QCD critical point: a historical perspective

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Summary

- **Motivations**
- **Order parameters**
- **First attempts of model calculations**
- **Universality arguments**
- **Lattice calculations**
- **Isospin chemical potential**
- **Conclusions**

Motivations

- Important to explore the entire QCD phase diagram: Understanding of

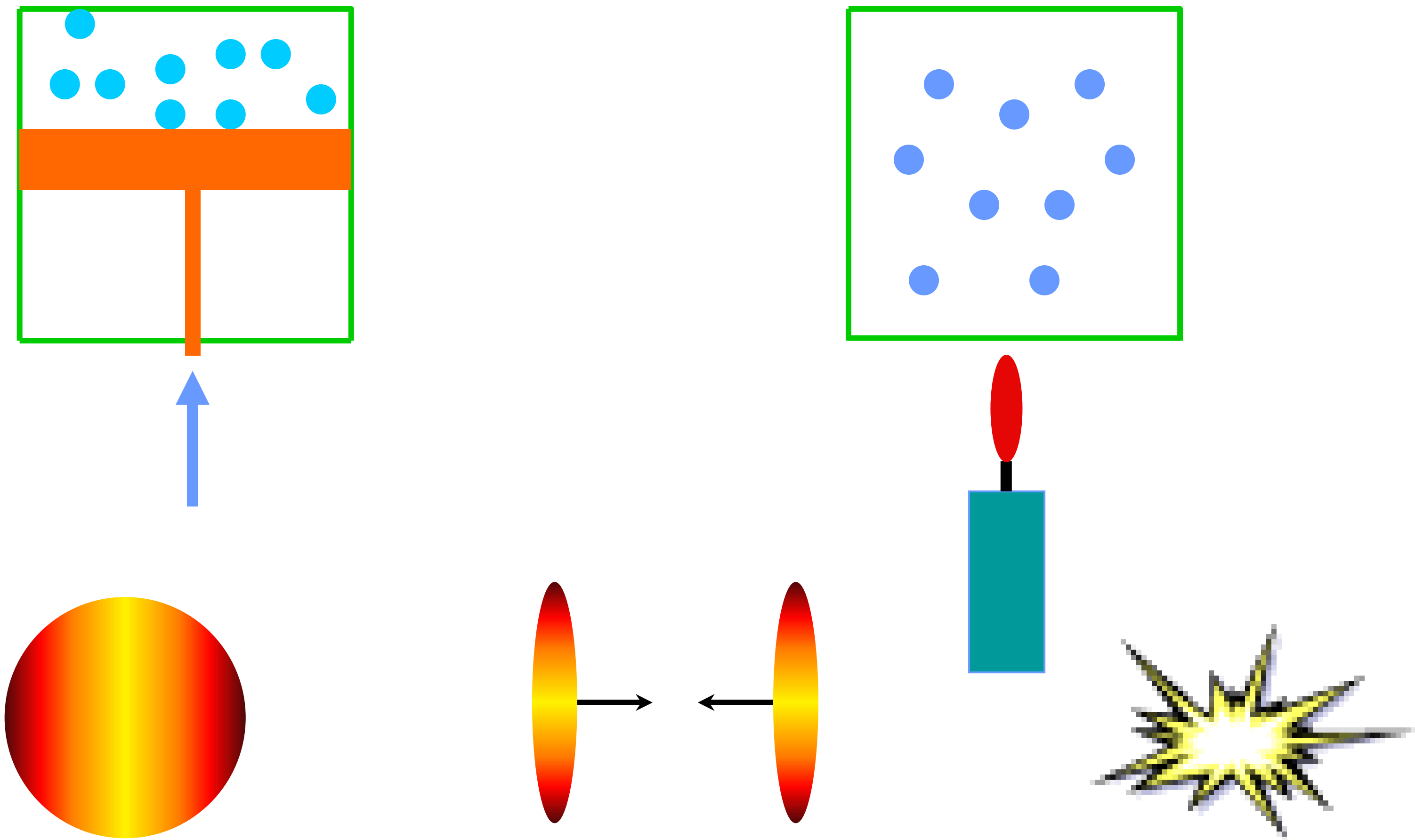
Hadrons → QCD-vacuum

↓

Understanding of its modifications

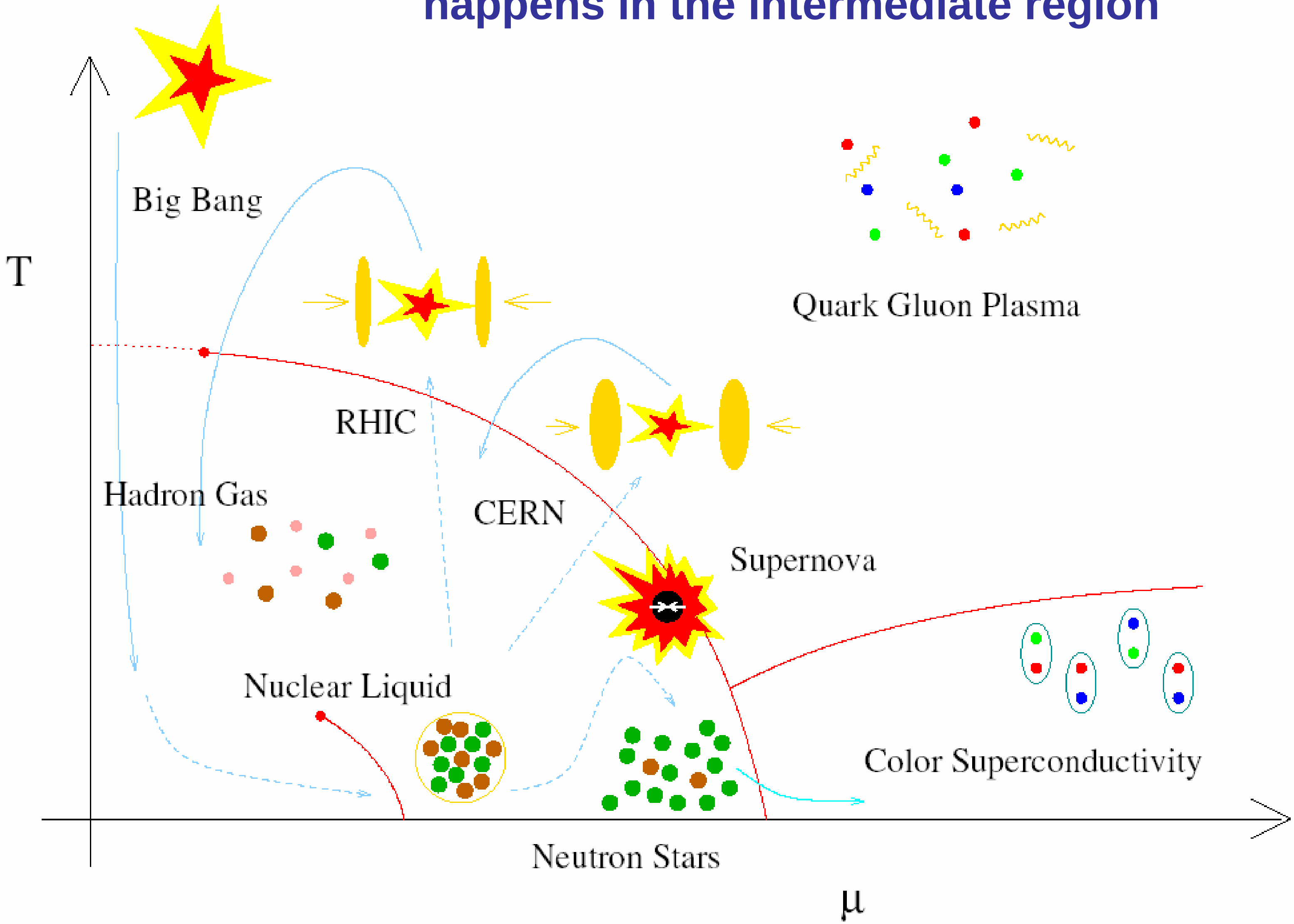
- Extreme Conditions in the Universe:
  Neutron Stars, Big Bang
- QCD simplifies in extreme conditions:

Studying the QCD vacuum under different and extreme conditions may help our understanding



Neutron star   Heavy ion collision   Big Bang

However it is important to understand what happens in the intermediate region



Order parameters

With no quarks consider the Polyakov loop:

$$L(\vec{x}) = \mathrm{tr}\Omega(\vec{x}), \quad \Omega(\vec{x}) = P \exp\left(i\int_0^\beta dt A_0(\vec{x},t)\right)$$

The free energy of a static quark is (F_0 energy of the vacuum)

$$\beta(F_q - F_0) = -\log\langle L(\vec{x})\rangle \quad \left(\langle L\rangle \approx \lim_{r\to\infty} e^{-\beta V(r)}\right)$$

Confining phase: $\langle L\rangle = 0$

Deconfined phase: $\langle L\rangle \neq 0$

L is an order parameter characterizing the breaking of $Z(N_c)$. From asymptotic freedom we expect that at some critical temperature:

$$\langle L \rangle = 0, \quad T < T_c, \qquad \langle L \rangle \neq 0, \quad T > T_c$$

Corresponding to a phase transition from a $Z(N_c)$ symmetric phase (confined) to a broken one (deconfined). If m_q not infinite the string breaks producing a hadron of mass M_h. We expect that L does not vanish

$$\langle L \rangle \approx e^{-\beta M_h}$$

With quarks another order parameter is $\langle\bar{\psi}\psi\rangle$

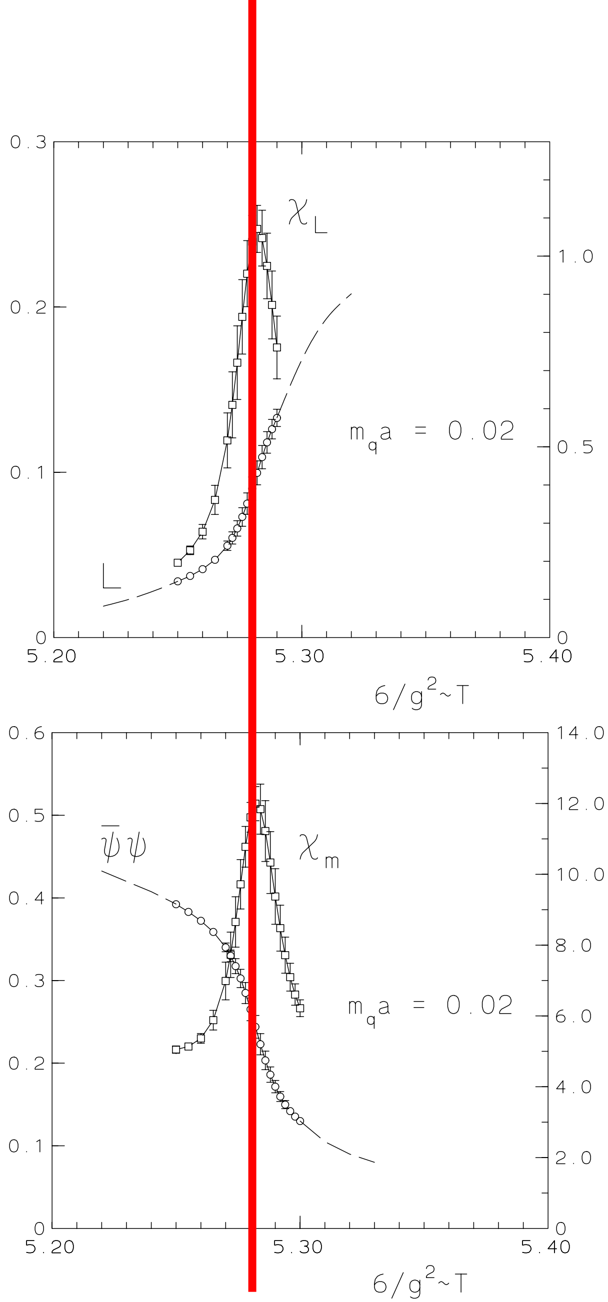
For m_q=0 we expect (breaking χS):

$$\langle\bar{\psi}\psi\rangle = 0 \quad \text{for} \quad T \to \infty, \qquad \langle\bar{\psi}\psi\rangle \neq 0 \quad \text{for} \quad T \to 0$$

In the real world this order parameter never vanishes but it will have a sharp variation or a crossover close to the transition. These quantities and the corresponding susceptibilities have been evaluated on the lattice:

$$\chi_L \approx \langle L^2\rangle - \langle L\rangle^2, \qquad \chi_m \approx \frac{\partial\langle\bar{\psi}\psi\rangle}{\partial m_q}$$

Results from lattice (Karsch & Laermann, PRD 50, 6954, 1994)



$$\chi_L \approx \langle L^2 \rangle - \langle L \rangle^2$$

$$\chi_m \approx \frac{\partial \langle \bar{\psi}\psi \rangle}{\partial m_q}$$

The critical temperature is the same for both transitions!

