_V \otimes U(1)_A$$

and under the total symmetry group

$$X \to g_c X g_L^T \exp(-2i\alpha - 2i\beta)$$

$$Y \to g_c Y g_R^T \exp(-2i\alpha + 2i\beta)$$

Being X and Y the phases of the condensates $(\bar{3}, \bar{3})$ we have

$$X, Y \in U(3)$$

The number of fields is

$$\#X + \#Y = (1+8) + (1+8) = 18$$

8 of these fields give mass to the gluons. **The physical NGB are 10** correponding to the breaking of the global symmetry

$$SU(3)_L \otimes SU(3)_R \otimes U(1)_V \otimes U(1)_A$$
$$\Downarrow$$
$$SU(3)_{L+R} \otimes Z_2 \otimes Z_2$$

Here we have included also $U(1)_A$ which is not really a symmetry, but it gets restored at asymptotic densities. The associated boson becomes massless for $\mu \to \infty$. It is convenient to separate the $U(1)$ factors defining

$$X = \hat{X} \exp(2i\phi + 2i\theta), \quad Y = \hat{Y} \exp(2i\phi - 2i\theta)$$

with $\hat{X}, \hat{Y} \in SU(3)$, also

$$\det(X) = \exp(6i\phi + 6i\theta)$$

$$\det(Y) = \exp(6i\phi - 6i\theta)$$

The transformation properties are

$$\hat{X} \to g_c \hat{X} g_L^T, \quad \hat{Y} \to g_c \hat{Y} g_R^T$$

$$\phi \to \phi - \alpha, \quad \theta \to \theta - \beta$$

The breaking of the global symmetry can be also described by gauge invariant order parameters given by

$$\Sigma_j^i = (\hat{Y}_\alpha^j)^* \hat{X}_\alpha^i \to \Sigma = \hat{Y}^\dagger \hat{X}$$

$$d_X = \det(X), \qquad d_Y = \det(Y)$$

These are the 8+2 NGB's corresponding to the breaking of the global symmetry and

$$\Sigma \to g_R^* \, \Sigma \, g_L^T$$

Σ^{T} transforms as the usual chiral field

The effective lagrangian

Invariant terms can be built starting from the currents (ignoring for a while the local color symmetry)

$$J_X^\mu = \hat{X}\partial^\mu \hat{X}^\dagger, J_Y^\mu = \hat{Y}\partial^\mu \hat{Y}^\dagger$$

$$J_\phi^\mu = U\partial^\mu U^\dagger, J_\theta^\mu = V\partial^\mu V^\dagger$$

with

$$\hat{X} = e^{i\tilde{\Pi}_X^a T_a}, \hat{Y} = e^{i\tilde{\Pi}_Y^a T_a}, U = e^{i\phi/f_T^V}, V = e^{i\theta/f_T^A}$$

and $T_a \in$ Lie SU(3). The transformation properties under the total symmetry G are

$$J_{X,Y}^\mu \to g_c J_X^\mu g_c^\dagger, \qquad J_{\phi,\theta}^\mu \to J_{\phi,\theta}^\mu$$

The most general invariant lagrangian under G the space rotation group $O(3)$ and Parity ($X \leftrightarrow Y$, $U \leftrightarrow U$, $V \leftrightarrow V^\dagger$) is (R. C. and G. Gatto, Phys. Lett. **B464** (1999) 111)

$$\mathcal{L} = -\frac{F_T^2}{4} Tr[(J_X^0 - J_Y^0)^2] - \alpha_T \frac{F_T^2}{4} Tr[(J_X^0 + J_Y^0)^2]$$

$$-\frac{{f_T^V}^2}{2}(J_\phi^0)^2 - \frac{{f_T^A}^2}{2}(J_\theta^0)^2$$

$$+\frac{F_S^2}{4} Tr[(\vec{J}_X - \vec{J}_Y)^2] + \alpha_S \frac{F_S^2}{4} Tr[(\vec{J}_X + \vec{J}_Y)^2]$$

$$+\frac{{f_S^V}^2}{2}(\vec{J}_\phi)^2 + \frac{{f_S^A}^2}{2}(\vec{J}_\theta)^2$$

With the definition

$$\Pi_X = \frac{\sqrt{\alpha_T} F_T}{2}(\tilde{\Pi}_X + \tilde{\Pi}_Y), \; \Pi_Y = \frac{F_T}{2}(\tilde{\Pi}_X - \tilde{\Pi}_Y)$$

and using $Tr[T_a T_b] = \delta_{ab}/2$ we get the properly normalized kinetic term for the 18 Goldstone bosons

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\dot{\Pi}_X^a)^2 + \frac{1}{2}(\dot{\Pi}_Y^a)^2 + \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\theta})^2$$

$$-\frac{v_X^2}{2}|\vec{\nabla}\Pi_X^a|^2 - \frac{v_Y^2}{2}|\vec{\nabla}\Pi_Y^a|^2 - \frac{v_\phi^2}{2}|\vec{\nabla}\phi|^2 - \frac{v_\theta^2}{2}|\vec{\nabla}\theta|^2$$

$$v_X^2 = \frac{\alpha_S}{\alpha_T}\frac{F_S^2}{F_T^2}, \; v_Y^2 = \frac{F_S^2}{F_T^2}, \; v_\phi^2 = \frac{{f_S^V}^2}{{f_T^V}^2}, \; v_\theta^2 = \frac{{f_S^A}^2}{{f_T^A}^2}$$

Let us consider now the local color invariance. To keep this into account use covariant derivatives (g_μ are the gluon fields)

$$\partial_\mu \hat{X} \to D_\mu \hat{X} = \partial_\mu \hat{X} - g_\mu \hat{X}$$

$$\partial_\mu \hat{Y} \to D_\mu \hat{Y} = \partial_\mu \hat{Y} - g_\mu \hat{Y}$$

$$g_\mu = i g_s g_\mu^a T^a/2 \in \text{Lie } SU(3)_c$$

$$J_X^\mu \to J_X^\mu = \hat{X}\partial^\mu \hat{X}^\dagger + g^\mu, \; J_Y \to J_Y^\mu = \hat{Y}\partial^\mu \hat{Y}^\dagger + g^\mu$$

We obtain the invariant lagrangian

$$\mathcal{L} = -\frac{F_T^2}{4} Tr[(X\partial^0 X^\dagger - Y\partial^0 Y^\dagger)^2]$$

$$-\alpha_T \frac{F_T^2}{4} Tr[(X\partial^0 X^\dagger + Y\partial^0 Y^\dagger + 2\, g^0)^2]$$

$$-\frac{{f_T^V}^2}{2}(J_\phi^0)^2 - \frac{{f_T^A}^2}{2}(J_\theta^0)^2$$

$+$ spatial terms and kinetic part for g^μ

Using gauge invariance, choose $\hat{X} = \hat{Y}^\dagger$ (unitary gauge), where

$$\tilde{\Pi}_X = -\tilde{\Pi}_Y, \quad \text{or} \quad \Pi_X = 0, \quad \Pi_Y = F_T \tilde{\Pi}_X$$

8 Goldstone bosons disappear to give mass to the 8 gluons. The gluons g_0^a and g_i^a acquire Debye and Meissner masses respectively

$$m_D^2 = \alpha_T\, g_s^2\, \frac{F_T^2}{4}, \quad m_M^2 = v_X^2\, \alpha_T\, g_s^2\, \frac{F_T^2}{4}$$

The true mass of the gluons is not given by the previous expressions since, in general, the **gluon kinetic term appearing in the effective lagrangian is renormalized by the in-medium interactions** (see later)

The effective lagrangian is supposed to be a valid description below the gap Δ. Since the gluons (as to be seen later on) acquire a mass of order Δ, when $E \ll \Delta$ the gluons decouple and they can be expressed as

$$g_\mu = -\frac{1}{2}(\hat{X}\partial_\mu \hat{X}^\dagger + \hat{Y}\partial_\mu \hat{Y}^\dagger)$$

The lagrangian becomes

$$\mathcal{L} = -\frac{F_T^2}{4} Tr[(\hat{X}\partial^0 \hat{X}^\dagger - \hat{Y}\partial^0 \hat{Y}^\dagger)^2]$$

$$-\frac{{f_T^V}^2}{2}(J_\phi^0)^2 - \frac{{f_T^A}^2}{2}(J_\theta^0)^2 + \text{spatial terms}$$

This can be easily expressed in terms of the chiral field Σ

$$\mathcal{L} = \frac{F_T^2}{4}\left(Tr\,[\dot{\Sigma}\dot{\Sigma}^\dagger] - v_Y^2\, Tr[\vec{\nabla}\Sigma \cdot \vec{\nabla}\Sigma^\dagger]\right)$$

$$-\frac{{f_T^V}^2}{2}\left((J_\phi^0)^2 - v_\phi^2 |\vec{J}_\phi|^2\right) - \frac{{f_T^A}^2}{2}\left((J_\theta^0)^2 - v_\theta^2 |\vec{J}_\theta|^2\right)$$

Notice that the first term coincides with the chiral lagrangian except for the breaking of the Lorentz invariance

