Notes on Algebras, Derivations and Integrals

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Chapter 1

Introduction

1.1 Motivations

The very idea of supersymmetry leads to the possibility of extending ordinary classical mechanics to more general cases in which ordinary configuration variables live together with Grassmann variables. More recently the idea of extending classical mechanics to more general situations has been further emphasized with the introduction of quantum groups, non-commutative geometry, etc. In order to quantize these general theories, one can try two ways: i) the canonical formalism, ii) the path-integral quantization. In refs. [1, 2] classical theories involving Grassmann variables were quantized by using the canonical formalism. But in this case, also the second possibility can be easily realized by using the Berezin's rule for integrating over a Grassmann algebra [3]. It would be desirable to have a way to perform the quantization of theories defined in a general algebraic setting. In this paper we will make a first step toward this construction, that is we will give general rules allowing the possibility of integrating over a given algebra. Given these rules, the next step would be the definition of the path-integral. In order to define the integration rules we will need some guiding principle. So let us start by reviewing how the integration over Grassmann variables come about. The standard argument for the Berezin's rule is translational invariance. In fact, this guarantees the validity of the quantum action principle. However, this requirement seems to be too technical and we would rather prefer to rely on some more physical argument, as the one which is automatically satisfied by the path integral representation of an amplitude, that is the combination law for probability amplitudes. This is a simple consequence of the factorization properties of the functional measure and of the additivity of the action. In turn, these properties follow in a direct way from the very construction of the path integral starting from the ordinary quantum mechanics. We recall that the construction consists in the computation of the matrix element $\langle q_f, t_f | q_i, t_i \rangle$, $(t_i < t_f)$ by inserting the completeness relation

$$\int dq \, |q, t\rangle\langle q, t| = 1 \tag{1.1}$$

inside the matrix element at the intermediate times t_a ($t_i < t_a < t_f$, $a = 1, \cdots, N$), and taking the limit $N \to \infty$ (for sake of simplicity we consider here the quantum mechanical case of a single degree of freedom). The relevant information leading to the composition law is nothing but the completeness relation (1.1). Therefore we will assume the completeness as the basic principle to use in order to define the integration rules over a generic algebra. In this paper we will limit our task to the construction of the integration rules, and we will not do any attempt to construct the functional integral in the general case. The extension of the relation (1.1) to a configuration space different from the usual one is far from being trivial. However, we can use an approach that has been largely used in the study of non-commutative geometry [4] and of quantum groups [5]. The approach starts from the observation that in the normal case one can reconstruct a space from the algebra of its functions . Giving this fact, one lifts all the necessary properties in the function space and avoids to work on the space itself. In this way one is able to deal with cases in which no concrete realization of the space itself exists. We will see in Section 2 how to extend the relation (1.1) to the algebra of functions. In Section 3 we will generalize the considerations of Section 2 to the case of an arbitrary algebra. In Section 4 we will discuss numerous examples of our procedure. The approach to the integration on the Grassmann algebra, starting from the requirement of completeness was discussed long ago by Martin [6].

1.2 The algebra of functions

Let us consider a quantum dynamical system and an operator having a complete set of eigenfunctions. For instance one can consider a one-dimensional free particle. The hamiltonian eigenfunctions are

$$\psi_k(x) = \frac{1}{\sqrt{2\pi}} \exp(-ikx) \tag{1.2}$$

Or we can consider the orbital angular momentum, in which case the eigenfunctions are the spherical harmonics $Y_\ell^m(\Omega)$. In general the eigenfunctions satisfy orthogonality relations

$$\int \psi_n^*(x)\psi_m(x)\, dx = \delta_{nm} \tag{1.3}$$

(we will not distinguish here between discrete and continuum spectrum). However $\psi_n(x)$ is nothing but the representative in the $\langle x|$ basis of the eigenkets $|n\rangle$ of the hamiltonian

$$\psi_n(x) = \langle x|n\rangle \tag{1.4}$$

Therefore the eq. (1.3) reads

$$\int \langle n|x\rangle\langle x|m\rangle\, dx = \delta_{nm} \tag{1.5}$$

which is equivalent to say that the $|x\rangle$ states form a complete set and that $|n\rangle$ and $|m\rangle$ are orthogonal. But this means that we can implement the completeness in the $|x\rangle$ space by means of the orthogonality relation obeyed by the eigenfunctions defined over this space. On the other side, given this equation, and the completeness relation for the set $\{|\psi_n\rangle\}$, we can reconstruct the completeness in the original space $\mathbb{R}^1$, that is the integration over the line. Now, we can translate the completeness of the set $\{|\psi_n\rangle\}$, in the following two statements

1. The set of functions $\{\psi_n(x)\}$ span a vector space.

2. The product $\psi_n(x)\psi_m(x)$ can be expressed as a linear combination of the functions $\psi_n(x)$, since the set $\{\psi_n(x)\}$ is complete.