The conclusion is that we get a well defined phase structure:

$$T < T_c \quad \text{confined phase} \quad \langle L \rangle \approx 0, \quad \langle \bar{\psi}\psi \rangle \neq 0$$

$$T > T_c \quad \text{deconfined phase} \quad \langle L \rangle \neq 0, \quad \langle \bar{\psi}\psi \rangle \approx 0$$

The associate symmetries Z_3 and the χS become exact in the following limits:

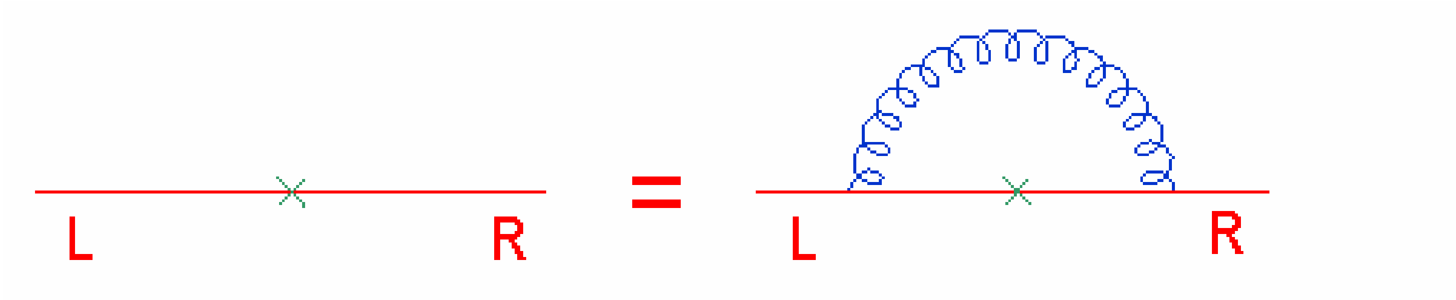
$$Z_3 \quad \text{for} \quad m_q \to \infty$$

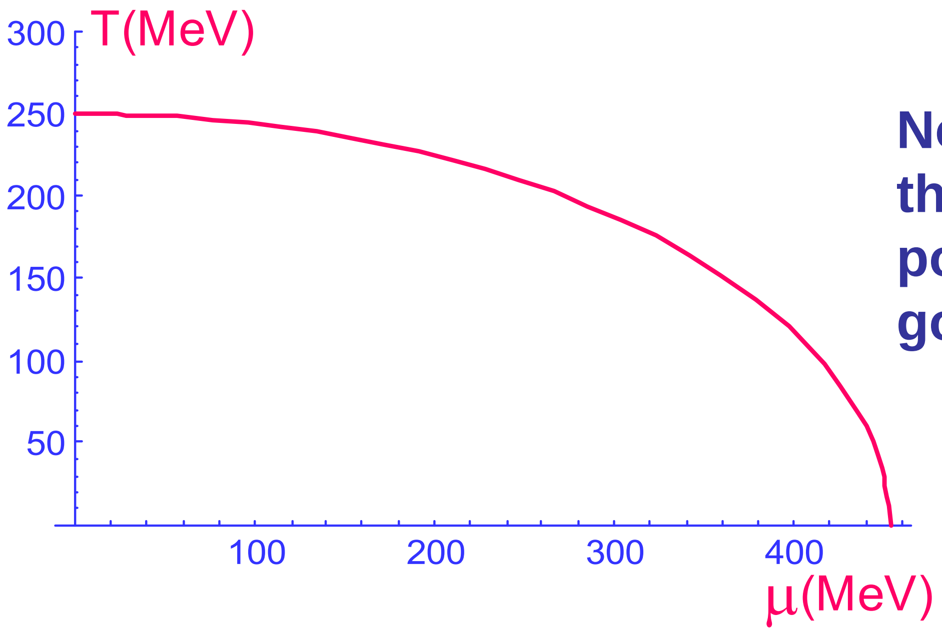
$$\chi S \quad \text{for} \quad m_q \to 0$$

From now on we will consider only the χSB transition.

First attempts of model calculations

One of the first attempts of evaluating the (μ,T) phase diagram in QCD due to Bailin, Cleymans and Scadron, PR D31, 164, 1985, within the approximation of one-gluon exchange:





No indication about the nature of the transition, defined as the point where chiral condensate goes to zero.

Attempts using the Coulomb gauge for the SD equation and neglecting retardation effects in the gluon propagator (close to non-relativistic description)

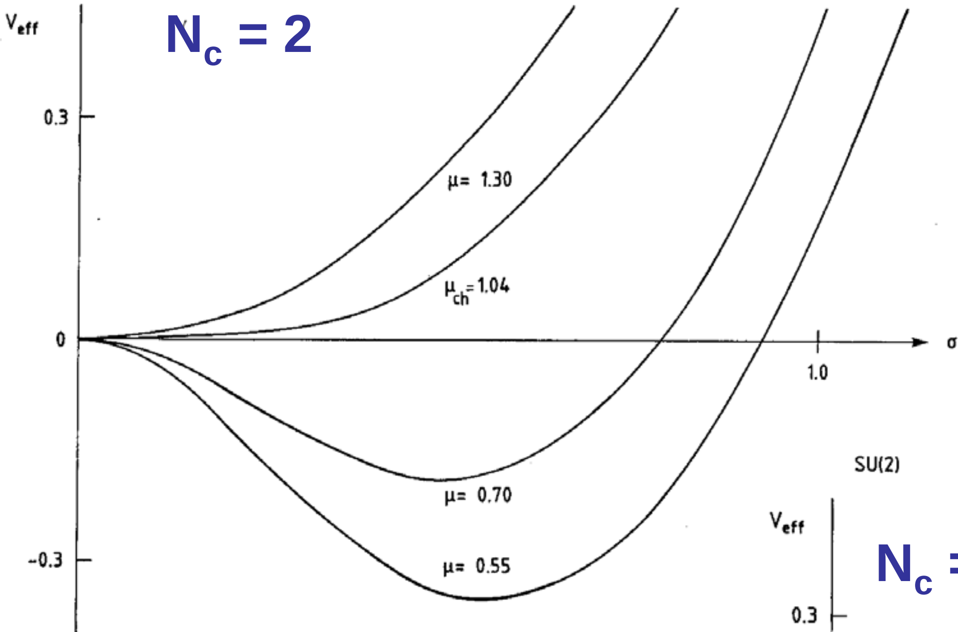
$$\Sigma(p) \approx \sum_{(n, q_0 = i\omega_n + \mu)} \int d^3q V(p-q)\gamma^0 S(q)$$

$$\omega_n = (2n+1)\pi/\beta, \quad S^{-1}(q) = \hat{q} - \Sigma$$

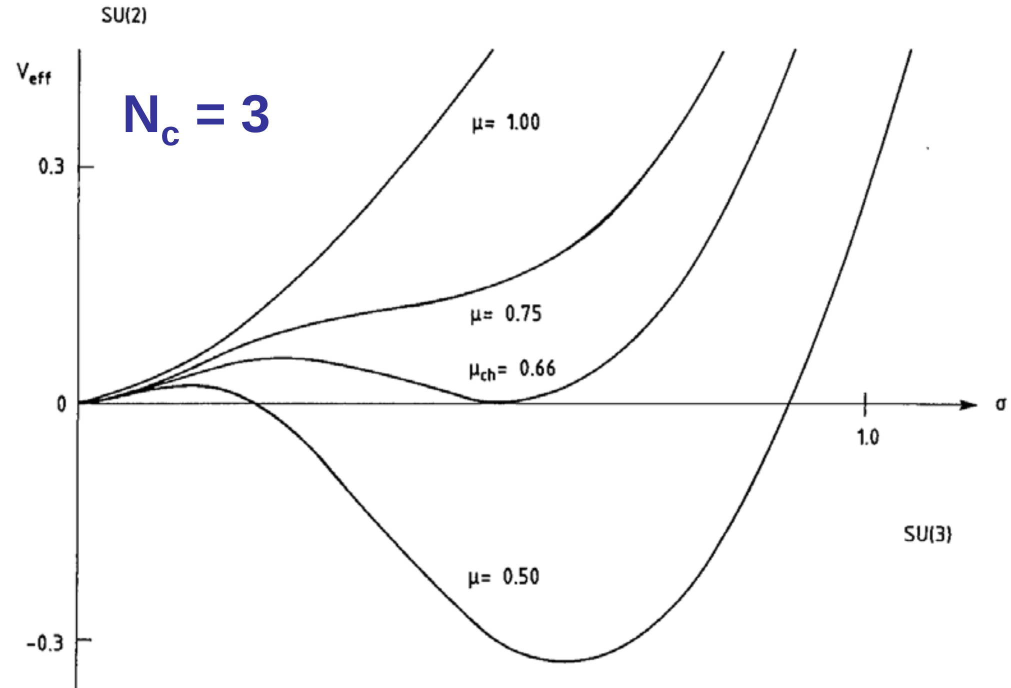
Kocic, PRD 33, 1785, 1986 and Galina & Viswanathan, PRD 38, 1988 tried different choices of V, δ-like, Coulomb, confining, they always found 2nd order transitions in the plane (μ, T).