Gauging $\mathrm{U}(1)_{\text{em}}$

The em interaction is included noticing that $U(1)_{\text{em}} \subset SU(3)_L \otimes SU(3)_R$ and extending once more the covariant derivative

$$D_\mu \hat{X} = \partial_\mu \hat{X} - g_\mu \hat{X} - \hat{X} Q A_\mu$$

$$D_\mu \hat{Y} = \partial_\mu \hat{Y} - g_\mu \hat{Y} - \hat{Y} Q A_\mu$$

The condensate breaks $U(1)_{\text{em}}$ but leaves invariant a combination of Q and of the color generator:

$$Q_{SU(3)_c} \equiv -2T_8/\sqrt{3} = \text{diag}(+2/3, -1/3, -1/3)$$

In fact

$$Q_{SU(3)_c} \langle \hat{X} \rangle + \langle \hat{X} \rangle Q \to (Q_{SU(3)_c})_{ab} \delta_{bi} + \delta_{aj} Q_{ji} = 0$$

The in-medium conserved electric charge is

$$\tilde{Q} = 1 \otimes Q - Q \otimes 1$$

The eigenvalues of $\tilde{Q}$ are $0, \pm 1$ as in the old Han-Nambu model

The in-medium em field A_μ and the gluon field g_μ^8 get rotated to new fields $\tilde{A}_\mu$ and $\tilde{G}_\mu$

$$A_\mu = \tilde{A}_\mu \cos\theta + \tilde{G}_\mu \sin\theta$$

$$g_\mu^8 = -\tilde{A}_\mu \sin\theta + \tilde{G}_\mu \cos\theta$$

with new interactions (defining $g_\mu = i g_s g_\mu^a T_a$)

$$g_s g_\mu^8 T_8 \otimes 1 + e A_\mu 1 \otimes Q \;\rightarrow\; \tilde{e}\tilde{Q}\tilde{A}_\mu + g_s' \tilde{G}\tilde{T}$$

where

$$\tan\theta = \frac{2}{\sqrt{3}}\frac{e}{g_s}, \quad \tilde{e} = e\cos\theta, \quad g_s' = \frac{g_s}{\cos\theta}$$

$$\tilde{T} = \frac{\sqrt{3}}{2}\left[(\cos^2\theta)\, Q \otimes 1 + (\sin^2\theta)\, 1 \otimes Q\right]$$

Mass terms for the NGB's

The QCD mass terms have the form

$$\bar{\psi}_L M \psi_R \ + \ \text{c.c.}$$

They are invariant if we transform at the same time the fields and the mass matrix

$$\psi_L \to g_L \, e^{i(\alpha+\beta)} \psi_L, \quad \psi_R \to g_R \, e^{i(\alpha-\beta)} \psi_R$$

$$M \to e^{2i\beta} g_L M g_R^\dagger$$

with $e^{i\alpha} \in U(1)_V$, $e^{i\beta} \in U(1)_A$. It is convenient to introduce the field

$$\tilde{\Sigma} = Y^\dagger X = e^{4i\theta} \Sigma$$

transforming as

$$\tilde{\Sigma}^T \to e^{-4i\beta} g_L \tilde{\Sigma}^T g_R^\dagger$$

whereas ($d_X = \det(X)$, $d_Y = \det(Y)$)

$$d_X \to e^{-6i(\alpha+\beta)} d_X, \quad d_Y \to e^{-6i(\alpha-\beta)} d_Y$$

and

$$\det(M) \to e^{6i\beta} \det M$$

$U(1)_V$ invariance requires dependence on the combination $d_X d_{Y^\dagger} = \det(\tilde{\Sigma})$

$$\det(\tilde{\Sigma}) \to e^{-12i\beta} \det(\tilde{\Sigma})$$

Using the Cayley identity for 3×3 matrices it is not difficult to prove that at the lowest order in M there are only three invariant terms, quadratic in M. This follows from the Z_2 symmetry acting on the left-handed fermion fields, under which $M \to -M$. One finds (D.T. Son and M.A. Stephanov, Phys. Rev. **D61** (2000) 074012, hep-ph/9910491, (E) *ibid.* **D62** (2000) 059902, hep-ph/0004095)

$$\mathcal{L}_{masses} = -c \left[\det(M)\, Tr[M^{-1}\tilde{\Sigma}^T] + \text{h.c.}\right]$$

$$-c' \left[\det(\tilde{\Sigma})\, (Tr[M\Sigma^*])^2 + \text{h.c.}\right]$$

$$-c'' \left[Tr[M\Sigma^*]\, Tr[M^\dagger\Sigma^T]\right]$$

In a weak coupling calculation the coefficient c' turns out to be very small compared with c, whereas c'' is zero at the leading order

Perturbative calculations

Once taken into account the diquark condensation, it is possible to do perturbative calculations at very high density taking advantage of asymptotic freedom. We will follow the following steps

- We go from $\mathcal{L}_{QCD}$ at high density to an effective theory describing gapped fermionic excitations close to the Fermi surface.
- We couple Goldstone and gluons in an invariant way to the fermions at the Fermi surface and evaluate the relevant $n-$point functions. This allows the determination of the couplings appearing in the effective lagrangian for NG bosons and gluons

We start describing the effective theory around the Fermi surface (the physics has been described by J. Polchinski, TASI 1992, hep-th/9210046, see also: D.K. Hong, Phys. Lett. **B473** (2000) 118, hep-ph/9812510 and Nucl. Phys. **B582** (2000) 451, hep-ph/9905523; S.R. Beane, P.F. Bedaque, M.J. Savage, Phys. Lett. **B483** (2000) 131, hep-ph/0002209)

We consider QCD at finite density, with a chemical potential μ ($a = 1, \cdots, 8$)

$$\mathcal{L}_{QCD} = \bar\psi i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \mu \bar\psi \gamma_0 \psi$$

For $\mu \gg \Lambda_{QCD}$ quarks are almost free . We have, ($\vec\alpha = \gamma_0 \vec\gamma$)

$$(\not{p} + \mu\gamma_0)\psi(p) = 0 \;\; \Rightarrow \;\; (p^0 + \mu)\psi = \vec\alpha \cdot \vec p\, \psi$$

The energy eigenvalues are

$$p^0 = E_\pm = -\mu \pm |\vec p|$$

with eigenstates $|\pm\rangle$. For momenta close to the Fermi momentum $|\vec p| \approx \mu$ only the states $|+\rangle$ close to the Fermi surface ($E_+ \approx 0$) can be excited. The states $|-\rangle$ with $E_- \approx -2\mu$ decouple at large μ. More formally write

$$p^\mu = \mu v^\mu + \ell^\mu, \quad v^\mu = (0, \vec v_F), \quad |\vec v_F| = 1$$

The Hamiltonian is ($\vec\alpha = \gamma^0 \vec\gamma$)

$$H = -\mu + \vec\alpha \cdot \vec p \;\; \to \;\; H = -\mu(1 - \vec\alpha \cdot \vec v_F) + \vec\alpha \cdot \vec\ell$$

Introducing the projectors

$$P_\pm = \frac{1 \pm \vec\alpha \cdot \vec v_F}{2}$$

and $|\pm\rangle = P_\pm \psi$, we get

$$H\,|+\rangle = \vec\alpha \cdot \vec\ell\, |+\rangle, \quad H|-\rangle = (-2\mu + \vec\alpha \cdot \vec\ell)\,|-\rangle$$

We decompose the fields with $P_\pm$ and integrate out all the modes with $|\vec{\ell}| > \mu$

$$\psi(x) = \sum_{\vec{v}_F} e^{-i\mu v\cdot x}\left[\psi_+(x) + \psi_-(x)\right]$$

$$\psi_\pm(x) = e^{i\mu v\cdot x} P_\pm\, \psi(x) = \int_{|\vec{\ell}|<\mu} \frac{d^4\ell}{(2\pi)^4} e^{-i\ell\cdot x}\psi_\pm(\ell)$$

Substituting inside the lagrangian we get (**off-diagonal terms in the velocity are cancelled by the exponential oscillations for $\mu \to \infty$**)

$$\mathcal{L} = \sum_{\vec{v}_F}\Big[\psi_+^\dagger iV\cdot D\,\psi_+ + \psi_-^\dagger(2\mu + i\tilde{V}\cdot D)\psi_-$$

$$+\ (\bar{\psi}_+ i\not{D}_\perp \psi_- + \text{h.c.})\Big]$$

$$V^\mu = (1, \vec{v}_F), \qquad \tilde{V}^\mu = (1, -\vec{v}_F)$$

$$\not{D}_\perp = D_\mu \gamma_\perp^\mu, \quad \gamma_\perp^\mu = P_\perp^{\mu\nu}\gamma_\nu$$

$$P_\perp^{\mu\nu} = (2g^{\mu\nu} - V^\mu\tilde{V}^\nu - \tilde{V}^\mu V^\nu)$$

Fields inside $\mathcal{L}$ are evaluated at the same Fermi velocity, we have

Fermi velocity selection rule

For large chemical potential the field ψ_- decouple and it can be eliminated through its equation of motion. At the leading order

$$iV \cdot D\,\psi_+ = 0, \quad \psi_- = -\frac{i}{2\mu}\gamma_0 \not{D}_\perp \psi_+$$

For fixed $\vec{v}_F$ only energy and momentum along the Fermi velocity are relevant. Due to the velocity selection rule we have

infinite copies of 2-d physics

At the next-to-leading order the effective action for the field ψ_+

$$\mathcal{L} = \sum_{\vec{v}_F} \left[\psi_+^\dagger iV \cdot D\psi_+ - \frac{1}{2\mu}\psi_+^\dagger (\not{D}_\perp)^2 \psi_+ \right]$$

The $1/\mu$ term may contribute to one-loop diagrams giving rise to an extra μ factor (see later).

