NC Geometry

and

Hopf Algebras

Integration over algebraic structures

Roberto Casalbuoni

Università di Firenze

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Motivations

Classical phase space

$$\Big\downarrow{\scriptstyle QM}$$

Noncommutative algebra

Notion of classical phase space is retained in the path-integral approach, but *Supersymmetry* changes this picture. In 1973 Volkov and Akulov noticed that SUSY transformations could be obtained by requiring the invariance of the following differential form

$$\omega_\mu = dx_\mu - i\bar{\theta}\sigma_\mu d\theta + id\bar{\theta}\sigma_\mu\theta$$

where θ is a Grassmann spinor. This suggests the description of a superparticle:

$$S = -\int\sqrt{dx^2} \to S = -\int\sqrt{\omega^2}$$

requiring to extend the path-integral to a non-commuting phase space.

Done via the Berezin integration over the Grassmann variables

$$\int d\theta\theta = 1, \quad \int d\theta 1 = 0$$

What is the motivation for this rule ?

The standard argument is translational invariance. Because this implies the Schwinger's quantum action principle:

$$0 = \int_{q_i,t_i}^{q_f,t_f} d\mu[q] \frac{\delta}{\delta q(t)} \left(F[q] e^{iS[q]} \right)$$

$$\downarrow$$

$$\langle q_f, t_f | T \left(F[q] \frac{\delta S}{\delta q(t)} \right) | q_i, t_i \rangle = i \langle q_f, t_f | T \left(\frac{\delta F[q]}{\delta q(t)} \right) | q_i, t_i \rangle$$

$F[q] = 1 \rightarrow$ equations of motion

$F[q] = q \rightarrow$ canonical commutators

But why should we insist on this principle when looking at possible generalizations as, for instance, quantum groups, M-theory, etc. ?

A more fundamental property seems to us the combination law for the probability amplitudes In the path-integral approach this follows from the factorization property of the measure, implying ($t_i \leq t \leq t_f$)

$$\langle q_f, t_f | q_i, t_i \rangle = \int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle$$

This property is equivalent to the *completeness* of the states $|q, t\rangle$

$$\int dq |q, t\rangle \langle q, t| = 1$$

On the other hand the path-integral quantization can be reconstructed by using repeatedly the completeness relation.

We will define the integration over an algebraic structure by requiring the validity of the completeness relation

No attempt to a complete mathematical discussion, only a set of operational rules. Also, no path-integral construction.

To define integration over a noncommutative space we follow the spirit of noncommutative geometry (Connes) based on Gel'fand-Naimark theorem:

classical space S

(locally compact Hausdorff space)

$$\downarrow$$

algebra of functions $f : S \to C$

These functions form a commutative algebra. Going to a noncommutative algebra one can define a geometry although, in general, there is no underlying classical space.

To implement these ideas in our context we show how to lift up the completeness relation in the configuration space to a relation on the space of its functions.

Start with

$$\int |x\rangle\langle x|dx = 1$$

and with an orthonormal set $\{|\psi_n\rangle\}$ of states. We can convert the completeness into the orthogonality relation for the functions $\psi_n(x) = \langle x|\psi_n\rangle$

$$\int \langle\psi_m|x\rangle\langle x|\psi_n\rangle dx = \int \psi_m^*(x)\psi_n(x)dx = \delta_{mn}$$

On the other hand, given this relation, and the completeness for the states, one can reconstruct the completeness in the x-space. The set $\{\psi_n(x)\}$ has the following properties:

- The set $\{\psi_n(x)\}$ spans a vector space.

- The product $\psi_m(x)\psi_n(x)$ can be expressed as a linear combination of $\psi_p(x)$ since the set is complete.

That is

The set $\{\psi_n(x)\}$ is an algebra

From completeness

$$\psi_n^*(x) = \sum_m C_{nm}\psi_m(x)$$

The conditions on C will be discussed later. The orthogonality becomes

$$\int \sum_m C_{nm}\psi_m(x)\psi_p(x)dx = \delta_{np}$$

For an arbitrary algebra with elements x_i the previous relation becomes

$$\int_{(x)} \sum_j C_{ij}x_j x_k = \delta_{ik}$$

This is the relation that we will require to define the integration over an algebra.