Effective lagrangian for different gauge groups (N_f=1) from strong coupling limit

(Damgaard, Hochberg & Kawamoto, PL B158, 239, 1985, see also Ilgenfritz & Kripfganz, Z. Phys. C29, 79, 1985)



– Figure 1 –



– Figure 2 –

Asakawa and Yazaki, NP A504, 668, 1989, studied chiral restoration within a NJL model for N_f=2 and N_c=3 by using an interaction given by

$$\mathcal{L}_I = \mathcal{L}_{sym} + \mathcal{L}_{det}$$

$$\mathcal{L}_{sym} = \frac{1}{2} g_1 \left[(\bar\psi\psi)^2 + (\bar\psi i\gamma_5 \vec\tau\psi)^2 + (\bar\psi i\gamma_5\psi)^2 + (\bar\psi\vec\tau\psi)^2 \right]$$

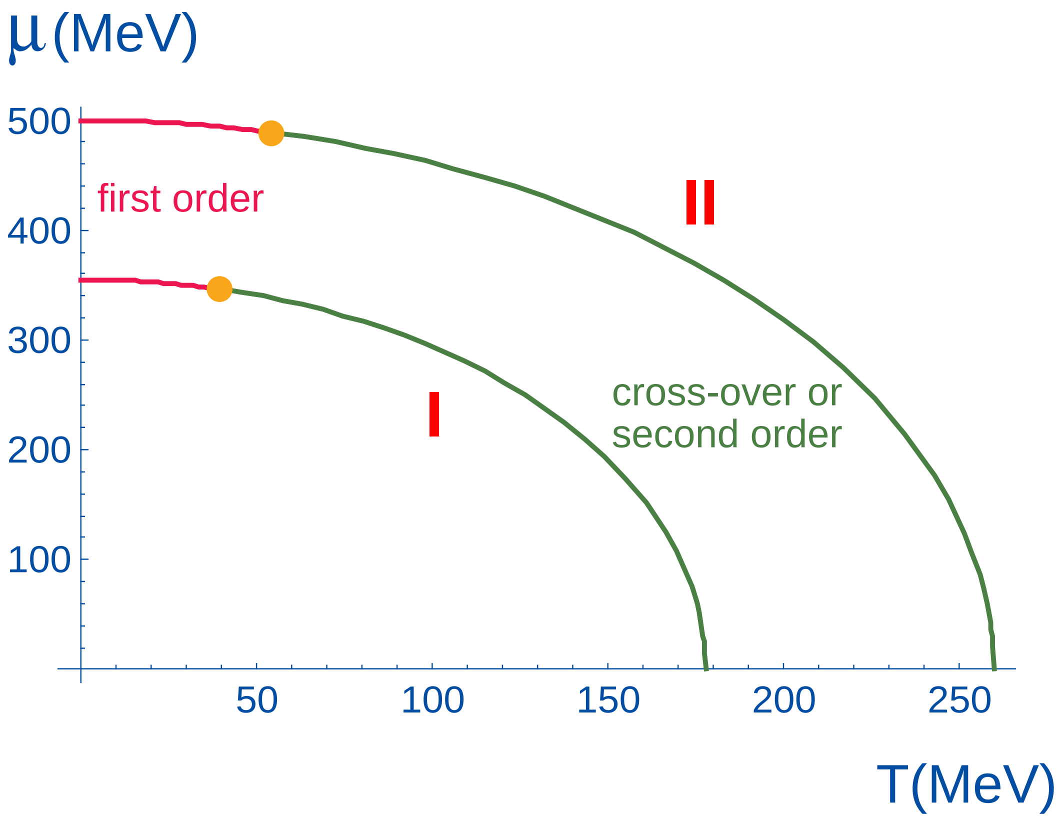
$$\mathcal{L}_{det} = \frac{1}{2} g_2 \left[(\bar\psi\psi)^2 + (\bar\psi i\gamma_5 \vec\tau\psi)^2 - (\bar\psi i\gamma_5\psi)^2 - (\bar\psi\vec\tau\psi)^2 \right]$$

Where L_{det} is the 't Hooft determinant breaking $U(1)_A$

$$\mathcal{L}_{det} = \frac{1}{2} g_2 \left[\mathrm{det}\{\bar\psi(1+\gamma_5)\psi\} + h.c. \right]$$

The authors use $g_1 = g_2$

The model depends on 3 parameters, g, the quark mass m and the cutoff Λ (necessary in NJL), fixed by the phenomenology at zero temperature and density.



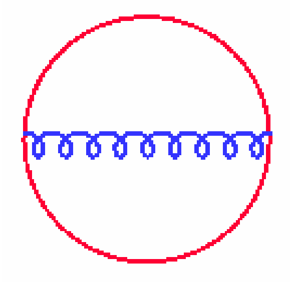
Results:

- **1st order transition at finite μ**
- **crossover at μ=0 and finite T**
- **existence of a critical point at a locatione depending on the parameters**

Parameters adjusted to the physical values of m_π, f_π and a reasonable condensate.

	m(MeV)	g(Mev^{-2})	Λ(MeV)	$<\bar{u}u>^{1/3}$ (MeV)
I	5.5	5.074×10^{-6}	631	- 247
II	5.0	2.337×10^{-6}	925	- 359

In 1989 Barducci, RC, De Curtis, Gatto and Pettini, PL 231B, 463, 1989, studied χSB in QCD through several approximations (Ladder-QCD). The CJT potential was evaluated at two-loop level

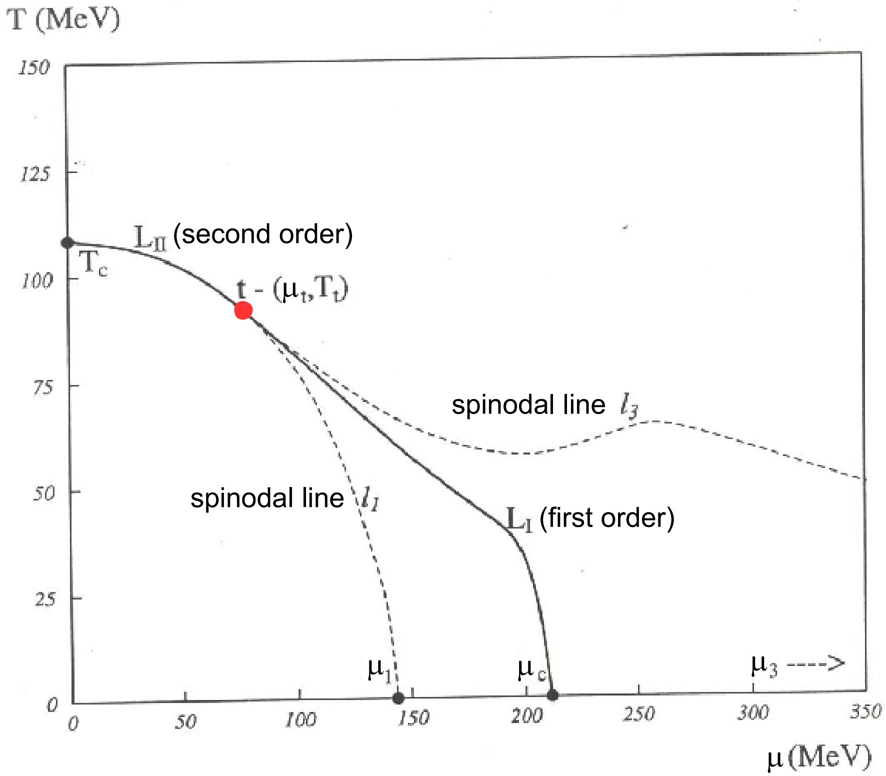


A coupling running with T and μ was used and the self-energy was parameterized as (to be consistent with asymptotic freedom)

$$\Sigma(p,T,\mu) = \chi(T,\mu)\frac{\Lambda}{\Lambda^2 + p^2}$$

$$\langle\bar{\psi}\psi\rangle_{T,\mu} = 3\frac{\Lambda^3}{g^2(T,\mu)}\chi(T,\mu)$$

The parameters at zero T and μ are fixed by the corresponding phenomenolgy

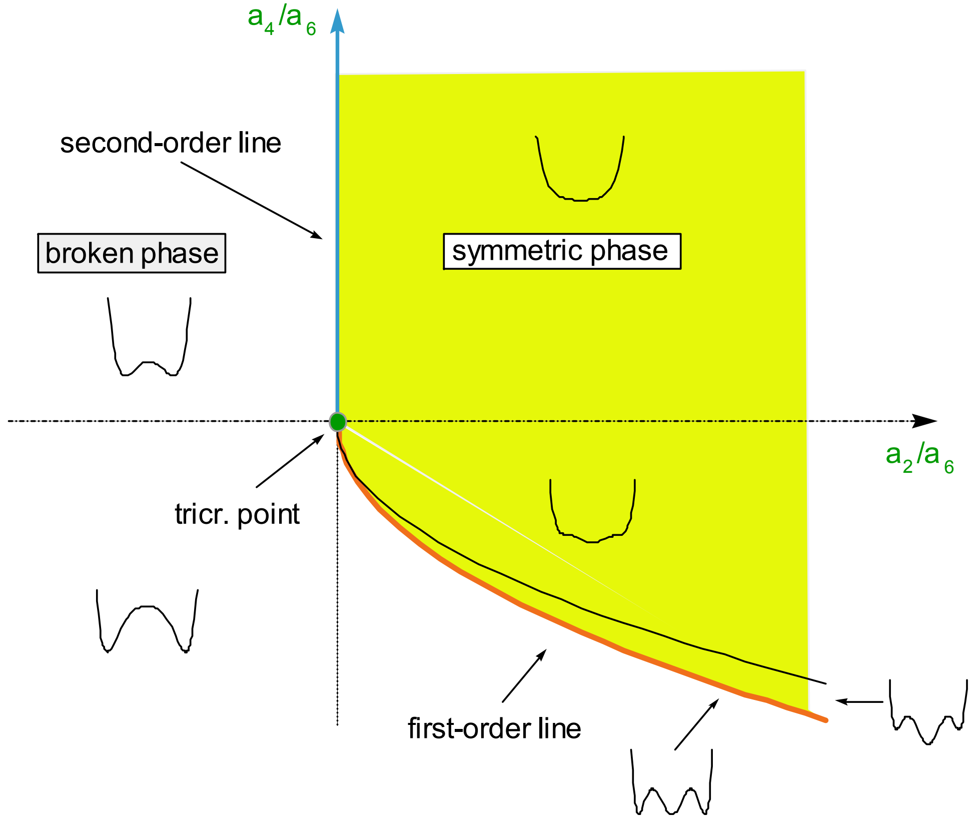


Results very similar to Asakawa:

- **1st order transition at finite μ, L_I**
- **2nd order at μ = 0 and finite T, L_{II} (zero quark mass)**
- **existence of a critical point at a location depending on the choice of the parameters**

In a next paper (Barducci et al. PRD, 1757, 1990) the effective potential was studied near the tricritical point by performing a GL expansion up to 6th order in the condensate χ. The expansion reads

$$V(\chi,T,\mu) = V(0,T,\mu) + a_2(T,\mu)\chi^2 + a_4(T,\mu)\chi^4 + a_6(T,\mu)\chi^6$$



The second order line is defined by $a_2 = 0,\ a_4 > 0$ whereas the tricritical point is where $a_2 = a_4 = 0$

An evaluation of the critical exponents is possible. Let us intoduce a quark mass term $h\chi$:

(Barducci et al. PRD, 426, 1994) close to the tricritical point (μ_c,T_c) one can expand a_2 and a_4 in the form

$$a_i(T,\mu) \approx a_{iT}\left|\frac{T-T_c}{T_c}\right| + a_{i\mu}\left|\frac{\mu-\mu_c}{\mu_c}\right|, \quad i=2,4$$

Then from the minimum condition:

$$h = 2a_2\chi + 4a_4\chi^3 + 6a_6\chi^5$$

If we denote by θ either μ or T one has:

$$\langle\chi\rangle_{m_q=0,\theta\to\theta_c} \approx \left|1-\frac{\theta}{\theta_c}\right|^{1/4} \to \beta = \frac{1}{4}$$

$$\langle\chi\rangle_{m_q\to 0,\theta=\theta_c} \approx m_q^{1/5} \to \delta = \frac{1}{5}$$

$$\frac{\partial\langle\chi\rangle}{\partial m_q}_{m_q=0,\theta\to\theta_c} \approx \left|1-\frac{\theta}{\theta_c}\right|^{-1} \to \gamma = 1$$

From these exponents, using the scaling relations for a 3d system (the finite temperature cutoff the time-like modes):

$$\alpha = 2-3\nu, \quad \beta = \frac{\nu}{2}(1+\eta), \quad \gamma = (2-\eta)\nu, \quad \delta = \frac{5-\eta}{1+\eta}$$