Couplings to Goldstone bosons

We have seen that the NG fields, $\hat{X}$ ($\hat{Y}$), transform under G as $q_L(q_R)$, for instance

$$q_L \to g_c\, q_L g_L^T, \qquad \hat{X} \to g_c \hat{X} g_L^T$$

There are two possible invariant couplings with the NGB's, and similar for $\hat{Y}$ and ψ_R, corresponding to the two channels $(\bar{3}, \bar{3})$ and $(6, 6)$

$$\gamma_1\, Tr\, [q_L^T \hat{X}^\dagger] C Tr[q_L \hat{X}^\dagger] + \gamma_2\, Tr\, [q_L^T C \hat{X}^\dagger q_L \hat{X}^\dagger]$$

$$+ \text{ h.c.}$$

Since in the fundamental state $\langle \hat{X} \rangle = \langle \hat{Y} \rangle = 1$, the two couplings reproduce the correct breaking of the symmetry in the CFL phase. For simplicity we will take

$$\gamma_1 = -\gamma_2 \propto \frac{\Delta}{2}$$

corresponding to a condensate in the representation $(\bar{3}, \bar{3})$

In this case the coupling can be written as

$$-\frac{\Delta}{2}\sum_{I=1,2,3} Tr\,[(q_L\hat{X}^\dagger)^T C\epsilon_I(q_L\hat{X}^\dagger)\epsilon_I]$$

with$(\epsilon_I)_{ab} = \epsilon_{Iab}$. It is convenient to define (λ_a $a = 1,\cdots,8$ are the Gell-Mann matrices, $\lambda_0 = \sqrt{2/3}\,1$, and $\Delta_a = \Delta$, $\Delta_9 = -2\Delta$)

$$\hat{X} = 1 + (\hat{X} - 1) \equiv 1 + X_1$$

$$\psi_\pm = \frac{1}{\sqrt{2}}\sum_{A=1}^{9}\lambda_A\psi_\pm^A$$

In terms of the velocity decomposition we get the lagrangian (R.C., R. Gatto and G. Nardulli, Phys. Lett. **B498** (2001) 179, hep-ph/0010321)

$$\mathcal{L} = \sum_{\vec{v}_F}\frac{1}{2}\Big[\sum_{A=1}^{9}\Big(\psi_+^{A\dagger} iV\cdot D\psi_+^A + \psi_-^{A\dagger} i\tilde{V}\cdot D\psi_-^A - \Delta_A(\psi_-^{A^T}C\psi_+^A + \text{h.c.})\Big) - \Delta\sum_{I=1,3}\Big(Tr[(\psi_- X_1^\dagger)^T C\epsilon_I(\psi_+ X_1^\dagger)\epsilon_I] + \text{h.c.}\Big)\Big]$$

Goldstone and gap terms couple fields with opposite Fermi velocity (Cooper pairs). ψ_- is obtained from ψ_+ sending $v_F \to -v_F$

Formalism neater introducing Nambu-Gorkov fields

$$\chi = \begin{pmatrix} \psi_+ \\ C\psi_-^* \end{pmatrix}$$

Making explicit the average over the Fermi velocity with a further 1/2 to take into account the doubling from the Nambu-Gorkov fields by

$$\sum_{\vec{v}_F} \Rightarrow \int \frac{d\vec{v}_F}{8\pi}$$

We get, for the quadratic part of the lagrangian

$$\boxed{\mathcal{L}_0 = \int \frac{d\vec{v}_F}{8\pi}\, \frac{1}{2} \sum_{A=1}^{9} \chi^{A\dagger} \begin{bmatrix} iV\cdot D & \Delta^A \\ \Delta^A & i\tilde{V}\cdot D^* \end{bmatrix} \chi^A}$$

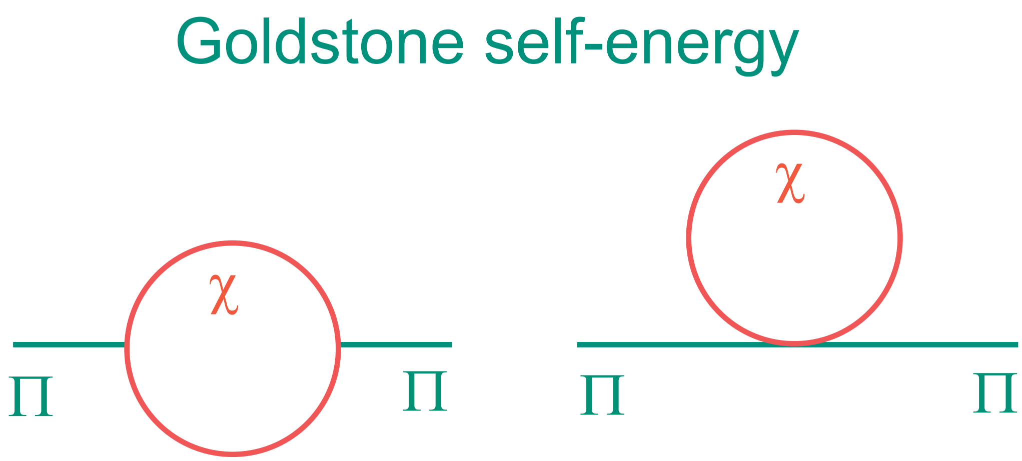
and the propagator

$$S_{AB}(p) = \frac{2\delta_{AB}}{(V\cdot p\,\tilde{V}\cdot p) - \Delta_A^2} \begin{bmatrix} \tilde{V}\cdot p & -\Delta_A \\ -\Delta_A & V\cdot p \end{bmatrix}$$

Expanding the Goldstone fields $\hat{X}$ and $\hat{Y}$ in the gauge $\hat{X} = \hat{Y}^\dagger$

$$\hat{X} = \exp i \left(\frac{\lambda_a \Pi^a}{2F} \right), \qquad a = 1, \cdots, 8$$

we get vertices $\Pi\chi\chi$ and $\Pi\Pi\chi\chi$. The Goldstone self-energy is given by the diagrams



Expanding to $\mathcal{O}(p^2)$ we get

$$i\mu^2 \frac{21 - 8\ln 2}{72\pi^2 F^2} \int \frac{d\vec{v}_F}{4\pi} \sum_{a=1}^{8} \Pi^a V \cdot p \tilde{V} \cdot p \Pi^a$$

from which

$$\mathcal{L}_{\text{eff}}^{\text{kin}} = \frac{\mu^2(21 - 8\ln 2)}{72\pi^2 F^2} \sum_{a=1}^{8} \left(\dot{\Pi}^a \dot{\Pi}^a - \frac{1}{3} |\vec{\nabla} \Pi_a|^2 \right)$$

To get the proper normalization we must have

$$F^2 = \frac{\mu^2(21 - 8\ln 2)}{36\pi^2}$$

Comparing with the effective lagrangian we see that

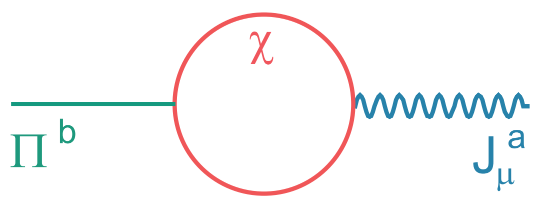
$$F_T = F, \quad F_S = \frac{F_T}{\sqrt{3}} \quad \Rightarrow \quad v_Y = \frac{1}{\sqrt{3}}$$

Therefore the pions satisfy the dispersion relation

$$(p^0)^2 - \frac{1}{3}|\vec{p}|^2 = 0 \quad \mapsto \quad p^0 = \pm\frac{1}{\sqrt{3}}|\vec{p}|$$

The same result has been obtained through the evaluation of the Debye and Meissner masses of the gluons (D. T. Son and M. A. Stephanov, Phys. Rev. **D61**, 074012 (2000), hep-ph/9910491; erratum, *ibid.* **D62**, 059902 (2000), hep-ph/0004095; M. Rho, A. Wirzba and I. Zahed, Phys. Lett. **B473**, 126 (2000), hep-ph/9910550; D. K. Hong, T. Lee and D. Min, Phys. Lett. **B477**, 137 (2000), hep-ph/9912531; C. Manuel and M. H. Tytgat, Phys. Lett. **B479**, 190 (2000), hep-ph/0001095; M. Rho, E. Shuryak, A. Wirzba and I. Zahed, Nucl. Phys. **A676**, 273 (2000), hep-ph/0001104; S. R. Beane, P. F. Bedaque and M. J. Savage, Phys. Lett. **B483**, 131 (2000), hep-ph/0002209; C. Manuel and M. Tytgat, hep-ph/0010274)