Example:

Consider a one-dimensional free particle:

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

The algebraic product rules are

$$\psi_k(x)\psi_{k'}(x) = \frac{1}{\sqrt{2\pi}} \psi_{k+k'}(x)$$

The conjugation gives

$$\psi_k^*(x) = \int dk' \delta(k+k')\psi_{k'}(x)$$

or

$$C_{kk'} = \delta(k+k')$$

Up to here the space x have played no role. We *define* the integration over x by requiring

$$\int_{(x)} \left(\int dk' \delta(k" + k')\psi_{k'}(x) \right) \psi_k(x) = \delta(k - k")$$

Using the algebra

$$\frac{1}{\sqrt{2\pi}} \int_{(x)} \psi_{k-k"}(x) = \frac{1}{2\pi} \int_{(x)} e^{i(k-k")x} = \delta(k - k")$$

Algebras

ALGEBRA $\mathcal{A}$: A vector space equipped with a bilinear mapping:

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

Basis of $\mathcal{A}$: $\{x_i,\ i = 0, 1, \cdots, n\}$ (possibly $n \to \infty$, or a continuous index).

Structure constants:

$$\boxed{x_i x_j = f_{ijk} x_k}$$

characterize completely the algebra. It is convenient to introduce vectors $\langle x| = (x_0, x_1, \cdots, x_n)$ and $|x\rangle$.

A very important tool is the *algebra of the left and right multiplications*. The associated matrices are:

$$\boxed{R_i|x\rangle = |x\rangle x_i, \qquad \langle x|L_i = x_i\langle x|}$$

We will use also:

$$L_i^T|x\rangle = x_i|x\rangle$$

Left and right multiplications encode the properties of the structure constants. From their definition we have:

$$(R_i)_{jk} = f_{jik}, \quad (L_i)_{jk} = f_{ikj}$$

Since left and right multiplications are linear operators we have

$$R_a = \sum_{i=1}^{n} a_i R_i \quad \text{if} \quad a = \sum_{i=1}^{n} a_i x_i \in \mathcal{A}$$

Examples:

Abelian algebras:

$$R_i|x\rangle = \underbrace{|x\rangle x_i = x_i|x\rangle}_{x_i x_j = x_j x_i} = L_i^T|x\rangle \Rightarrow \boxed{R_i = L_i^T}$$

Associative algebras:

$$(|x\rangle x_i)x_j = R_i|x\rangle x_j = R_i R_j|x\rangle$$

$$|x\rangle(x_i x_j) = f_{ijk}|x\rangle x_k = f_{ijk}R_k|x\rangle$$

$$\boxed{R_i R_j = f_{ijk}R_k}$$

in analogous way

$$\boxed{L_i L_j = f_{ijk}L_k, \qquad [R_i, L_j^T] = 0}$$

R_i and L_i are representations of $\mathcal{A}$ (*regular*).

Integration rules

In the following we will consider associative algebras with identity. If the representations of $\mathcal{A}$ generated by R_i and L_i happen to be equivalent, there exists a matrix C such that

$$\boxed{R_i = C^{-1} L_i C}$$

In this case one has

$$L_i|Cx\rangle = |Cx\rangle x_i, \qquad |Cx\rangle = C|x\rangle$$

as it follows from

$$R_i|x\rangle = C^{-1} L_i C|x\rangle = |x\rangle x_i$$

We define the integration over the algebra in such a way that the following identity is satisfied:

$$\boxed{\int_{(x)} |Cx\rangle\langle x| = 1}$$

In components

$$\int_{(x)} C_{ij} x_j x_k = C_{ij} f_{jkp} \int_{(x)} x_p = \delta_{ik}$$

Trouble !!!! $(n+1)^2$ equations for $n+1$ unknown quantities

$$\int_{(x)} x_p$$

But taking $x_k = x_0 = I$, we get

$$\int_{(x)} C_{ij} x_j = \delta_{i0} \Rightarrow \int_{(x)} x_i = (C^{-1})_{i0}$$

This is the **general solution**.