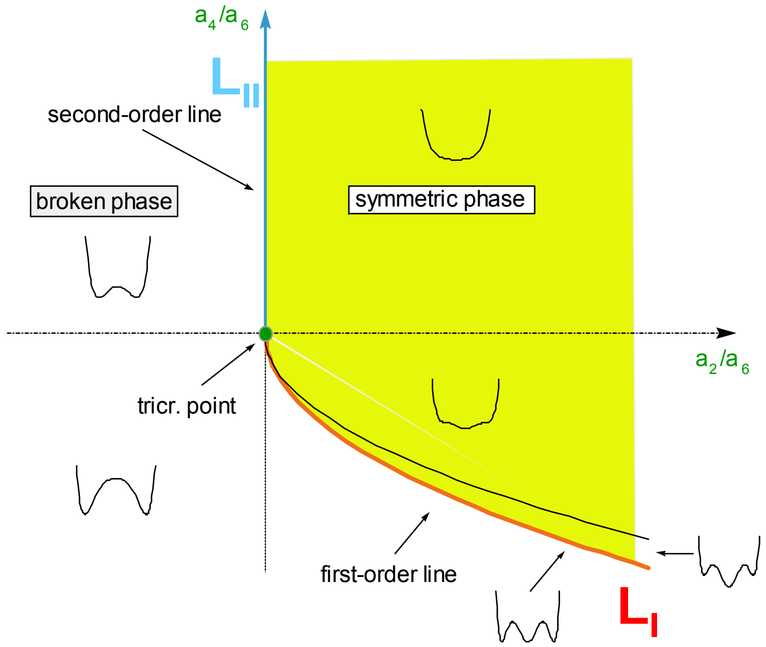
One finds $\alpha = \frac{1}{2}, \quad \nu = \frac{1}{2}, \quad \eta = 0$

With α, ν and η related to the behaviour of the specific heat, the correlation length and of the correlation function at zero momentum:

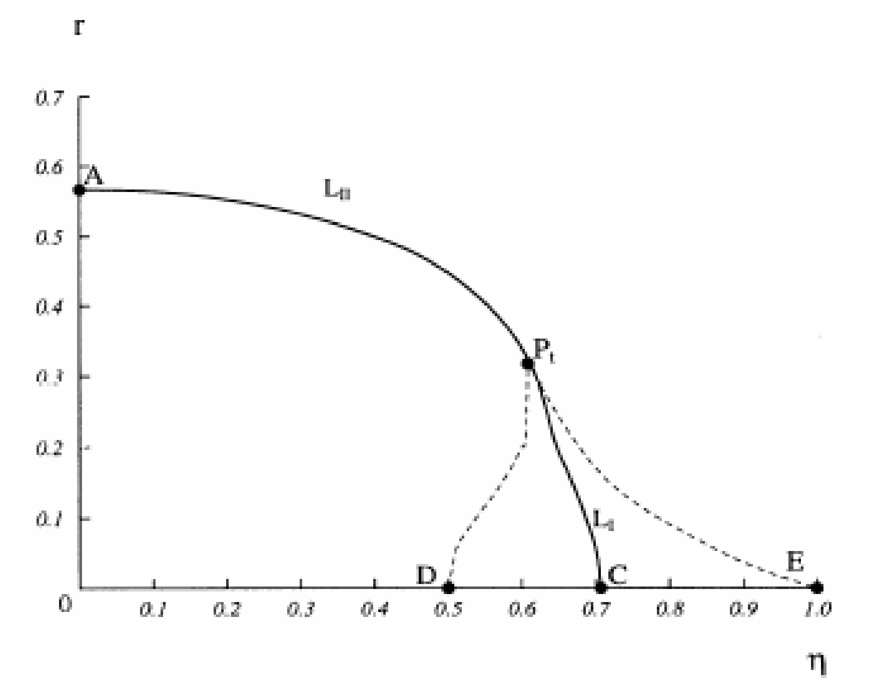
$$C(\theta) \approx \left|1 - \frac{\theta}{\theta_c}\right|^{-\alpha}$$

$$\xi \approx \left|1 - \frac{\theta}{\theta_c}\right|^{-\nu}, \quad G_{\alpha\beta}(k \to 0) \approx k^{-2+\eta}$$

In 1990 we were not very well acquainted with the tricritical behaviour. However, since 1985 (Wolff, PLB 157, 303, 1985) it was known that the 2d Gross-Neveu model had a similar behaviour. This was rather interesting considering the many similarities of the GN model with QCD. Also this phase diagram is very common in physics. In fact it is nothing but the vapor-liquid phase diagram. Consider the plane (1/n,p) or (V,p). In this plane the two degenerate minima of the first order line L_I correspond to different densities. So the line L_I splits in two different lines.

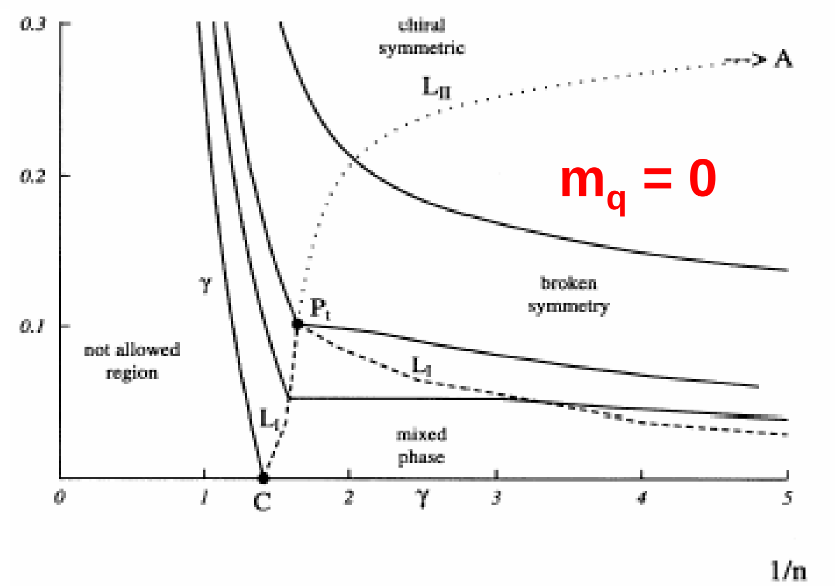


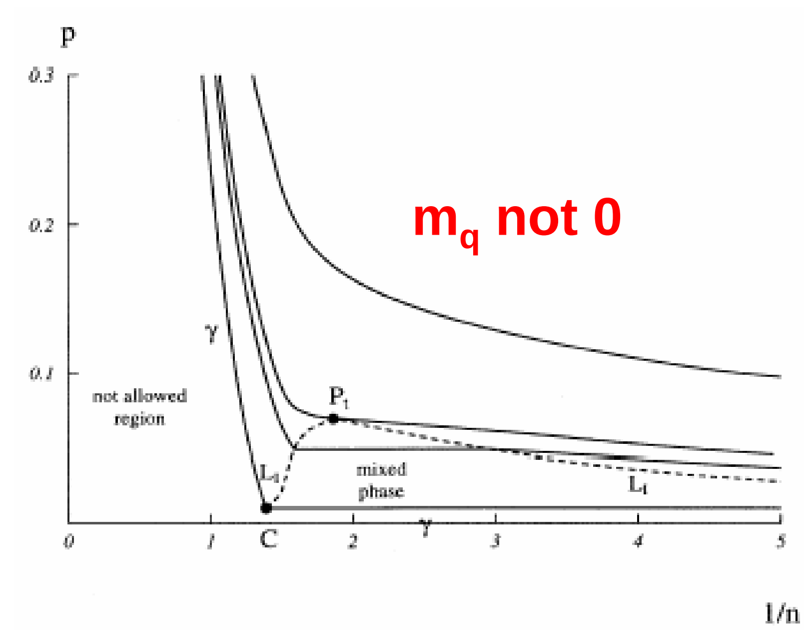
Plotting the isotherm lines one gets the standard diagram for the vapor-liquid transition:



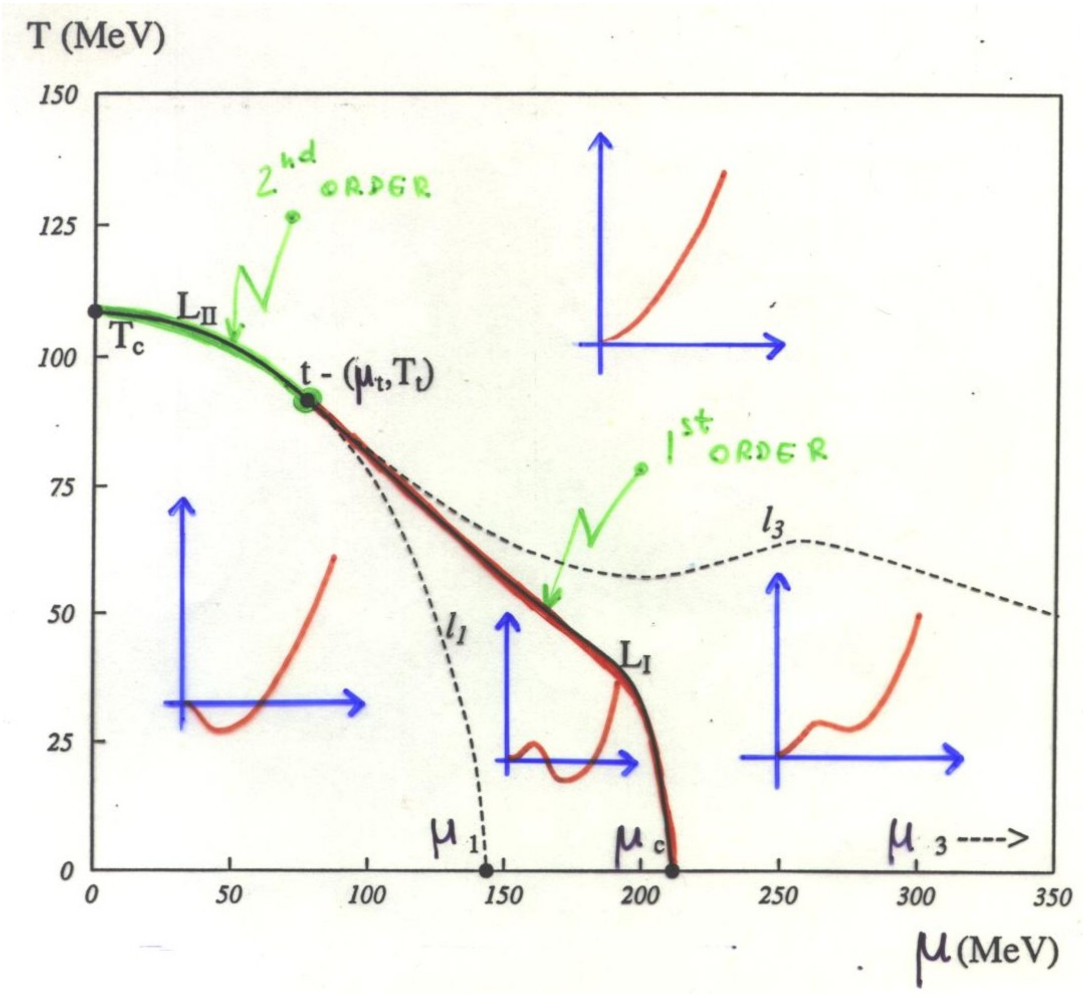
Pictures from GN model

(Barducci et al. PRD 51, 3042, 1995)





These papers went practically unknow. Most of the attention dedicated to the transition at zero chemical potential as a function of the number of light flavors (key to the critical end point).