To the same result one can arrive through the evaluation of the diagram (R.C., R. Gatto and G. Nardulli, Phys. Lett. **B498** (2001) 179)



giving

$$\langle 0|J^a_\mu|\Pi^b\rangle = iF\delta_{ab}\tilde{p}_\mu, \quad \tilde{p}^\mu = \left(p^0, \frac{1}{3}\vec{p}\right)$$

This relation shows the **current conservation**, due to the dispersion relation for the pions

$$p\cdot\tilde{p} = (p^0)^2 - \frac{1}{3}|\vec{p}|^2 = 0$$

Similar calculations can be done for the NG fields ϕ and θ

Allowing for quark masses, one can evaluate the coupling c

$$c = \frac{3\Delta^2}{2\pi^2}$$

and the NGB's masses. For instance

$$m_{\pi^\pm}^2 = \frac{2c}{F^2} m_s(m_u + m_d)$$

$$m_{K^\pm}^2 = \frac{2c}{F^2} m_d(m_u + m_s)$$

showing that

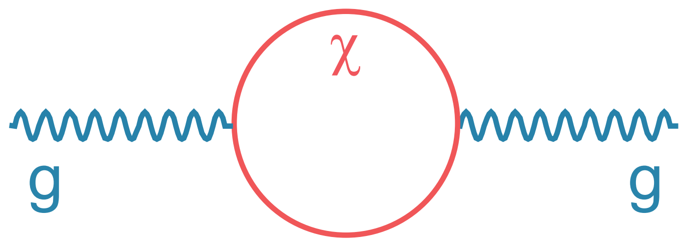
$$\frac{m_{K^\pm}^2}{m_{\pi^\pm}^2} \approx \frac{m_d}{m_u + m_d}$$

Kaons lighter than pions

Couplings to the gluons

By the same techniques we can evaluate the gluon self-energy. The couplings of the gluons to fermions is given by the covariant derivative. This gives rise to the diagram

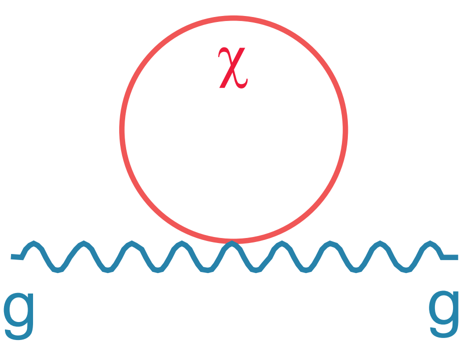
Gluon self-energy



We have also the tadpole contribution arising from the term

$$-\frac{1}{2\mu}\psi_+^\dagger(\not{D}_\perp)^2\psi_+$$

Contribution to the Meissner mass



The loop integration gives an extra μ factor compensating the one in the denominator. From the constant part of the diagrams we get Debye and Meissner masses

$$m_D^2 = g_s^2 F^2 = \frac{\mu^2 g_s^2}{36\pi^2}(21 - 8\log 2)$$

$$m_M^2 = \frac{\mu^2 g_s^2}{108\pi^2}\left(-33 - 8\log 2 + \underbrace{54}_{\text{tadpole}}\right) = \frac{m_D^2}{3}$$

Comparison with the effective lagrangian shows

$$\alpha_T = \alpha_S = 1$$

The tadpole term is also essential in order to satisfy the Ward identity ($\Pi_{ab}^{\mu\nu}$ is the gluon self-energy)

$$p_\mu \Pi_{ab}^{\mu\nu} \propto \langle 0|J^\nu|\Pi^b\rangle = iF\delta_{ab}\tilde{p}^\nu$$

The Meissner and the Debye masses are not the physical masses of the gluons. This comes from the wave-function renormalization proportional to $\mu^2 g_s^2/\Delta^2$ making the **effective square masses proportional to** Δ^2 rather than to $g_s^2\mu^2$($m_{\text{gluon}} \approx 3\Delta$). This changes also $g_s \to g_s\Delta/(g_s\mu) = \Delta/\mu$

Wave function renormalization of order $g_s^2\mu^2/\Delta^2$ for the gluons appears to be a rather general phenomenon. For instance, consider the 2SC phase. The low energy degrees of freedom are 3 gluons and the almost free quarks of color 3. The symmetries determining the effective lagrangian are: the gauge symmetry $SU(2)_c$ and rotation invariance (Lorentz is broken being at finite density). For the gluons one gets (Rischke, Son, Stephanov, 2000)

$$\mathcal{L}_{\text{eff}} = \frac{\epsilon}{2}\vec{E}^a \cdot \vec{E}^a - \frac{1}{2\lambda}\vec{B}^a \cdot \vec{B}^a$$

with a propagation velocity for the gluons given by $v = 1/\sqrt{\epsilon\lambda}$. values of ϵ and λ different from 1 originate from wave function renormalization. One finds

$$\epsilon = 1 + \frac{g_s^2\mu^2}{18\pi^2\Delta^2} \approx \frac{g_s^2\mu^2}{18\pi^2\Delta^2}, \quad \lambda = 1$$

The strong coupling constant gets modified

$$\alpha_s \;\to\; \alpha_s' = \frac{g_{\rm eff}^2}{4\pi v} = \frac{g_s^2}{4\pi\sqrt{\epsilon}} = \frac{3}{2\sqrt{2}}\frac{g_s\Delta}{\mu}$$

due to the changes in the propagation velocity and in the Coulomb force

$$g_s^2/r \to g_s^2/(\epsilon r) \;\;\Rightarrow\;\; g_s^2 \to g_{\rm eff}^2 = g_s^2/\epsilon$$

Similar results hold for the massive gluons of type 4, 5, 6 and 7 which acquire a mass of order Δ. Exceptions are the spatial components (but not the time one) of the gluon 8. In this case there is no wave function renormalization of the time derivative and the mass is of order $g_s\mu$ (R.C., R. Gatto, M. Mannarelli and G. Nardulli, in preparation). Also the em dielectric constant gets modified by the in-medium effects both in the CFL and in the 2SC phases (D.F. Litim and C. Manuel, hep-ph/0105165)

$$\tilde{\epsilon} = 1 + \frac{r}{18\pi^2}\frac{\tilde{e}^2\mu^2}{\Delta^2}$$

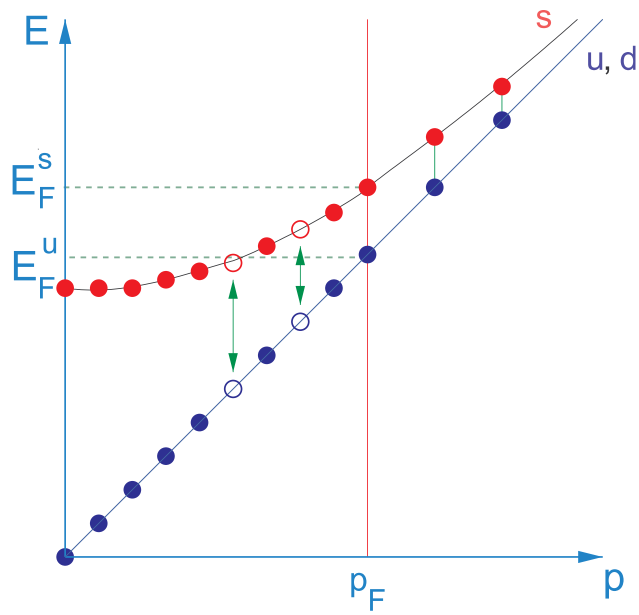
$\tilde{e}$ the in-medium rotated electric charge and

$$r = 4 \text{ in CFL}, \quad r = 1 \text{ in 2SC}$$

The LOFF phase

BCS condensation happens for pairs of opposite momenum. In presence of mass difference between quarks of different flavor, if the corresponding energy difference exceeds the gap the condensate gets disrupted.