The reason is:

$$\int_{(x)} C_{ij} x_j x_k = C_{ij} f_{jkp} \int_{(x)} x_p = C_{ij} (R_k)_{jp} (C^{-1})_{p0}$$

implying

$$\int_{(x)} C_{ij} x_j x_k = (C R_k C^{-1})_{i0} = (L_k)_{i0} = f_{k0i} = \delta_{ik}$$

since

$$x_i\, x_0 = x_i \longrightarrow f_{i0j} = \delta_{ij}$$

If $C^T = C$ holds, we have also

$$\int_{(x)} |x\rangle\langle Cx| = \int_{(x)} |C^{-1}Cx\rangle\langle x| C^T = C^{-1} C^T = 1$$

This allows the definition of a scalar product. Put

$$f(x) = \sum_i f_i x_i, \quad f^\star(x) = \sum_{ij} \bar{f}_i C_{ij} x_j$$

and

$$|f\rangle = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ f_n \end{pmatrix}, \qquad \langle f| = (\bar{f}_0, \bar{f}_1, \cdots, \bar{f}_n)$$

Then

$$\langle f|g\rangle = \int_{(x)} \langle f|Cx\rangle\langle x|g\rangle = \int_{(x)} f^\star(x) g(x) =$$

$$= \sum_i \bar{f}_i \, g_i$$

Examples:

Paragrassmann algebras

$\mathcal{G}_1^p$: $(1,\theta) \longrightarrow \theta^{p+1} = 0$, $x_i = \theta^i$, $f_{ijk} = \delta_{i+j,k}$

$$\theta^i\theta^j = \theta^{i+j}, \qquad i,j,i+j = 0,1,\cdots,p$$

$$= 0 \qquad\qquad \text{otherwise}$$

C matrix: $\theta^i \longrightarrow \theta^{p-i} \longrightarrow C_{ij} = \delta_{i+j,p}$

$$C^T = C, \qquad C^2 = 1$$

$$\boxed{\int_{(\theta)} \theta^i = (C^{-1})_{0i} = \delta_{i,p}}$$

$$\int_{(\theta)} 1 = \int_{(\theta)} \theta = \cdots \int_{(\theta)} \theta^{p-1} = 0$$

$$\int_{(\theta)} \theta^p = 1$$

To find C look for eigenbras of L_i:

$$L_i|Cx\rangle = x_i|Cx\rangle \Rightarrow (L_i)_{jk}(Cx)_k = x_i(Cx)_j$$

In this case ($x_i = \theta^i$):

$$f_{ikj}(Cx)_k = \delta_{i+k,j}(Cx)_k = \theta^i(Cx)_j$$

This equation is satisfied by

$$(Cx)_k = \theta^{p-k}$$

in fact

$$\delta_{i+k,j}\theta^{p-k} = \theta^{p-(j-i)}$$

$$\theta^i\theta^{p-j} = \theta^{p-j+i}$$

It follows

$$C_{ij}\theta^j = \theta^{p-i} \Rightarrow C_{ij} = \delta_{i+j,p}$$

Matrix algebra

Algebra $\mathcal{A}_N$ of the $N \times N$ matrices.

$$A = \sum_{n,m=1}^{N} e^{(nm)} a_{nm}$$

$$e^{(nm)}_{ij} = \delta^n_i \delta^m_j, \qquad e^{(nm)} e^{(pq)} = \delta_{mp} e^{(nq)}$$

C matrix: $e^{(nm)} \to e^{(mn)} \to C_{(mn)(rs)} = \delta_{ms} \delta nr$

$$C^T = C, \quad C^2 = 1$$

$$I = \sum_{n=1}^{N} e^{(nn)}$$

$$\boxed{\int_{(e)} e^{(rs)} = \sum_n (C^{-1})_{(nn)(rs)} = \sum_n \delta_{ns} \delta_{nr} = \delta_{rs}}$$

$$\boxed{\int_{(e)} A = \sum_{n,m=1}^{N} a_{nm} \int_{(e)} e^{(nm)} = Tr(A)}$$

In this case C is an involution. Define

$$e^{(mn)^\star} = e^{(pq)} C_{(pq)(mn)} = e^{(nm)}$$

and

$$A^\star = \sum_{n,m=1}^{N} e^{(mn)^\star} \bar{a}_{mn} = \sum_{n,m=1}^{N} e^{(nm)} \bar{a}_{mn} = A^\dagger$$

Derivations

A derivation is a linear mapping on the algebra $D : \mathcal{A} \to \mathcal{A}$, such that

$$D(ab) = (Da)b + a(Db), \quad a, b \in \mathcal{A}$$

We define the matrix d associated to D as

$$Dx_i = d_{ij}x_j$$

If D is a derivation,

$$S = e^{\alpha D}$$

is an algebra automorphism:

$$e^{\alpha D}(ab) = (e^{\alpha D}a)(e^{\alpha D}b)$$

On the contrary if $S(\alpha)$ is a continuous automorphism, then

$$D = \lim_{\alpha \to 0} \frac{S(\alpha) - 1}{\alpha}$$

is a derivation.