Barducci & al. Proceedings of the LHC Conference in Aachen, 1990

No real understanding why the expansion up to 6th order and no universality argument

Universality arguments

QCD study through an effective theory (Pisarski & Wilczek, PRD 29, 338, 1984). Introduce light fields transforming as

$$\Phi \approx \bar{\psi}_L \psi_R$$

under the group $G = U_A(1) \otimes SU(N)_L \otimes SU(N)_R$ $\quad \Phi \to e^{i\alpha} U_L \Phi U_R$

If we parameterize $\quad \Phi = \phi U, \quad U \in SU(N)$

The light fields (close to a 2nd order transition) are the $N^2 - 1$ Goldstone fields U, and the condensate ϕ. The effective G-invariant lagrangian is

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}(\partial_\mu \Phi^\dagger \partial^\mu \Phi) - \frac{1}{2} m^2 \mathrm{tr}\Phi^\dagger\Phi - g_1 \left(tr(\Phi^\dagger\Phi)\right)^2 - g_2 \mathrm{tr}(\Phi^\dagger\Phi)^2$$

plus another term breaking the anomalous $\quad U_A(1) \to Z_A(N)$

$$\mathcal{L}' = c\left(\det\Phi + \det\Phi^\dagger\right)$$

At T = 0 the breaking is to $SU(N)_{L+R}$ enforced by the expectation value

$$\langle \Phi \rangle = \Phi_0 \cdot 1$$

The RG study of the beta-functions associated to g_1 and g_2 shows that for

$$N \geq 3$$

the transition to the symmetric phase should be first order. This is proved by using the expansion in 4-ε dimensions and noticing that at T not zero the time-like modes are cut off and one has an effective 3-dim theory. Extrapolation to ε=1 gives the previous result. Notice that at $d=3$, $[\Phi]=1/2$, and a term of the type Φ^6 would be marginal (see later). In the case N = 2, there is a second order transition or none (crossover) for massive quarks

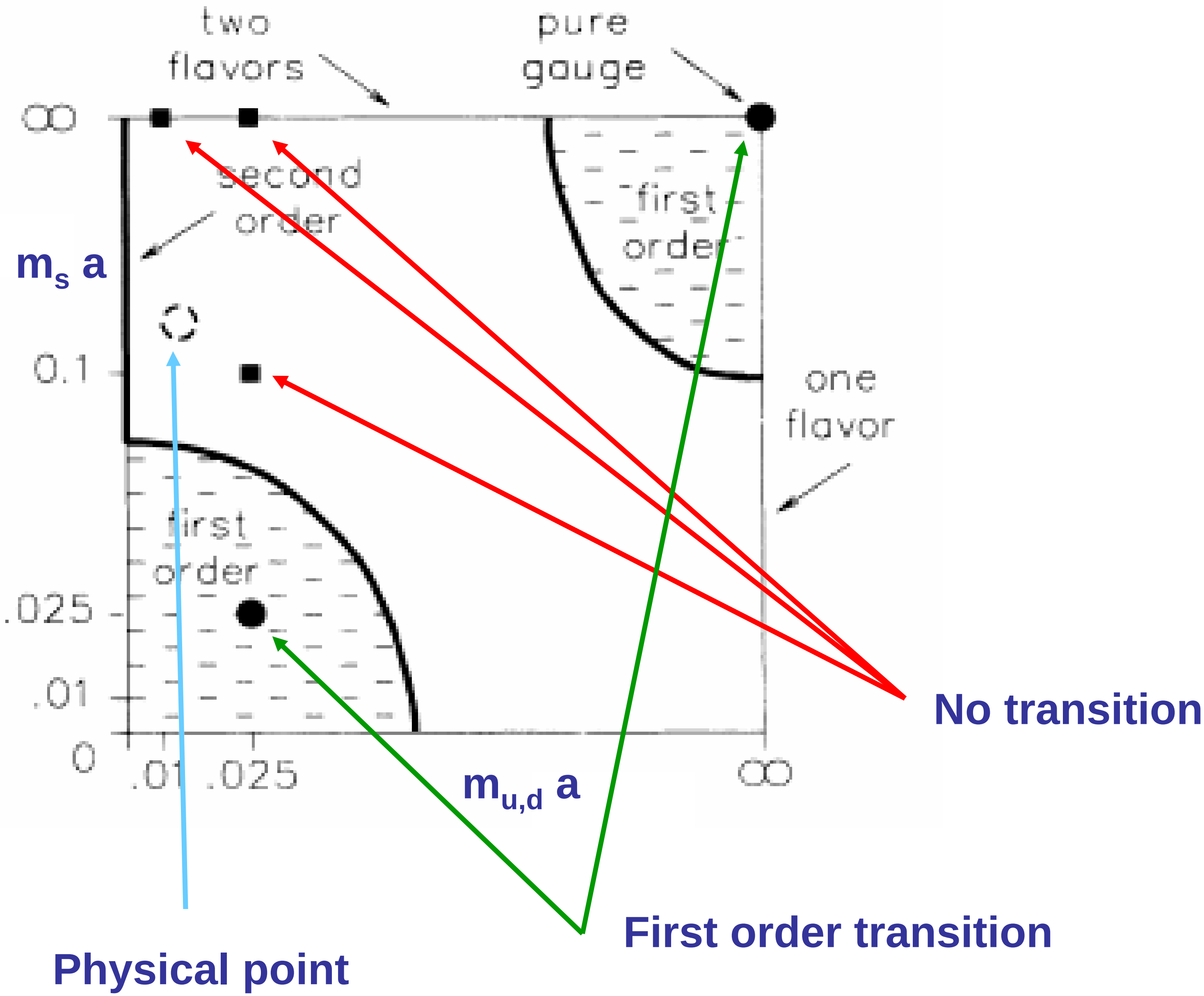
Notice also that close to the critical point the effective potential for the condensate ϕ from the previous lagrangian will be of the form

$$\frac{M^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

with M and λ depending on m^2, g_1, g_2, c and N. In particular the critical temperature is determined by the condition

$$M(T_c) = 0$$

Confirmations also from lattice calculations. In particular, see the work by Brown et al. PRL 65, 2491, 1990


two flavors
pure gauge
∞
second order
first order
$m_s a$
0.1
one flavor
first order
.025
.01
0
.01
.025
$m_{u,d} a$
∞
No transition
First order transition
Physical point

A lattice compilation

Date	Authors	2f	3f	4f	6f	Size
1987	Gottlieb	xover		1^{st}		$(8,10)^3$x4
1990	Gottlieb	xover				12^3x8
1990	Fukugita	xover		1^{st}		12^3x4
1990	Kogut	xover	1^{st} ?	1^{st}		12^3x4
1990	Brown	xover	1^{st}			16^3x4
1992	Barnard	xover				12^3x6
1994	Zhu	xover				$(16,32)^3$x8
1995	Iwasaki	xover	1^{st}		1^{st}	12^3x(6,18), 18^2x24x(6,18)

How do we go from a first to a second order transition varying the strange quark mass? Wilczek, Int. J. Mod. Phys. A7,3911, 1992; E A7,6951, 1992, Rajagopal & Wilczek, NPB 399, 395, 1993 proposed the following solution. Adding a massive quark to the two flavor theory does not change the low energy theory (the light fields are unchanged), therefore the effective theory close to the critical point in d = 3 should be always of the type

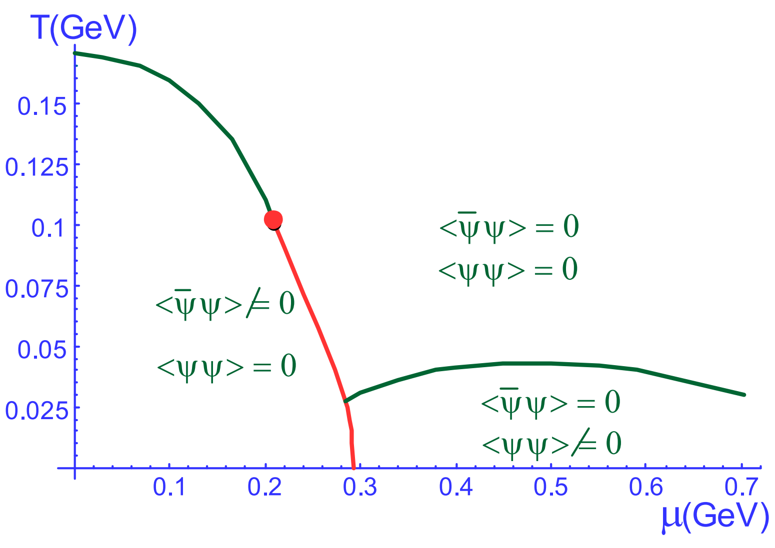
$$L = \int d^3x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{M^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right)$$

However the presence of the massive quark renormalizes the coupling. The changes in M^2 will modify T_c, but it could modify also λ in such a way to make it negative. Then the expansion should be extended up to the sixth order adding a term

$$\frac{\delta}{6}\phi^6$$

In this case as we have seen the transition jumps from second to first order at the tricritical point.

Revival of interest at $\mu \neq 0$ at the end of the 90's in the context of color superconductivity (Alford, Rajagopal & Wilczek, PL B422, 247, 1997 and NP B537, 443, 1999; Rapp, Schafer, Shuriak & Velkowsky PRL 81, 53, 1998), that is diquark condensation, $<\psi\psi>$. At the same time Berges & Rajagopal NP B538, 215, 1999 studied the coexistence of the chiral and diquark condensates within a NJL model. They found also the existence of the tricritical point justifying it using an argument very close to the one used by Wilczek and Wilczek & Rajagopal for the existence of such a point for the strange quark case.