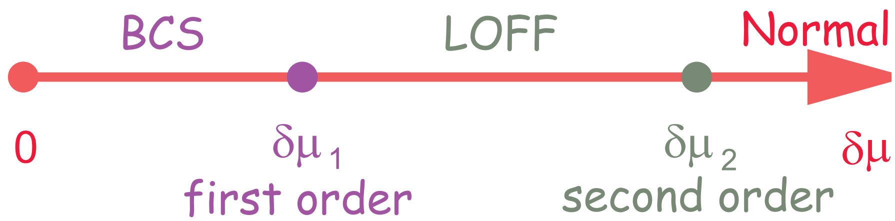
(M. Alford, hep-ph/0102047)



Simulated in a simple model (M. Alford, J.A. Bowers and K. Rajagopal, hep-ph/0008208) with two species of quarks, say up and down, with different μ's

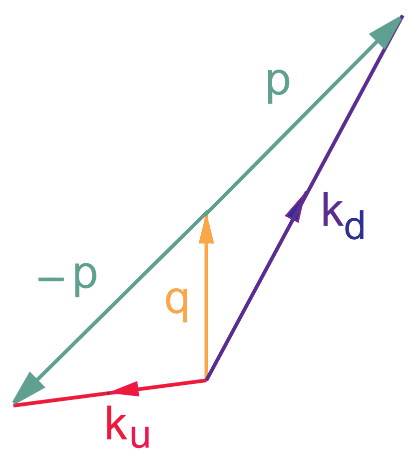
$$\mu_u = \bar{\mu} - \delta\mu, \quad \mu_d = \bar{\mu} + \delta\mu$$

Two critical values of $\delta\mu$:



For $\delta\mu_1 < \delta\mu < \delta\mu_2$ condensation happens between pairs of non-vanishing total momentum $2\vec{q}$

$$\vec{k}_u = \vec{p} + \vec{q}, \quad \vec{k}_d = -\vec{p} + \vec{q}$$



This state (**LOFF state**) is somewhat favored since quarks can stay close to their own Fermi surface

In the LOFF ground state both translational and rotational invariance are spontaneously broken. Here the simplest kind of condensate is assumed

$$\langle\psi(\vec{x})\psi(\vec{x})\rangle \approx \Delta e^{2i\vec{q}\cdot\vec{x}}$$

In weak coupling one finds $\delta\mu_1 = \Delta_0/\sqrt{2} \approx 0.71\Delta_0$ with Δ_0 = BCS gap. The window $(\delta\mu_1, \delta\mu_2)$ is relatively narrow for a contact 4-fermi interaction

$\bar{\mu}(MeV)$	$\Delta_0(MeV)$	$\delta\mu_2$
400	40	$0.744\Delta_0$

it opens up for one-gluon exchange (A.K. Leibovich, K. Rajagopal and E. Shuster, hep-ph/0104073)

$\bar{\mu}(MeV)$	$\Delta_0(MeV)$	$\delta\mu_2$
400	107	$1.24\Delta_0$
1000	37.5	$3.63\Delta_0$

and $\delta\mu_2/\Delta_0 \to \infty$ for $\bar{\mu} \to \infty$

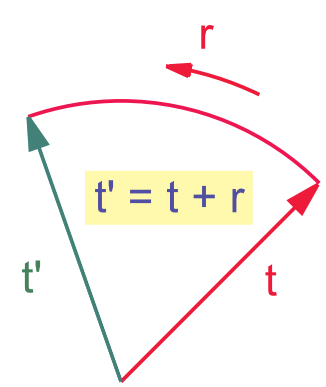
In the LOFF phase two condensates, the scalar

$$\langle\epsilon^{\alpha\beta 3}\epsilon_{ij}\psi^T_{i\alpha}(x)C\psi_{i\beta}(x)\rangle = \Gamma^{(s)}\, e^{2i\vec{q}\cdot\vec{x}}$$

and the vector ($\vec{n} = \vec{q}/|\vec{q}\,|$)

$$\langle\epsilon^{\alpha\beta 3}\psi_{i\alpha}(x)\sigma^1_{ij}C(\vec{\alpha}\cdot\vec{n})\psi_{j\beta}(x)\rangle = \Gamma^{(v)}\, e^{2i\vec{q}\cdot\vec{x}}$$

Both terms break space symmetries. To count the number of NGB's notice that



Three NGB's at most. We introduce a vector field $\vec{R}(x)$, and a scalar field $T(x)$ to account for the variation of the condensates under the space group.

$\vec{R}(x)$ takes into account variations $\perp$ to $\vec{n}$ (assumend along the 3-direction)

$$\vec{R}(x) = \left[e^{i(\xi_1 L_1 + \xi_2 L_2)}\right]_{i3}, \quad (L_i)_{jk} = -i\epsilon_{ijk}$$

$$|\vec{R}(x)|^2 = 1, \quad \langle \vec{R}(x) \rangle = \vec{n}$$

Using this vector field we can compensate the variations of the condensates under rotations

$$\vec{\alpha} \cdot \vec{n} \;\rightarrowtail\; \vec{\alpha} \cdot \vec{R}(x)$$

$$e^{iq\vec{n}\cdot\vec{x}} \;\rightarrowtail\; e^{iq\vec{R}(x)\cdot\vec{x}}$$

Since under the translation, $\vec{x} \to \vec{x} + \vec{a}$

$$e^{iq\vec{R}(x)\cdot\vec{x}} \;\rightarrowtail\; e^{iq\vec{R}(x)\cdot\vec{x} + 2iq\vec{R}(x)\cdot\vec{a}}$$

Introducing a field $T(x)$ such that

$$T \;\rightarrowtail\; T - 2q\vec{R}(x)\cdot\vec{a}$$

we make invariant the quantity

$$e^{i\Phi(x)} = e^{2iq\vec{R}(x)\cdot\vec{x} + iT(x)}$$

we have also to require

$$\langle T(x) \rangle = 0$$

Since

$$\langle \Phi(x) \rangle = 2q\vec{n}\cdot\vec{x}$$

we define

$$\Phi(x) = \langle \Phi(x) \rangle + \phi/f$$

with

$$\phi(x)/f = 2q(\vec{R}(x) - \vec{n})\cdot\vec{x} + T(x)$$

Physically we must have $\phi(x)/f \ll 1$ for any $\vec{x}$. This is possible only if $\vec{R}(x)$ and $T(x)$ are related, that is $\vec{R}(x) \equiv \vec{R}(T(x))$. The solution is

$$\vec{R}(x) = \frac{\vec{\nabla}\Phi(x)}{|\vec{\nabla}\Phi(x)|}$$

In the LOFF phase only one NGB is present: the phonon. $\phi(x)$ is the physical field, but the invariant lagrangian is more easily obtained in terms of Φ

$$\mathcal{L} = \frac{f^2}{2}\left[\dot{\Phi}^2 - \sum_{n=1}^{\infty} c_n(|\vec{\nabla}\Phi|^2)^n\right]$$

Gradient expansion in Φ is not valid since

$$\langle \vec{\nabla}\Phi \rangle = 2\vec{q}$$

of order Δ. However the expansion is feasible for ϕ. Noticing that

$$|\vec{\nabla}\Phi|^2 = 4q^2 + \frac{4q}{f}\vec{n}\cdot\vec{\nabla}\phi + \frac{1}{f^2}|\vec{\nabla}\Phi|^2$$

at two spatial derivatives order we get

$$\mathcal{L} = \frac{1}{2}\left[\dot{\phi}^2 - v_{\|}^2(|\vec{\nabla}_{\|}\phi|^2) - v^2(4qf\vec{\nabla}_{\|}\phi + |\vec{\nabla}\phi|^2)\right]$$

with

$$\vec{\nabla}_{\|}\phi = \vec{n}\cdot\vec{\nabla}\phi$$

The phonon satisfies an anisotropic dispersion relation. The lack of rotational invariance in $\mathcal{L}(\phi)$ follows from the gradient expansion. Similar to chiral case.

Conclusions

- In high density QCD **color superconducting phases** are formed with different features according to m_s

  - $m_s = 0 \ \Rightarrow$ **CFL**
  - $m_s = \infty \Rightarrow$ **2SC**

- An **effective lagrangian** description for the CFL phase has been proposed (for 2SC see R.C., Z. Duan , F. Sannino, Phys. Rev. **D62** (2000) 094004)

- Asymptotic freedom allows for weak coupling calculations of the parameters of $\mathcal{L}_{\text{eff}}$. Use is made of a formalism describing **excitations close to the Fermi surface** which simplifies calculations a lot. Equivalent to

  ∞ **copies of 2-dim physics**

♦ The effective description of a **superconductive crystalline state** has been also considered. This state **spontaneously breaks space symmetries**. The low lying excitation is a **phonon** with a peculiar **anisotropic dispersion relation**

Appendix 1:Quark-Hadron continuity

The effective lagrangian description of CFL suggests strongly complementarity (see later) between the hadron and the CFL phase. However, ignoring $U(1)_A$ which is only an asymptotic symmetry,

$$\text{CFL phase:} \quad U(1)_V \quad \text{broken} \to NGB$$

$$\text{hadr. phase at } T=\mu=0: \; U(1)_V \text{ unbroken}$$

The NGB makes the CFL phase a superfluid. For 3-flavors a dibaryon condensate, H, of the type $(udsuds) \approx \det(X)$ is possible (R.L. Jaffe, Phys. Rev. Lett. **38** 195, 617(E), 1977). This may arise **at μ such that the Fermi momenta of the baryons in the octet are similar** allowing pairing in strange, isosinglet dibaryon states of the type ($p\Xi^-$, $n\Xi^0$, $\Sigma^+\Sigma^-$, $\Sigma^0\Sigma^0$, $\Lambda\Lambda$) (all of the type $udsuds$). This would be again a **superfluid phase**. The symmetries of this phase, called **hypernuclear matter** are the same as the ones in CFL. Therefore there is **no need of phase transition between hypernuclear matter and CFL phase** (T. Schäfer and F. Wilczek, Phys. Rev. Lett. **82**, 3956, 1999, hep-ph/9811473). This is strongly suggested by complementarity idea.