The following theorem generalizes the usual integration by part formula

THEOREM:

If D is such that $\int_{(x)} Df(x) = 0$, then the integral is invariant under the related automorphism $\exp(\alpha D)$ *and viceversa.* Also under the hypothesis of the theorem one has the following identities:

$$\boxed{CdC^{-1} = -d^T}$$

And by exponentiation

$$\boxed{C \exp(\alpha d) C^{-1} = \exp(-\alpha d^T)}$$

here $s(\alpha) = \exp(\alpha d)$ is the matrix of the automorphism $S(\alpha) = \exp(\alpha D)$.

Proof:

If $\int_{(x)} Df(x) = 0$, then

$$0 = \int_{(x)} D(|Cx\rangle\langle x|) =$$

$$= \int_{(x)} Cd|x\rangle\langle x| + \int_{(x)} C|x\rangle\langle Dx| =$$

$$= CdC^{-1} + \int_{(x)} |Cx\rangle\langle x|d^T$$

$$\Downarrow$$

$$\boxed{CdC^{-1} = -d^T}$$

By exponentiation

$$\boxed{C\,\mathrm{exp}(\alpha d)C^{-1} = \mathrm{exp}(-\alpha d^T)}$$

here $s(\alpha) = \mathrm{exp}(\alpha d)$ is the matrix of the automorphism $S(\alpha) = \mathrm{exp}(\alpha D)$.

Using the previous relation ($Cs = s^{T-1}C$)

$$1 = \int_{(x)} s^{T-1}|Cx\rangle\langle x|s^T = \int_{(x)} Cs|x\rangle\langle x|s^T =$$

$$= \int_{(x)} C|Sx\rangle\langle Sx| = \int_{(x)} |Cx'\rangle\langle x'|$$

where $x' = Sx$. But for any automorphism of the algebra we have (see next slide)

$$\int_{(x')} |Cx'\rangle\langle x'| = 1$$

Therefore

$$\boxed{\int_{(x')} = \int_{(x)}}$$

On the contrary if the measure is invariant, then

$$1 = \int_{(x')} |Cx'\rangle\langle x'| = \int_{(x)} |Cx'\rangle\langle x'| =$$

$$= \int_{(x)} Cs|x\rangle\langle x|s^T = CsC^{-1}s^T$$

implying $CsC^{-1} = s^{T-1}$ and equivalently

$$CdC^{-1} = -d^T \rightarrow \int_{(x)} Df(x) = 0$$

Since an algebra automorphism $a \to a'$ leaves invariant the product rules:

$$ab = c \Rightarrow a'b' = c'$$

the linear mappings defined by the right and left multiplications (and their matrices) are unchanged. Therefore *under automorphisms*

Integration rules are invariant

In other words,

$$\boxed{\int_{(x')} f(x') = \int_{(x)} f(x)}$$

One can prove also the **THEOREM:**

An associative self-conjugated algebra with identity, and with $C^T = C$, has a set of derivations leading to an invariant integration measure, the inner derivations.

In this case the inner derivations are given by

$$D_a x_i = [x_i, a], \quad a \in \mathcal{A}$$

or

$$d_a = R_a - L_a^T$$

$$(d_a|x\rangle = (R_a - L_a^T)|x\rangle = |x\rangle a - a|x\rangle)$$

It follows

$$C d_a C^{-1} = C(R_a - L_a^T)C^{-1} =$$

$$L_a - (C^{T-1} L_a C^T)^T = L_a - R_a^T = -d_a^T$$

For a matrix algebra

$$\int_{(e)} D_B A = \int_{(e)} [A, B] = Tr\,[[A, B]] = 0$$

Integration over a subalgebra

Given self-conjugated algebra $\mathcal{A}$ and subalgebra $\mathcal{B}$, we would like to recover the integration over $\mathcal{B}$ in terms of the integration over $\mathcal{A}$. Given the decomposition

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$$

look for $P \in \mathcal{A}$ such that

$$\int_{\mathcal{A}} \mathcal{C}P = 0, \quad \int_{\mathcal{A}} \mathcal{B}P = \int_{\mathcal{B}} \mathcal{B}$$

or

$$\int_{\mathcal{A}} \mathcal{A}P = \int_{\mathcal{A}} \mathcal{B}P = \int_{\mathcal{B}} \mathcal{B}$$

These are as many conditions as the basis elements of $\mathcal{A}$. P is uniquely determined.