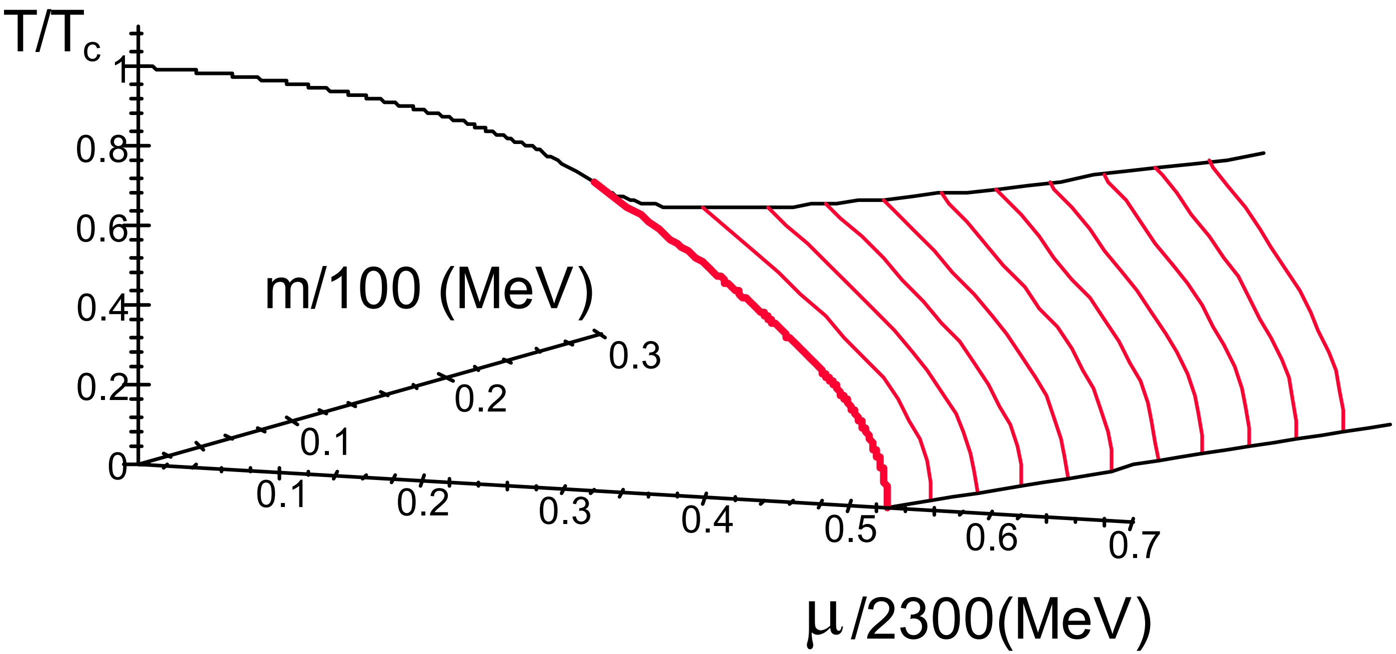


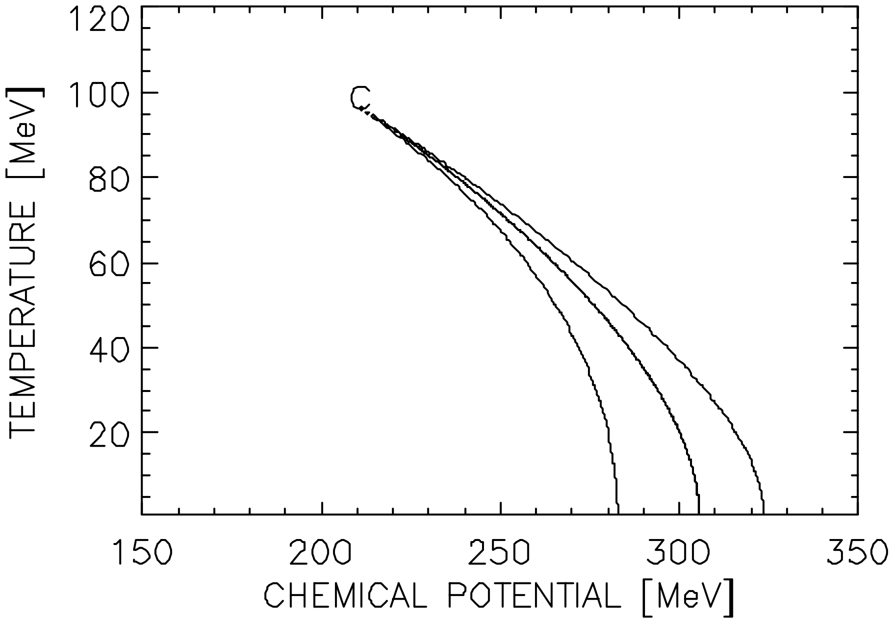
The idea is that at zero quark mass the theory belongs to the O(4) universality class (Ising Z_2 for m_q not 0) and this is not changed at finite density (Hsu & Schwetz, PL B432, 203,1998), but μ might change the coefficient of the quartic term forcing the introduction of a sixth order term in the condensate.

By doing so one gets a GL expansion of the effective 3d potential producing the tricritical point. Also, since the term ϕ^6 is marginal one expects corrections at most logarithmic to the critical exponents we have evaluated before.

After this paper by Berges and Rajagopal many other model calculations were made to verify the existence of the tricritical point:

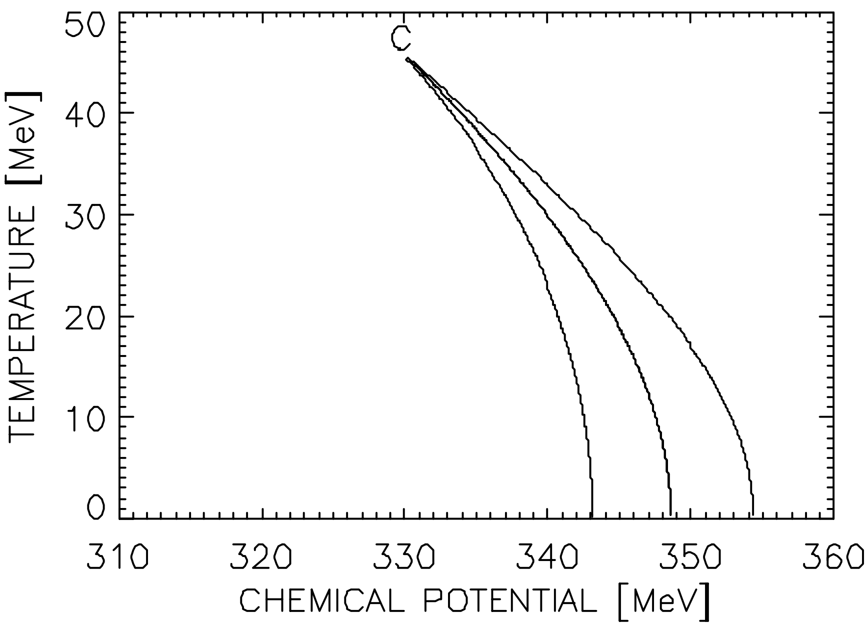
Halasz et al., PRD 58, 096007,1998 **through a random matrix model**



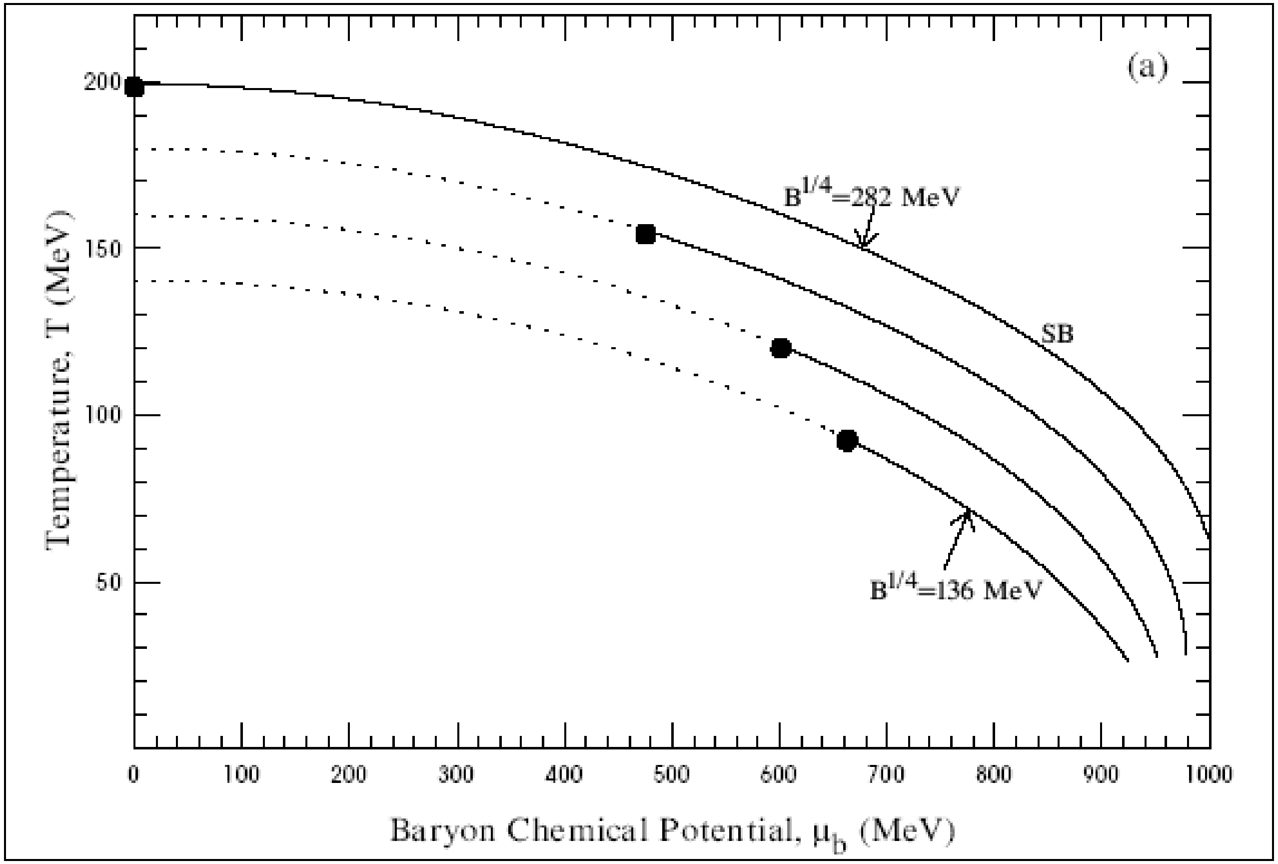


Linear σ-model

Scavenius et al., PRC 64, 045202, 2001

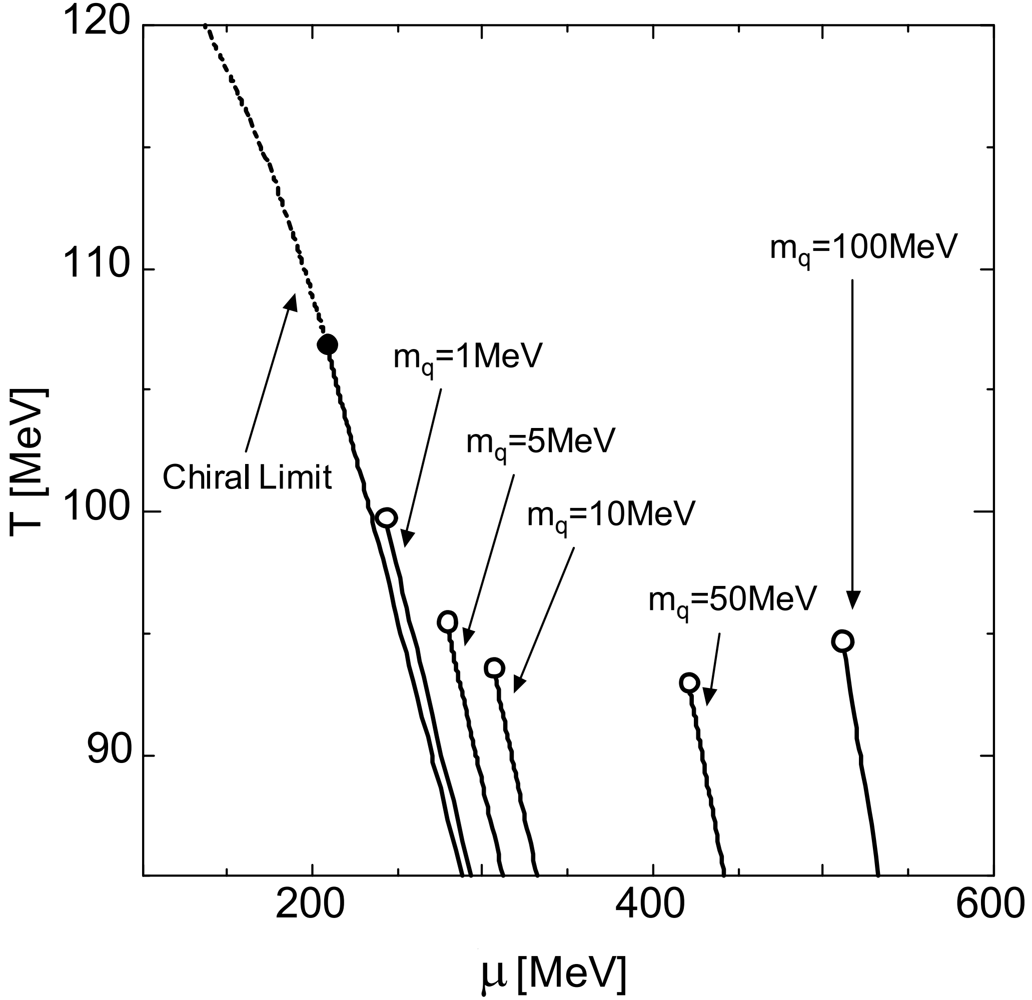


NJL model



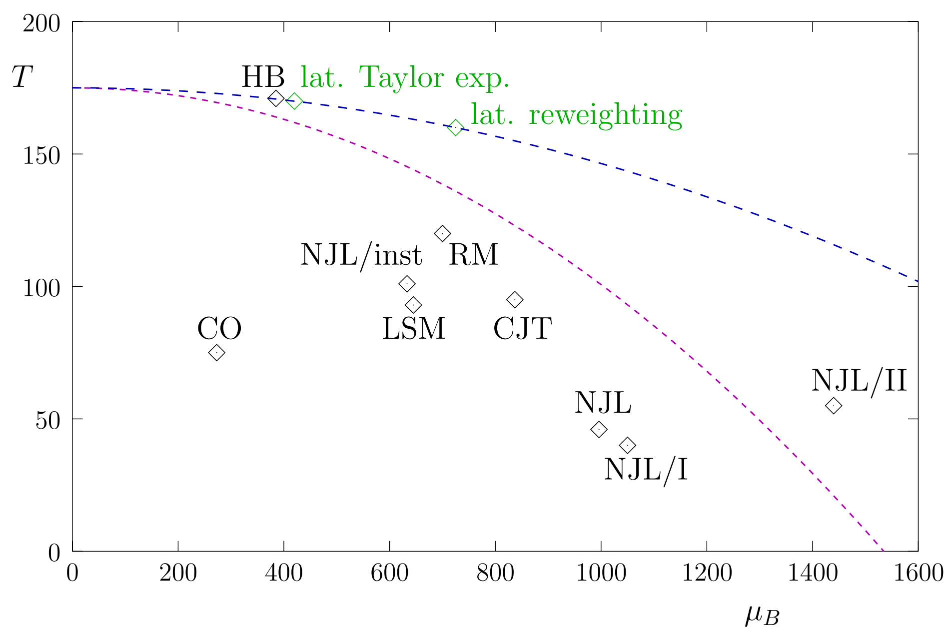
B = bag constant

Antoniou & Kapoyannis, PLB 563, 165, 2003 through statistical bootstrap (clusters made by clusters of the same mass spectrum)