Complementarity

Complementarity refers to gauge theories with a one-to-one correspondence between the spectra of the physical states in the Higgs and in the confined phases (T. Banks, E. Rabinovici, Nucl. Phys. **B160** (1979) 349; E. Fradkin and S.H. Shenker, Phys. Rev. **D19** (1979) 3682 ,for $U(1)$ theories and for $SU(2)$ G. 't Hooft, Cargèse (1979), S. Dimopoulos, S. Raby, L. Susskind, Nucl. Phys. **B173** (1980) 208; L.F. Abbott, E. Fahri, CERN TH3015 (1981)) Specific examples (see E. Fradkin, S.H. Shenker) show that the two phases are rigorously indistinguishable.

No phase transition but a smooth variation of the parameters characterizing the two phases

- A way to implement complementarity

$\psi_i \in R$ of G, elementary states
$Q_A \in \tilde{R}$ of $\tilde{G}$, composite states

R and G isomorphic to $\tilde{R}$ and $\tilde{G}$. Also effective Higgs fields (ϕ^i_A) mapping the two set of states (R.C. and R. Gatto, Phys. Lett. **103B** (1981) 113)

$$Q_A(x) = \psi_i(x)\phi^i_A(x), A \in \tilde{G}, i \in G$$

- In the broken phase, $\langle\phi^i_A\rangle \propto \delta^i_A$ implying that the states in the two phases are the same, except for a necessary redefinition of the conserved quantum numbers by the requirement that the Higgs fields should be neutral in the broken vacuum

- The gauge fields $(g_\mu)^j_i$ go into the vector mesons of the confined phase

$$(Z_\mu)^A_B = -\phi^{*j}_B \left[\partial_\mu - (g_\mu)^i_j\right] \phi^A_i$$

In the case of CFL phase and hypernuclear matter, we have $G = SU(3)_c$ and $\tilde{G} = SU(3)$, and three copies of the fundamental of G. The effective Higgs field is given by the diquark field $D_k^\gamma = \epsilon_{ijk}\epsilon^{\alpha\beta\gamma}\psi_\alpha^i\psi_\beta^j$, with the property $\langle D_k^\gamma\rangle \propto \delta_k^\gamma$

CFL phase	Hypernuclear phase
$\psi_\alpha^i\langle D_k^\alpha\rangle$	$B_k^i = \psi_\alpha^i D_k^\alpha$
$\langle (D^*)_\alpha^i\rangle g_\beta^\alpha \langle D_k^\beta\rangle$	$(D^*)_\alpha^i g_\beta^\alpha D_k^\beta$
Mesons = phases of $(D^*)_{\alpha L}^i D_{jR}^\alpha$	Mesons = phases of $\bar{\psi}_{jL}^\alpha \psi_{\alpha R}^i$

The two phases are very similar but there are several differences (T. Schäfer and F. Wilczek, Phys. Rev. Lett. **82** (1999) 3956, hep-ph/9811473)

- In the hypernuclear phase there is a nonet of vector bosons. However if the dibaryon H exists the singlet vector becomes unstable and does not need to appear in the effective theory

- In the CFL phase there are nine $(\mathbf{8} \oplus \mathbf{1})$ quark states, but the gap of the singlet is bigger than for the octet. Also in the quark model an unstable massive singlet could exist

The CFL phase is a concrete example of complementarity

In the following tables we show the electric charges of the various states

Electric charges

ψ^i_α = quark field

ψ^i_α	u	d	s
R	2/3	-1/3	-1/3
B	2/3	-1/3	-1/3
W	2/3	-1/3	-1/3

$D^\gamma_k = \epsilon_{ijk}\epsilon^{\alpha\beta\gamma}\psi^i_\alpha\psi^j_\beta$ = diquark field

D^γ_k	R	B	W
u	-2/3	-2/3	-2/3
d	1/3	1/3	1/3
s	1/3	1/3	1/3

$B^i_k = \psi^i_\gamma D^\gamma_k = \psi^i_\gamma \left(\epsilon_{rsk}\epsilon^{\alpha\beta\gamma}\psi^r_\alpha\psi^s_\beta\right)$ = baryon field

B^i_k	u	d	s
u	0	-1	-1
d	1	0	0
s	1	0	0

$G^i_k = (D^*)^i_\alpha g^\alpha_\beta D^\beta_k$ = vector meson field

G^i_k	u	d	s
u	0	-1	-1
d	1	0	0
s	1	0	0

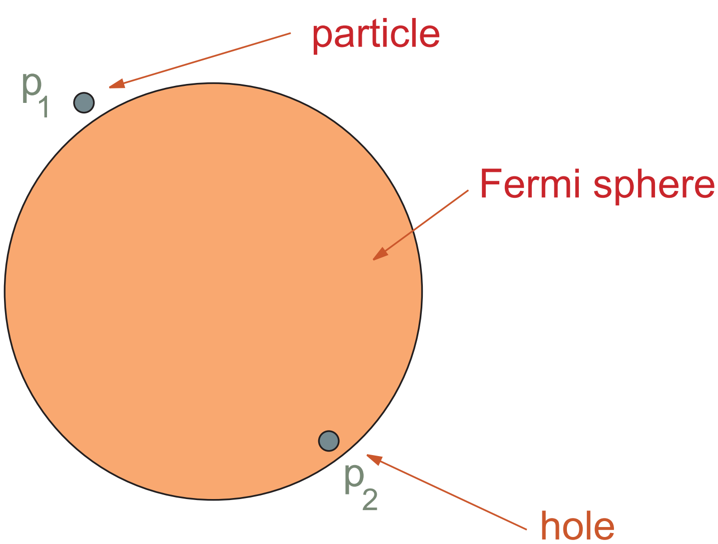
In the CFL phase the charge $\tilde{Q}$ of diquarks is zero whereas for quarks, ψ^i_α, and gluons, $g_{\alpha\beta}$, coincides with the charge Q of baryons, B^i_k, and of vector mesons, G^i_k

Appendix 2: Field theory at the Fermi surface

Ordinary superconductivity, like BCS theory, thought of in terms of a gas of almost free electrons (Landau). Main idea

quasiparticle ≈ dressed electrons

Quasiparticles are the excitations obtained by adding particles above the Fermi surface or removing particles from inside (holes).



Since we will consider particles and holes close to the Fermi sphere we can use a non-relativistic description (Polchinski, TASI 1992, hep-th/9210046)

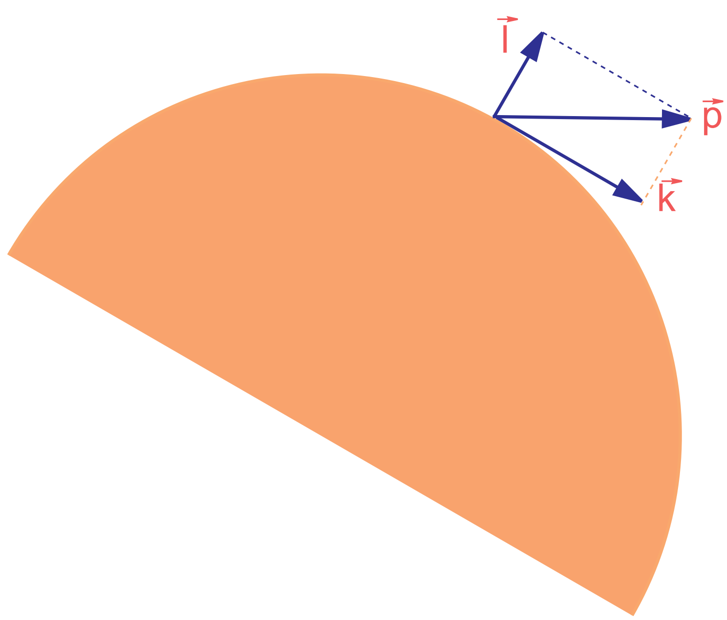
$$\int dt\, d^3\vec{p}\{i\psi_\sigma^\dagger(\vec{p})\partial_t\psi_\sigma(\vec{p}) - (\epsilon(\vec{p}) - \epsilon_F)\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})\}$$

($\sigma =$ spin index). The ground state is given by

states with	$\epsilon(\vec{p}) < \epsilon_F$	filled
states with	$\epsilon(\vec{p}) > \epsilon_F$	empty

The interesting question is about the behaviour of the fields when scaling down the energies by a factor $s < 1$, that is toward ϵ_F. In order to realize the scaling momenta have to scale toward the Fermi surface, that is

when $E \to sE$, then $\vec{l} \to s\vec{l}$, $\quad \vec{k} \to \vec{k}$



Expanding around ϵ_F for small l

$$\epsilon(\vec{p}) - \epsilon_F = \left|\frac{\partial\epsilon(\vec{p})}{\partial\vec{p}}\right|_{l=0} l + \mathcal{O}(l^2) \equiv v_F(\vec{k})l + \cdots$$

Under the scaling

$$dt \to s^{-1}dt, \quad d\vec{k} \to d\vec{k}, \quad d\vec{l} \to s\vec{l}$$
$$\partial_t \to s\partial_t, \quad l \to sl$$

we see that in the action

$$\int dt\, d^2\vec{k}\, d\vec{l}\{i\psi_\sigma^\dagger(\vec{p})\partial_t\psi_\sigma(\vec{p}) - lv_F(\vec{k})\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})\}$$

each term scales as s times the scaling of $\psi_\sigma^\dagger\psi_\sigma$

$$\psi_\sigma \approx s^{-1/2}$$

We can list the terms compatible with the symmetries of the problem

- Quadratic terms

$$\int dt\, d^2\vec{k}\, d\vec{l}\,\mu(\vec{k})\psi_\sigma^\dagger(\vec{p})\psi_\sigma(\vec{p})$$

scale as $s^{-1+1-2\times 1/2} = s^{-1}$. This behaves as a mass term and it is relevant. But it can go into the definition of $\epsilon(\vec{p})$ producing at most a modification of the Fermi surface.