EX: *Grassmann algebra*:

$\mathcal{G}_1$ can be embedded in $\mathcal{A}_2$ (2×2 matrices)

$$\theta \to \sigma_+, \qquad 1 \to 1_2$$

Then

$$P = \sigma_-$$

In fact, given

$$A = a + b\sigma_3 + c\sigma_+ + d\sigma_-$$

one has uniquely: $A = f(\theta) + C$

$$f(\theta) = a - b + c\sigma_+, \quad C = b(1 + \sigma_3) + d\sigma_-$$

and

$$\int_{\mathcal{A}_2} CP = Tr[C\sigma_-] = 0$$

$$\boxed{\int_\theta f(\theta) = Tr[f(\sigma_+)\sigma_-]}$$

that is

$$\int_\theta 1 = Tr[\sigma_-] = 0, \qquad \int_\theta \theta = Tr[\sigma_+\sigma_-] = 1$$

Projective group algebras

Given a group G and an arbitrary projective linear representation $\mathcal{A}(G)$:

$$a \rightarrow x(a), \quad a \in G, \quad x(a) \in \mathcal{A}(G)$$

$\mathcal{A}(G)$ has a natural algebra structure:

$$x(a)x(b) = e^{i\overbrace{\alpha(a,b)}^{\text{cocycle}}} x(ab)$$

The product for an arbitrary element of the algebra, $\sum_{a\in G} f(a)x(a)$, is defined by linearity. The cocycle is constrained by the associativity of the algebra. In particular, $\alpha(ab, b^{-1}) = \alpha(b^{-1}, a^{-1})$

Structure constants: $f_{abc} = \delta_{ab,c}\, e^{i\alpha(a,b)}$

C matrix: $C_{ab} = \delta_{ab,e}$ (e is the identity in G),with

$$C^T = C, \quad C^2 = 1$$

Also in this case C is an involution. Define

$$x(a)^\star = C_{ab}x(b) = x(a)^{-1} = x(a^{-1})$$

For a generic element of the algebra:

$$f^\star = \sum_{a\in G} \bar{f}_a x(a^{-1})$$

Integration Rule

$$\boxed{\int_{(x)} x(a) = C^{-1}_{a,e} = \delta_{a,e}}$$

Take $G = R^D$ and $\mathcal{A}(G)$ a *vector* representation, then

$$x(\vec{a}) = e^{i\vec{q}\cdot\vec{a}}$$

and

$$\int_{(x)} e^{i\vec{q}\cdot\vec{a}} = \delta^D(\vec{a}) \Rightarrow \int_{(x)} = \int \frac{d^D\vec{q}}{(2\pi)^D}$$

Consider the set

$\widehat{G}$= The set of equivalence classes of the unitary IR's of G.

and the mapping $\widehat{f} : \widehat{G} \to \mathcal{A}(G)$

$$\widehat{f}(\lambda) = \sum_{a \in G} f(a) x_\lambda(a)$$

This can be thought as the **Fourier Transform** (FT) of $f : G \to C$. The algebraic integration can be used to invert this formula

$$f(a) = \int_{(x_\lambda)} \widehat{f}(\lambda) x_\lambda(a^{-1})$$

Therefore $\int_{(x_\lambda)}$ must be related to a *sum over the representations* of G.