Hatta & Ikeda, PRD 67, 014028, 2003, ladder-QCD and CJT effective potential

Source	(T, μ_B), MeV	Comments	Label
MIT Bag/QGP	none	*only 1st order, no chiral symmetry*	—
Asakawa,Yazaki '89	(40, 1050)	NJL, CASE I	NJL/I
"	(55, 1440)	NJL, CASE II	NJL/II
Barducci, *et al* '89-94	$(75, 273)_{\mathrm{TCP}}$	composite operator	CO
Berges, Rajagopal '98	$(101, 633)_{\mathrm{TCP}}$	instanton NJL	NJL/inst
Halasz, *et al* '98	$(120, 700)_{\mathrm{TCP}}$	random matrix	RM
Scavenius, *et al* '01	(93,645)	linear σ-model	LSM
"	(46,996)	NJL	NJL
Fodor, Katz '01	(160, 725)	lattice reweighting	
Hatta, Ikeda, '02	(95, 837)	effective potential (CJT)	CJT
Antoniou, Kapoyannis '02	(171, 385)	hadronic bootstrap	HB
Ejiri, *et al* '03	(?,420)	lattice Taylor expansion	



Compilation by
Stephanov, Progr. Theor. Phys. Suppl. 153, 139, 2004

Lattice calculations

It would be very nice if we could test all these ideas on the lattice. However the usual sampling method, based on a positive definite measure, does not work in presence of a chemical potential since the fermionic determinant turns out to be complex in euclidean space. We define euclidean variables through the following substitutions

$$x_0 \to -i x_E^4, \quad x^i \to x_E^i, \quad \gamma_0 \to \gamma_E^4, \quad \gamma^i \to -i\gamma_E^i$$

The Dirac operator with a chemical potential term is

$$D(\mu) = \gamma_E^\mu D_E^\mu + \mu \gamma_E^4, \qquad D_E^\mu = \partial_E^\mu + i A_E^\mu$$

At $\mu=0$ the eigenvalues of $D(\mu)$ are pure imaginary and if $|\lambda>$ is an eigenvector, $\gamma_5|\lambda>$ corresponds to - λ_5, as follows from

$$D(0)^\dagger = -D(0), \qquad \gamma_5 D(0) \gamma_5 = -D(0)$$

Therefore

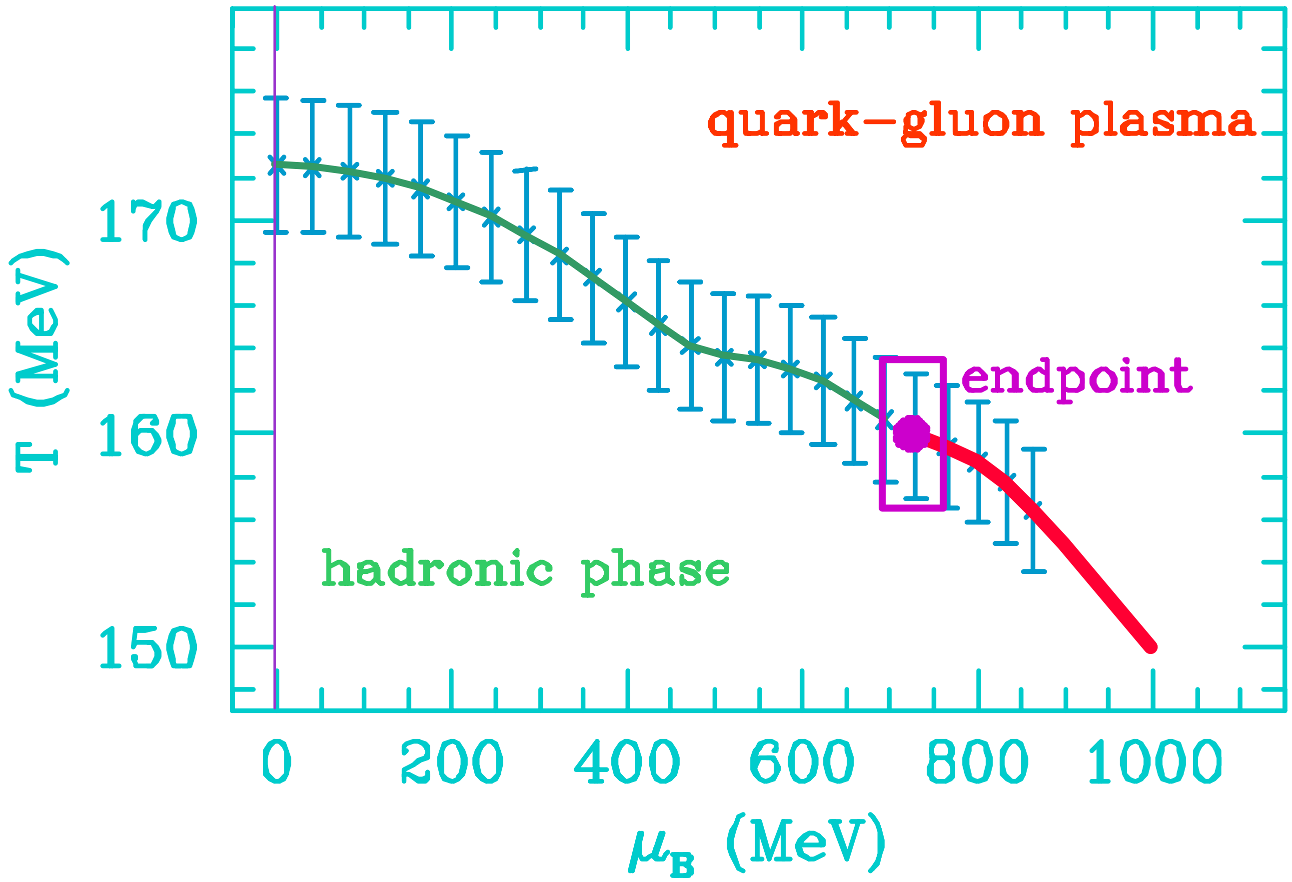
$$\mathrm{det}[D(0)] = \prod_{\lambda}(\lambda)(-\lambda) > 0$$

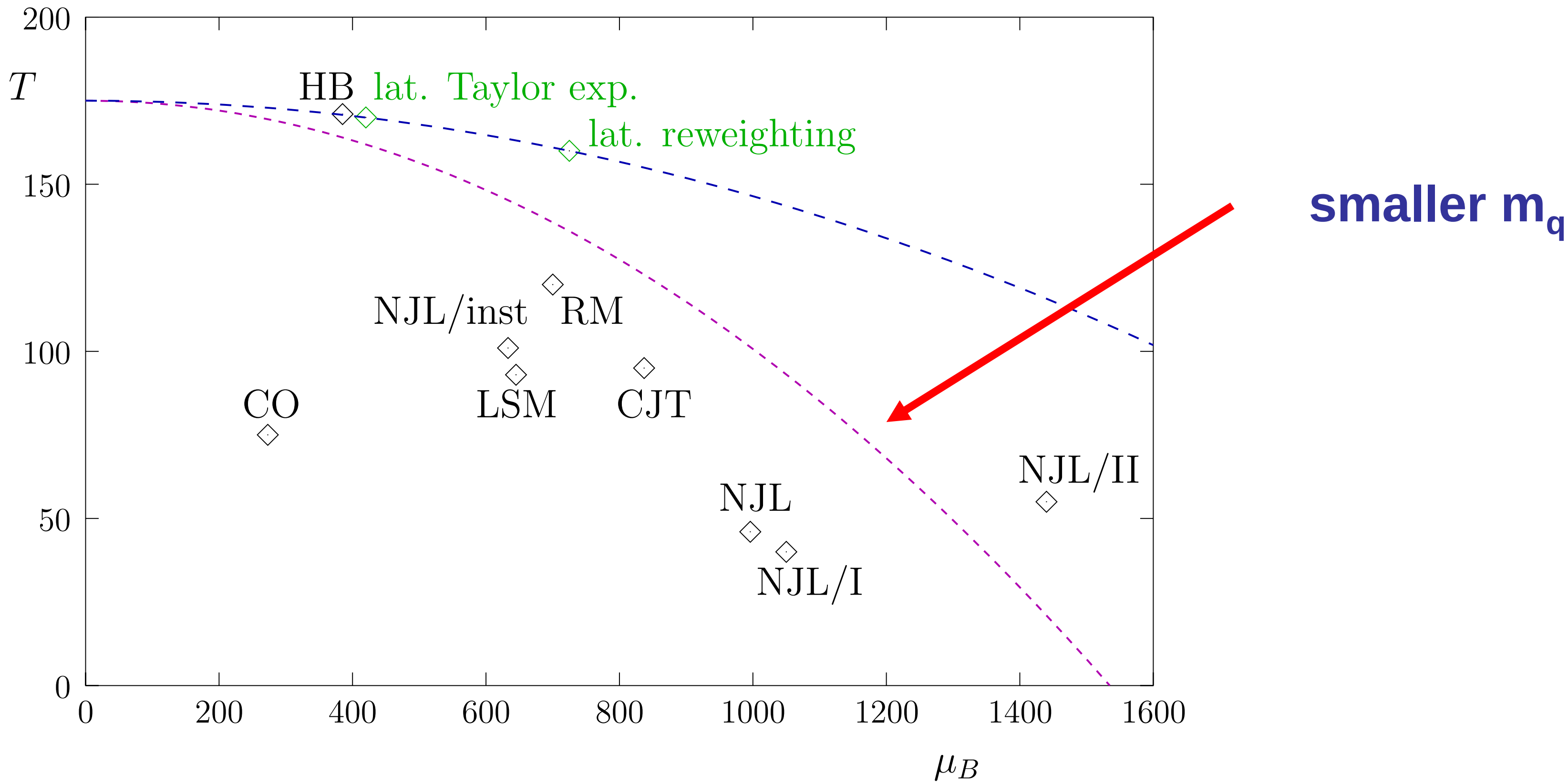
At μ not 0 this argument does not hold and the determinant is complex. This is true for the baryonic chemical potential, but if one considers the case of two degenerate flavors, u and d, and the isospin chemical potential associated to the conserved current τ_3, the positivity may be shown by using τ_1 in conjuction with the hermitian conjugate.