Adding one more time derivative or a term proportional to $|\vec{l}|$ makes the bilinear operators marginal as the ones already included. More time derivatives or $|\vec{l}|$ factors make the operator irrelevant.

- Quartic terms

$$\int dt\, d^2\vec{k}_1\, d\vec{l}_1\, d^2\vec{k}_2\, d\vec{l}_2\, d^2\vec{k}_3\, d\vec{l}_3\, d^2\vec{k}_4\, d\vec{l}_4$$
$$V(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\psi_\sigma(\vec{p}_1)^\dagger\psi_\sigma(\vec{p}_3)\psi_{\sigma'}(\vec{p}_2)^\dagger\psi_{\sigma'}(\vec{p}_4)$$
$$\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$

scale as $s^{-1+4-4\times 1/2} = s$ times the scaling of the delta-function. Generally one can neglect the longitudinal momenta inside the delta function getting

$$\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4) \approx \delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)$$

In this case the term is irrelevant. However consider the scattering $\vec{p}_1+\vec{p}_2 \to \vec{p}_3+\vec{p}_4$. Expanding

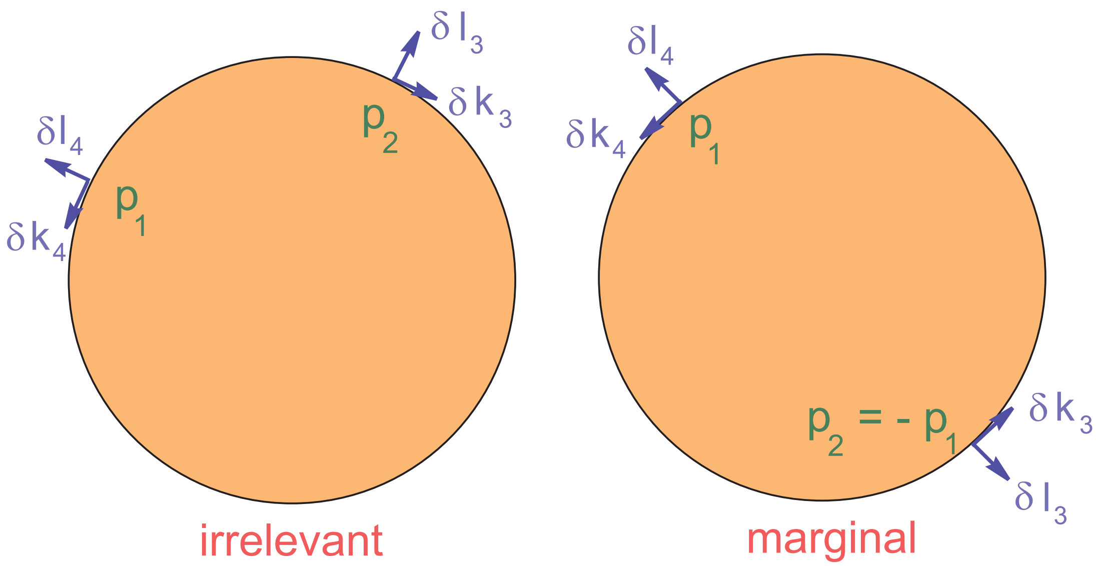
$$\vec{p}_3 = \vec{p}_1 + \delta\vec{k}_3 + \delta\vec{l}_3,\;\; \vec{p}_4 = \vec{p}_2 + \delta\vec{k}_4 + \delta\vec{l}_4$$

we get for the delta function

$$\delta^3(\delta\vec{k}_3 + \delta\vec{k}_4 + \delta\vec{l}_3 + \delta\vec{l}_4)$$

For arbitrary $\vec{p}_1$ and $\vec{p}_2$ the transverse momenta $\vec{\delta} k_3$ and $\vec{\delta} k_4$ span all the space. However for $\vec{p}_1 = -\vec{p}_2$ the delta function factorizes

$$\delta^2(\delta\vec{k}_3 + \delta\vec{k}_4)\delta(\delta\vec{l}_3 + \delta\vec{l}_4)$$

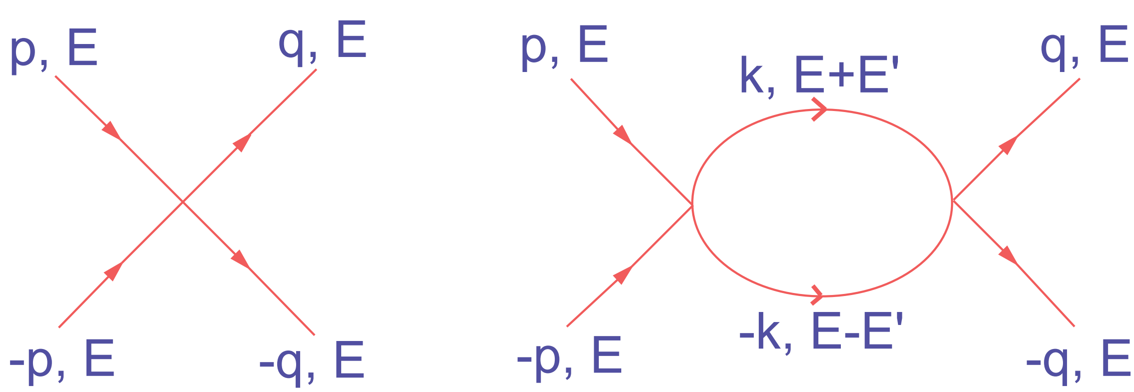


In this case the delta function scales as s^{-1} and the interaction is marginal. Notice that the previous arguments holds for any number of space dimensions. The exceptions are one-dimensional problems where quartic interactions are always marginal.

- Higher interactions

All interactions with a higher number of fermion fields are irrelevant. For instance, with 6 fermi fields we get a scaling factor $s^{-1+6-6\times1/2} = s^2$ times the scaling of the delta-function. For N fermi fields we get $s^{-1+N-N\times1/2} = s^{N/2-1}$ again times the scaling of the delta function.

The previous analysis shows that the excitations around the Fermi surface are essentially free, BUT one has to check the quantum corrections to the marginal operators



Assuming $V(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \equiv V$ as a constant one gets for the four-fermi coupling at one loop

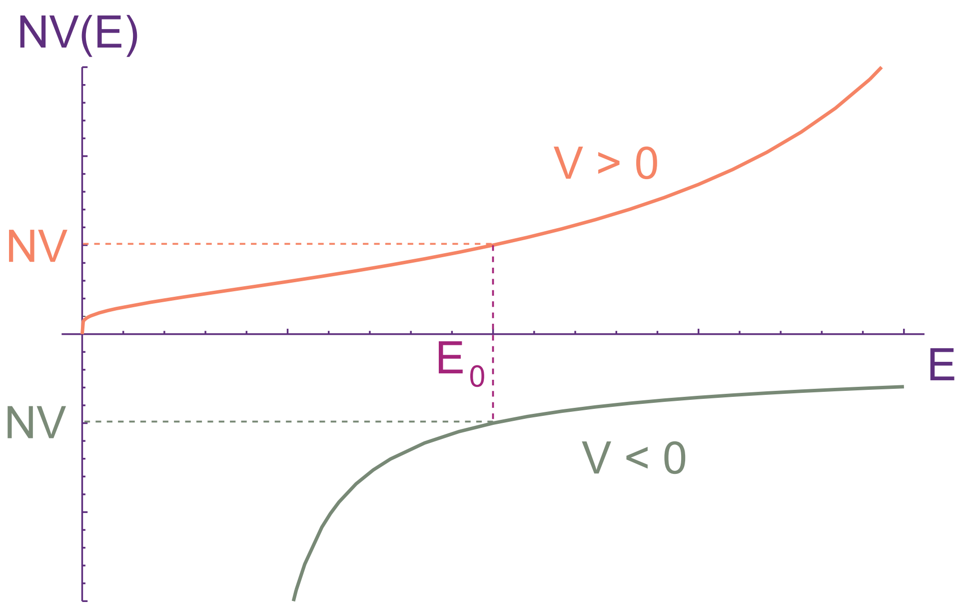
$$V(E) = V - NV^2 \log(E_0/E) + \mathcal{O}(V^3)$$

where E_0 is an upper cutoff and

$$N = \int \frac{d^2\vec{k}}{(2\pi)^3} \frac{1}{v_F(\vec{k})}$$

is the density of states at the Fermi energy. By using RG equations one gets

$$V(E) = \frac{V}{1 - NV \log(E/E_0)}$$



According to the sign of $V(E_0) = V$ we have

$V > 0$ repulsive $\to V(E)$ weaker for $E \to 0$
$V < 0$ attractive $\to V(E)$ stronger for $E \to 0$

An attractive four-fermi interaction, no matters how weak it is at some scale E_0 becomes stronger scaling toward the Fermi surface. The one-loop approximation does not hold any more. Higher orders are important and a BCS condensate $\langle \psi(\vec{p})\psi(-\vec{p})\rangle$ is formed. This is the physical origin of superconductivity.