For discrete groups one has the inversion formula

$$f(a) = \frac{1}{n_G} \sum_{\lambda \in \widehat{G}} d_\lambda tr \left[\hat{f}(\lambda) x_\lambda(a^{-1}) \right]$$

where n_G= *order of* G, d_λ = *dimension of the IR* λ. It follows:

$$\boxed{\int_{(x_\lambda)} (\cdots) = \frac{1}{n_G} \sum_{\lambda \in \widehat{G}} d_\lambda tr \left[\cdots \right]}$$

Also for compact groups:

$$\hat{f}(\lambda) = \int_G d\mu(a) f(a) x_\lambda(a)$$

where $d\mu(a)$ is the Haar measure, one has

$$f(a) = \sum_{\lambda \in \widehat{G}} d_\lambda tr \left[\hat{f}(\lambda) x_\lambda(a^{-1}) \right]$$

and again

$$\boxed{\int_{(x_\lambda)} (\cdots) = \sum_{\lambda \in \widehat{G}} d_\lambda tr \left[\cdots \right]}$$

Consider now *projective representations*. One has:

$$\widehat{f}_1(\lambda)\widehat{f}_2(\lambda) = \sum_{a\in G} h(a) x_\lambda(a)$$

where

$$h(a) = \sum_{b\in G} f_1(b) f_2(b^{-1}a) \underbrace{e^{i\alpha(b,b^{-1}a)}}$$

This proves the **Theorem**:

The FT of the *deformed* convolution product is equal to the product of the FT's.

Consider now the case of the abelian group $G = R^D$. Its vector IR's are given by the characters:

$$\chi_{\vec{q}}(\vec{a}) = e^{i\vec{q}\cdot\vec{a}}$$

In fact

$$\chi_{\vec{q}}(\vec{a})\chi_{\vec{q}}(\vec{b}) = \chi_{\vec{q}}(\vec{a} + \vec{b})$$

But also

$$\chi_{\vec{q}_1}(\vec{a})\chi_{\vec{q}_2}(\vec{a}) = \chi_{\vec{q}_1+\vec{q}_2}(\vec{a})$$

The space $\widehat{G}$ is an abelian group, the **dual** *of G*. In this case $\hat{G} = R^D$, for $G = Z^D$, $\hat{G} = T^D$ (the D-dimensional torus) ...

Usually one defines the FT in terms of the characters:

$$\tilde{f}(\vec{q}) = \sum_{a \in G} f(a) \chi_{\vec{q}}(a)$$

Then the FT of the deformed convolution product is:

$$\tilde{h}(\vec{q}) = \sum_{a \in G} h(a) \chi_{\vec{q}}(\vec{a}) =$$

$$= \sum_{a,b \in G} f_1(a) f_2(b) e^{i\alpha(a,b)} \chi_{\vec{q}}(\vec{a} + \vec{b}) =$$

$$= e^{-i\alpha(\partial_{\vec{q}_1}, \partial_{\vec{q}_2})} \tilde{f}_1(\vec{q}_1) \tilde{f}_2(\vec{q}_2) \Big|_{\vec{q}_1 = \vec{q}_2 = \vec{q}} \equiv \tilde{f}_1 \star \tilde{f}_2(\vec{q})$$

where the $\star$ is the Moyal product. On the contrary one gets the usual product defining the Fourier transform in terms of the projective representations $x_{\vec{q}}(a)$

Consider again *Projective Representations* of $G = R^D$. They can be obtained from

$$x_{\vec{q}}(\vec{a}) = e^{-i\vec{q}\cdot\vec{a}}$$

assuming $\vec{q}$ an operator such that:

$$[q_i, q_j] = i\eta_{ij}$$

with η_{ij} c-numbers related to the cocycle by

$$\alpha(\vec{a}, \vec{b}) = -\frac{1}{2}\eta_{ij}a_i b_j$$

which follows from BCH

$$e^{-i\vec{q}\cdot\vec{a}}e^{-i\vec{q}\cdot\vec{b}} = e^{-i\eta_{ij}a_i b_j/2}e^{-i\vec{q}\cdot(\vec{a}+\vec{b})}$$

This is a very simple example of noncommutative geometry that we can deal with our methods starting from:

$$q_i = \int d^D\vec{a}\left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right)x_{\vec{q}}(\vec{a})$$