However recently there have been many attempts of evaluating the phase diagram for non zero chemical potential on the lattice. A partial list is given in the following:

Reweighting method: in practice very difficult to estimate the accuracy

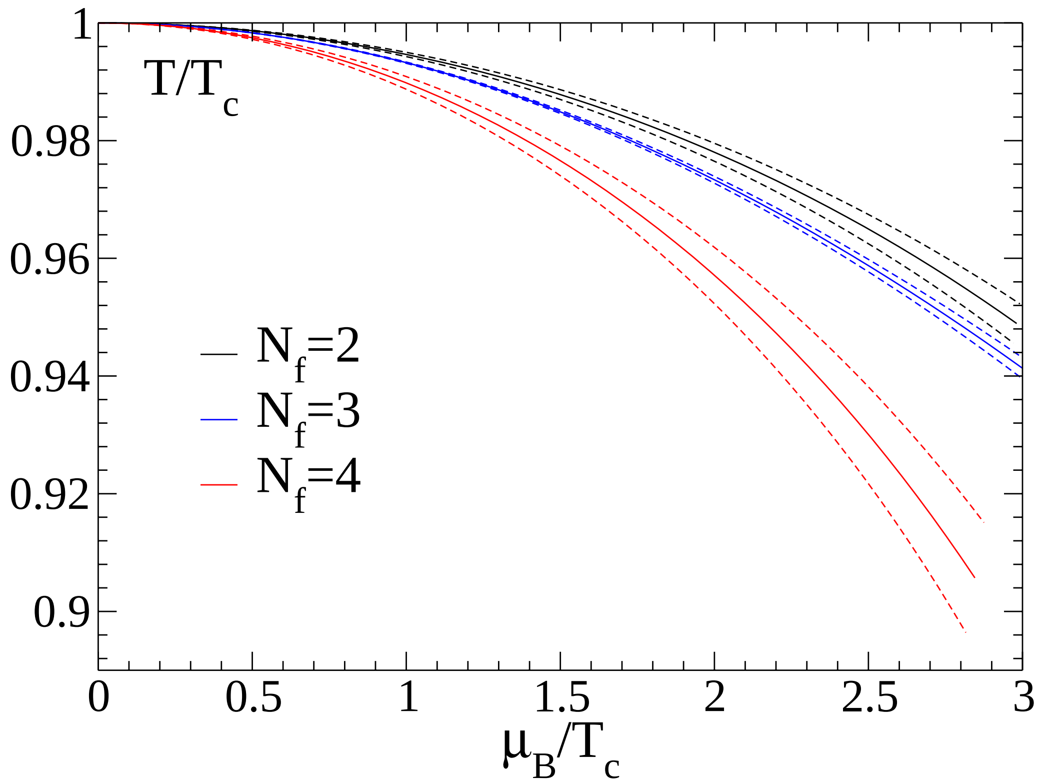
Fodor & Katz, JHEP 0203, 014, 2002



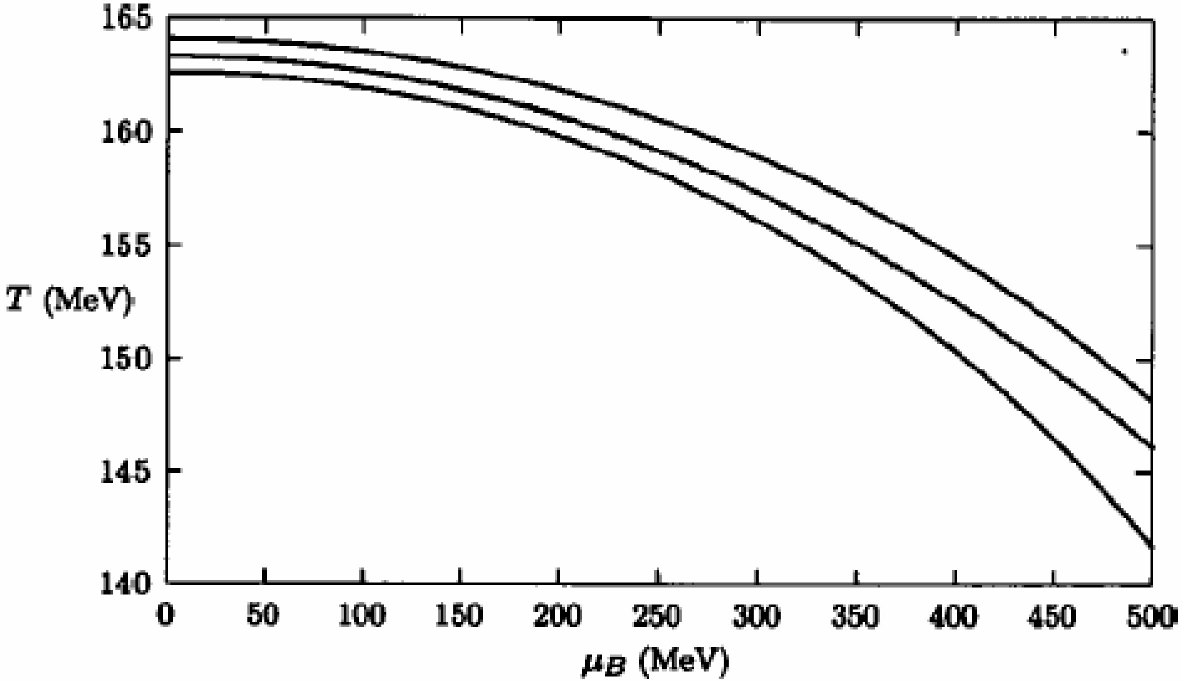


Taylor expansion for small μ, Ejiri &al., Prog. Theor. Phys. Suppl. 153, 118, 2004.

Use of immaginary values of the chemical potential



Forcrand & Philipsen, NPB 673, 170, 2003

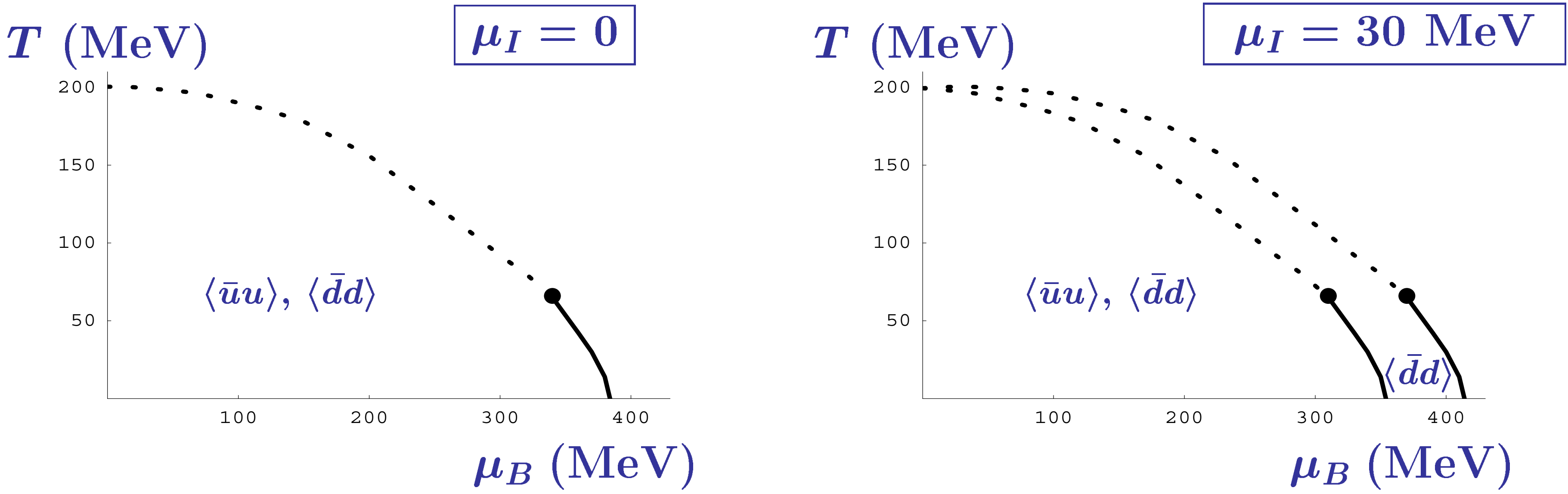


D'Elia & Lombardo, PRD 67, 014505, 2003

N_f =4

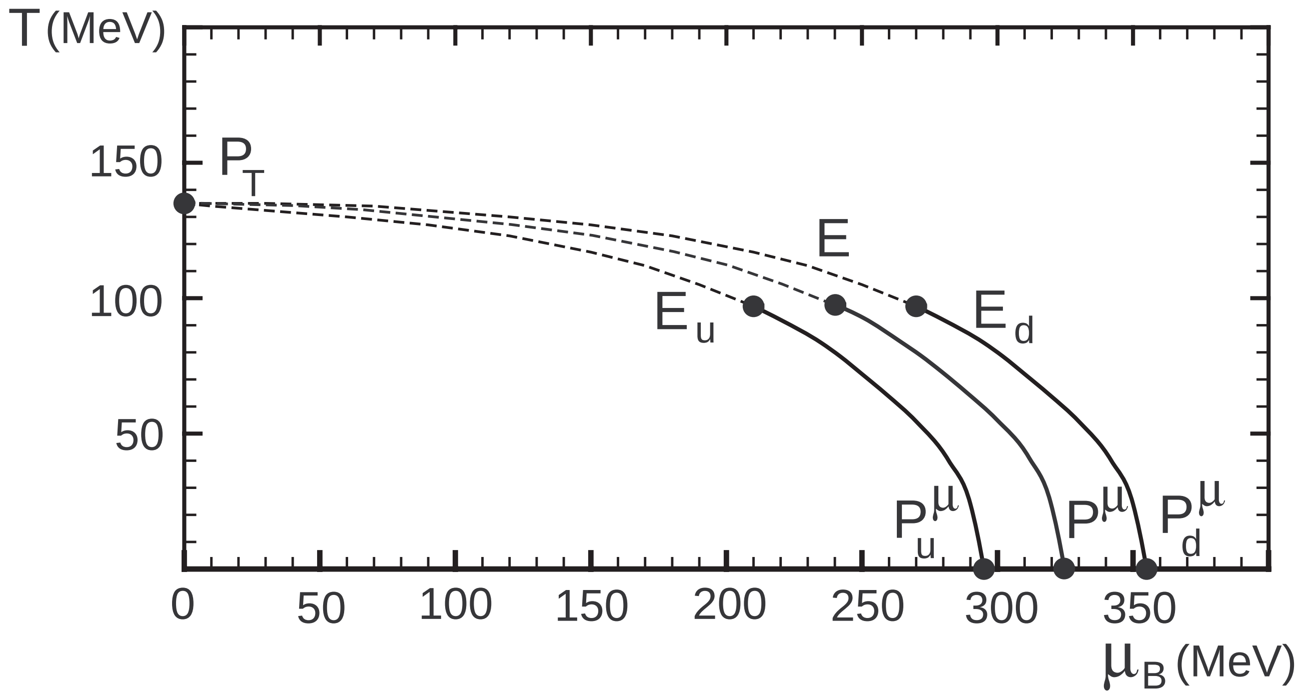
Isospin chemical potential

Klein, Toublan & Verbaarshot, PRD 68, 014009, 2003, using a random matrix model, and Toublan & Kogut, PLB 564, 212, 2003, found a splitting of the critical line



NJL

Also found in ladder QCD, Barducci & al. PLB 564, 217, 2003



However if the two-flavor mixing is too strong the effect could vanish (Frank, Buballa & Oertel, PLB 562, 221, 2003)

Conclusions

- **A lot of progress in understanding the phase diagram of QCD, but still at a qualitative level.**
- **For a more quantitative understanding need of big improvement on the lattice calculations (unless one might devise some controllable analytical method)**