Appendix 3: The axial anomaly

At large density, for $N_f = 3$, the coefficient of the axial anomaly is not modified (F. Sannino, Phys. Lett. **B480** (2000) 280; S. Hsu, F. Sannino and M. Schwetz, hep-ph/0006059). Therefore for the current $j_5^{\mu 3}$ associated to π^0 we get (R. Casalbuoni, Z. Duan and F. Sannino, Phys. Rev. **D63** (2001)114026, hep-ph/0011394)

$$\partial_\mu j_5^{\mu 3} = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\,Tr\left[(1\otimes T_3)(1\otimes Q)^2\right]$$

$$= -\frac{e^2 N_c}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\,Tr\left[T_3 Q^2\right]$$

$$= -\frac{e^2}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$$

Introducing the rotated fields we get

$$\partial_\mu j_5^{\mu 3} = -\frac{e^2}{32\pi^2}\left[\cos^2\theta\,\epsilon^{\alpha\beta\mu\nu}\tilde{F}_{\alpha\beta}\tilde{F}_{\mu\nu} + \cdots\right]$$

The amplitude for the in-medium decay of $\pi^0\tilde{\gamma}\tilde{\gamma}$ is the same as in vacuum, with $e \to \tilde{e} = e\cos\theta$. The other contributions correspond to anomalous couplings $\pi^0\tilde{\gamma}\tilde{G}$ and $\pi^0\tilde{G}\tilde{G}$

Appendix 4: Meson masses

The structure of the dominant term for the meson masses can be easily understood by iterating the quark mass term

$$(\bar{q}_L M q_R)^2 \approx (q^{\alpha *}_{iL}\, M^i_j\, q^j_{\alpha R})(q^{\beta *}_{kL}\, M^k_l\, q^l_{\beta R})$$

$$\approx \epsilon_{ikm}\epsilon^{\alpha\beta\gamma} X^m_\gamma M^i_j M^k_l \epsilon^{jlp}\epsilon_{\alpha\beta\delta} Y^{\delta *}_p$$

$$\approx \epsilon_{ikm}\epsilon^{jlp} X^m_\gamma Y^{\gamma *}_p M^i_j M^k_l \approx \epsilon_{ikm}\epsilon^{jlp} M^i_j M^k_l \tilde{\Sigma}^m_p$$

$$= \epsilon_{ikm}\epsilon^{jlp} M^i_j M^k_l M^m_a (M^{-1})^a_b \tilde{\Sigma}^b_p$$

$$\approx \mathrm{det}(M) Tr[M^{-1}\tilde{\Sigma}]$$

The most general invariant mass term will have the structure

$$I = (\mathrm{det}(\tilde{\Sigma}))^{a_1} (\mathrm{det}(M))^{a_2} (\mathrm{det}(M)^*)^{\bar{a}_2}$$

$$\times (Tr[M\Sigma^*])^{a_3} (Tr[M^\dagger \Sigma^T])^{\bar{a}_3}$$

$$\times (Tr[(M\Sigma^*)^2])^{a_4} (Tr[(M^\dagger \Sigma^T)^2])^{\bar{a}_4}$$

Asking for analiticity all the exponents have to be integer

Invariance under the global symmetry requires

$$-2a_1 + (a_2 - \bar{a}_2) + (a_3 - \bar{a}_3) + 2(a_4 - \bar{a}_4) = 0$$

Asking for I to be of order n in the masses, we get the equation

$$3(a_2 + \bar{a}_2) + (a_3 + \bar{a}_3) + 2(a_4 + \bar{a}_4) = n \quad (*)$$

Subtracting these two equations we find

$$2a_1 + 4a_2 + 2\bar{a}_2 + 2a_3 + 4a_4 = n$$

implying n to be even. Therefore from $(*)$ we see that $a_2 = \bar{a}_2 = 0$. Choosing $n = 2$ we then easily get that the only solutions are

$$a_1 = 1, \quad a_3 = 2, \quad \bar{a}_3 = 0, \quad a_4 = 0, \quad \bar{a}_4 = 0$$

$$a_1 = 1, \quad a_3 = 0, \quad \bar{a}_3 = 0, \quad a_4 = 1, \quad \bar{a}_4 = 0$$

with their complex conjugated and

$$a_1 = 1, \quad a_3 = 1, \quad \bar{a}_3 = 01, \quad a_4 = 0, \quad \bar{a}_4 = 0$$

The first solution corresponds to the term c', the second one to c and the third one to c''. To see that the second solution is indeed the term c, one has to use the Cayley identity from which it follows

$$Tr[M^{-1}\Sigma^T]\,\det(M)$$
$$= \frac{1}{2}\det(\tilde{\Sigma})\left\{(Tr[M\Sigma^*])^2 - Tr[(M\Sigma^*)^2]\right\}$$

Appendix 5: Gluon masses

The Meissner and the Debye masses are not the physical masses of the gluons. The origin is a wave-function renormalization factor proportional to $\mu^2 g_s^2/\Delta^2$ making the effective square masses proportional to Δ^2 rather than to $g_s^2\mu^2$ ($m_{\rm gluon} \approx 3\Delta$). More explicitly we get the following results for the different components of the gluon fields

$$g_0^a, \quad g_L^{ia} = \frac{\vec{p}\cdot\vec{g}^{\,a}}{|\vec{p}|^2}p^i, \quad g_T^{ia} = g^{ia} - g_L^{ia}$$

$$p^0 = \pm E_{g_{0,L,T}}$$

$$E_{g_0} = \frac{1}{\sqrt{3}}\sqrt{|\vec{p}|^2 + \frac{m_D^2}{\alpha_1}}, \;\; E_{g_L} = \frac{1}{\sqrt{3}}\sqrt{\frac{\alpha_2}{\alpha_1}|\vec{p}|^2 + \frac{m_D^2}{\alpha_1}},$$

$$E_{g_T} = \sqrt{\frac{\alpha_3}{\alpha_1}|\vec{p}|^2 + \frac{m_D^2}{3\alpha_1}}$$

with

$$\alpha_1 = \frac{\mu^2 g_s^2}{216\Delta^2\pi^2}\left(7 + \frac{16}{3}\ln 2\right)$$

$$\alpha_2 = -\ \frac{\mu^2 g_s^2}{3240\Delta^2\pi^2}\left(59 - \frac{688}{3}\ln 2\right)$$

$$\alpha_3 = -\frac{\mu^2 g_s^2}{3240\Delta^2\pi^2}\left(41 - \frac{112}{3}\ln 2\right)$$

It is possible define various type of masse scales as the rest mass, m^R, defined as the energy at $\vec{p} = 0$, and the inverse of the penetration lenght, m^P, as the ratio of the mass term to the coefficient of $|\vec{p}|$ for $E \to 0$. Defining

$$m^R = \frac{m_D}{\sqrt{3\alpha_1}}$$

we get

$$m^R_{g_0} = m^R_{g_L} = m^R_{g_T} = m_R \approx 2.94\Delta$$

$$m^P_{g_0} = \frac{m_D}{\sqrt{\alpha_1}} = \sqrt{3}m^R \approx 5.10\Delta$$

$$m^P_{g_L} = \frac{m_D}{\sqrt{\alpha_2}} = \sqrt{\frac{3\alpha_1}{\alpha_2}}m^R \approx 6.46\Delta$$

The inverse of the penetration lenght for the transverse field has no meaning since $\alpha_3 < 0$.

We could also define effective masses, m^*, from

$$\vec{v} = \frac{\partial E}{\partial \vec{p}} = \frac{\vec{p}}{m^*(|\vec{p}|)}$$

by taking the limit $\vec{p} \to 0$. We get

$$m^*_{g_0} = \sqrt{\frac{3}{\alpha_1}} m_D = 3m^R \approx 8.83\Delta$$

$$m^*_{g_L} = \frac{\sqrt{3\alpha_1}}{\alpha_2} m_D = 3\frac{\alpha_1}{\alpha_2} m^R \approx 14.17\Delta$$

$$m^*_{g_T} = \frac{\sqrt{\alpha_1}}{\sqrt{3}\alpha_3} m_D = \frac{\alpha_1}{\alpha_3} m^R \approx -31.24\Delta$$

The meaning of $m^*_{g_T} < 0$ is that the spectrum of the transverse gluons has a maximum at $\vec{p} = 0$. Therefore these particles are diffucult to be produced at small temperatures. This reminds the spectrum of the excitations of He^4