In this way q_i is an element of $\mathcal{A}(G)$ via the FT of a distribution over G. One gets*

$$q_i x_{\vec{q}}(\vec{a}) = i\nabla_i x_{\vec{q}}(\vec{a}), \qquad x_{\vec{q}}(\vec{a})q_i = i\bar{\nabla}_i x_{\vec{q}}(\vec{a})$$

$$\nabla_i = \frac{\partial}{\partial a_i} + i\alpha_{ij}a_j, \qquad \bar{\nabla}_i = \frac{\partial}{\partial a_i} - i\alpha_{ij}a_j$$

where $\alpha_{ij} = \alpha(\vec{e}_{(i)}, \vec{e}_{(j)})$ and $\vec{e}_{(i)}$ a basis in R^D. Using

$$\hat{f}(\vec{q}) = \int d^D\vec{a} f(\vec{a})x_{\vec{q}}(\vec{a})$$

*Remember that

$$x_{\vec{q}}(\vec{a})x_{\vec{q}}(\vec{b}) = e^{i\alpha(\vec{a},\vec{b})}x_{\vec{q}}(\vec{a}+\vec{b})$$

we get

$$[q_i, \hat{f}(\vec{q})] = \int d^D\vec{a}\,[-i\,(\bar{\nabla}_i - \nabla_i)\,f(\vec{a})]\,x_{\vec{q}}(\vec{a})$$

or

$$[q_i, \hat{f}(\vec{q})] = -2i\alpha_{ij}D_j\hat{f}(\vec{q})$$

Here D_j is a derivation defined, for any abelian group, by the equation

$$\vec{D}x_\lambda(\vec{a}) = -i\vec{a}x_\lambda(\vec{a})$$

From

$$D_j q_i = \int d^D\vec{a}\left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right)\left(-ia_j x_\lambda(\vec{a})\right)$$

one has

$$D_j q_i = \delta_{ij}$$

and

$$[q_i, q_j] = -2i\alpha_{ij} \Rightarrow \alpha_{ij} = -\frac{1}{2}\eta_{ij}$$

Another interesting result comes from

$$S(\vec{\alpha})x_{\vec{q}}(\vec{a}) = e^{\vec{\alpha}\cdot\vec{D}}x_{\vec{q}}(\vec{a}) =$$

$$= e^{-i\vec{\alpha}\cdot\vec{a}}x_{\vec{q}}(\vec{a}) = x_{\vec{q}+\vec{\alpha}}(\vec{a})$$

as it follows from

$$\int d^D\vec{a}\left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right)e^{\vec{\alpha}\cdot\vec{D}}x_{\vec{q}}(\vec{a}) = q_i + \alpha_i$$

Since

$$\int_{(x_\lambda)}\vec{D}x_{\vec{q}}(\vec{a}) = -i\vec{a}\int_{(x_\lambda)}x_{\vec{q}}(\vec{a}) = -i\vec{a}\delta^D(\vec{a}) = 0$$

follows that the measure is invariant under translations:

$$\int_{(\vec{q})} = \int_{(\vec{q}+\vec{\alpha})}$$

A similar analysis holds for $G = Z^D$ or $G = Z_n^D$ (Z_n the group of the n^{th} roots of unity).

There is a simple physical interpretation of the previous results. From the invariance of the measure one can show that

$$\hat{f}(\vec{q}+\vec{\alpha}) = e^{\vec{\alpha}\cdot\vec{D}}\hat{f}(\vec{q})$$

implying that on the group functions, one generates the gauge transformation

$$f(\vec{a}) \to e^{-i\vec{\alpha}\cdot\vec{a}} f(\vec{a})$$

On the other hand, the operator defined by

$$q_i x_{\vec{q}}(\vec{a}) = i\nabla_i x_{\vec{q}}(\vec{a})$$

can be interpreted as a covariant derivative operating upon the group functions

$$i\nabla_i = \frac{\partial}{\partial a_i} + i\alpha_{ij}a_j$$

Furthermore the previous gauge transformation induces the following transformation upon the gauge potential $\mathcal{A}_i = \alpha_{ij}a_j$

$$\mathcal{A}_i \to \mathcal{A}_i - \partial_i \Lambda$$

with $\Lambda = \vec{a}\cdot\vec{\alpha}$.

In the case $G = Z^D$, introduce a basis on the square lattice Z^D, $\vec{e}_{(i)}$, $i = 1, 2, \cdots, D$. Any element of the algebra can be made in terms of

$$U_i = x(\vec{e}_{(i)})$$

playing the same role of $\vec{q}$. The FT is defined by

$$\hat{f}(\vec{U}) = \sum_{\vec{m} \in Z^D} f(\vec{m}) x_{\vec{U}}(\vec{m})$$

Since the integration rules are

$$\int_{(\vec{U})} x_{\vec{U}}(\vec{m}) = \delta_{\vec{m},0}$$

The inverse FT is given by

$$f(\vec{m}) = \int_{(\vec{U})} \hat{f}(\vec{U}) x_{\vec{U}}(-\vec{m})$$

The commutator relation is now given by

$$U_i \hat{f}(\vec{U}) U_i^{-1} = e^{-2\alpha_{ij} D_j} \hat{f}(\vec{U})$$

where the derivation D_i acts as

$$D_i U_j = -i \delta_{ij} U_j$$

The analogue of the commutator among the components of $\vec{q}$ is now

$$U_i U_k U_i^{-1} U_k^{-1} = e^{2i\alpha_{ik}}$$

The automorphism group generated by $\vec{D}$ is

$$S(\vec{\phi}) x_{\vec{U}}(\vec{m}) = e^{\vec{\phi}\cdot\vec{D}} x_{\vec{U}}(\vec{m}) = e^{-\vec{\phi}\cdot\vec{m}} x_{\vec{U}}(\vec{m})$$

meaning $U_i \to S(\vec{\phi}) U_i = e^{-i\phi_i} U_i$. This corresponds to a trivial cocycle, and it can be seen by inverse FT that it generates a phase transformation on the group functions $f(\vec{m}) \to e^{i\vec{\phi}\cdot\vec{m}} f(\vec{m})$. This case might be also described in terms of $\vec{q}$, by $U_i = e^{-iq_i}$.

Compactification on noncommutative tori

$D0$-branes are point-like objects described by $N \times N$ hermitian matrices

$$X^{\mu}_{i_1,i_2},\ i_1, i_2 = 1, \cdots,\ \mu = 1, \cdots, 9$$

The compactification on a torus T^D, $(D < 9)$ difficult to require directly, since X^{μ} are dynamical variables. The solution is to notice that

$$T^D = R^D / Z^D$$

Then we describe the motion in R^D and mod out with Z^D. Technically take ∞ copies of the $D0$ brane via the extension

$$X^{\mu}_{i_1,i_2} \rightarrow X^{\mu}_{(i_1,a_1),(i_2,a_2)}$$

with

$$a_i = \sum_{\mu=1}^{D} m_i^{\mu} e_{(\mu)} \in Z^D$$

Therefore X^{μ} becomes a linear operator acting upon the regular representation of Z^D.

The quotient with Z^D is taken by requiring invariance with respect to

$$U(a)^{-1}X^\mu U(a) = X^\mu + a^\mu$$

In general

$$U(a)U(b) = e^{i\alpha(a,b)}U(a+b)$$

If $\alpha(a,b) \neq 0$ (mod trivial cocycles) we get a noncommutative torus. A solution to the compactification condition comes noticing that (follows from $D(R(x(a))|x\rangle) = D(|x\rangle x(a))$

$$R(x(a)^{-1})dR(x(a)) = d - R(x(a))^{-1}R(Dx(a))$$

where $x(a)$ is a IR and

$$R(x(a))|x\rangle = |x\rangle x(a)$$

and requiring

$$Dx(a) \propto x(a)$$

This is satisfied by

$$\vec{D}x(a) = -i\vec{a}x(a)$$

The most general solution is

$$X^\mu = -iD^\mu + A^\mu$$

with

$$[A^\mu, R(x(a))] = 0$$

Since the group algebra is associative, we recall that

$$[L^T(x(a)), R(x(b))] = 0$$

and therefore

$$X^\mu = -iD^\mu + \sum_{a \in Z^D} f^\mu(a) L^T(x(a))$$

It is easy to check

$$[X^\mu, X^\nu] = -L^T(F^{\mu\nu})$$

with

$$F^{\mu\nu} = D^\mu f^\nu - D^\nu f^\mu - [f^\mu, f^\nu]$$

and

$$f^\mu = \sum_{a \in Z^D} f^\mu(a) x(a) \in \mathcal{A}(Z^D)$$

Conclusions and outlook

- Definition of integration for a very large class of algebraic structures (associative self-conjugated algebras with identity).

- I have shown how to recover many known cases and in particular

  - Integration by part theorem can be generalized.

  - Integration over subalgebras is naturally defined.

- The method can be extended to other cases as the algebra of bosonic and q-oscillators, and also to nonassociative algebras as the algebra of octonions and the Jordan algebra over a bilinear form (Clifford algebra).

- The application to group algebras seems particularly interesting because it allows to define the calculus over some simple non-commutative space.

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