All this amounts to say that the set $\{\psi_n(x)\}$ is a basis of an algebra. The product rules for the eigenfunctions are

$$\psi_m(x)\psi_n(x) = \sum_p c_{nmp}\psi_p(x) \tag{1.6}$$

with

$$c_{nmp} = \int \psi_n(x)\psi_m(x)\psi_p^*(x)\; dx \tag{1.7}$$

For instance, in the case of the free particle

$$c_{kk'k''} = \frac{1}{\sqrt{2\pi}}\delta(k+k'-k'') \tag{1.8}$$

In the case of the angular momentum one has the product formula [7]

$$\begin{aligned} Y_{\ell_1}^{m_1}(\Omega)Y_{\ell_2}^{m_2}(\Omega) &= \sum_{L=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\sum_{M=-L}^{+L}\left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}\right] \\ &\times \langle\ell_1\ell_2 00|L0\rangle\langle\ell_1\ell_2 m_1 m_2|LM\rangle Y_L^M(\Omega) \end{aligned} \tag{1.9}$$

where $\langle j_1 j_1 m_1 m_2|JM\rangle$ are the Clebsch-Gordan coefficients. A set of eigenfunctions can then be considered as a basis of the algebra (1.6), with structure constants given by (1.7). Any function can be expanded in terms of the complete set $\{\psi_n(x)\}$, and therefore it will be convenient, for the future, to introduce the following ket made up in terms of elements of the set $\{\psi_n(x)\}$

$$|\psi\rangle = \begin{pmatrix} \psi_0(x) \\ \psi_1(x) \\ \cdots \\ \psi_n(x) \\ \cdots \end{pmatrix} \tag{1.10}$$

A function $f(x)$ such that

$$f(x) = \sum_n a_n\psi_n(x) \tag{1.11}$$

can be represented as

$$f(x) = \langle a|\psi\rangle \tag{1.12}$$

where

$$\langle a| = (a_0, a_1, \cdots, a_n, \cdots) \tag{1.13}$$

To write the orthogonality relation in terms of this new formalism it is convenient to realize the complex conjugation as a linear operation on the set $\{\psi_n(x)\}$. In fact, due to the completeness, $\psi_n^*(x)$ itself can be expanded in terms of $\psi_n(x)$

$$\psi_n^*(x) = \sum_n C_{nm}\psi_m(x) \tag{1.14}$$

or

$$|\psi^*\rangle = C|\psi\rangle \tag{1.15}$$

Defining a bra as the transposed of the ket $|\psi\rangle$

$$\langle\psi| = (\psi_0(x), \psi_1(x), \cdots(x), \psi_n(x), \cdots) \tag{1.16}$$

the orthogonality relation becomes

$$\int |\psi^*\rangle\langle\psi|\; dx = \int C|\psi\rangle\langle\psi|\; dx = 1 \tag{1.17}$$

Notice that by taking the complex conjugate of eq. (1.15), we get

$$C^*C = 1 \tag{1.18}$$

The relation (1.17) makes reference only to the elements of the algebra of functions and it is the key element in order to define the integration rules on the algebra. In fact, we can now use the algebra product to reduce the expression (1.17) to a linear form

$$\delta_{nm} = \sum_\ell \int \psi_n(x)\psi_\ell(x)C_{\ell m}\; dx = \sum_{\ell,p} c_{n\ell p}C_{\ell m} \int \psi_p(x)\; dx \tag{1.19}$$

If the set of equations

$$\sum_p A_{nmp} \int \psi_p(x)\; dx = \delta_{nm}, \quad A_{nmp} = \sum_\ell c_{n\ell p}C_{\ell m} \tag{1.20}$$

has a solution for $\int \psi_p(x)\; dx$, then we will be able to define the integration over all the algebra, by linearity. We will show in the following that indeed a solution exists for many interesting cases. For instance a solution always exists, if the constant function is in the set $\{\psi_p(x)\}$. Let us show what we get for the free particle. The matrix C is easily obtained by noticing that

$$\begin{aligned} \left(\frac{1}{\sqrt{2\pi}}\exp(-ikx)\right)^* &= \frac{1}{\sqrt{2\pi}}\exp(ikx) \\ &= \int dk'\delta(k+k')\frac{1}{\sqrt{2\pi}}\exp(-ik'x) \end{aligned} \tag{1.21}$$

and therefore

$$C_{kk'} = \delta(k + k') \tag{1.22}$$

It follows

$$A_{kk'k''} = \int dq\ \delta(k' + q)\frac{1}{\sqrt{2\pi}}\delta(q + k - k'') = \frac{1}{\sqrt{2\pi}}\delta(k - k' - k'') \tag{1.23}$$

from which

$$\delta(k - k') = \int dk'' \int A_{kk'k''}\psi_{k''}(x)dx = \int \frac{1}{2\pi}\exp(-i(k - k')x)dx \tag{1.24}$$

This example is almost trivial, but it shows how, given the structure constants of the algebra, the property of the exponential of being the Fourier transform of the delta-function follows automatically from the formalism. In fact, what we have really done it has been **to define the integration rules in the x space** by using only the algebraic properties of the exponential. As a result, our integration rules require that the integral of an exponential is a delta-function. One can perform similar steps in the case of the spherical harmonics, where the C matrix is given by

$$C_{(\ell,m),(\ell',m')} = (-1)^m \delta_{\ell,\ell'}\delta_{m,-m'} \tag{1.25}$$

and then using the constant function $Y_0^0 = 1/\sqrt{4\pi}$, in the completeness relation.

The procedure we have outlined here is the one that we will generalize in the next Section to arbitrary algebras. Before doing that we will consider the possibility of a further generalization. In the usual path-integral formalism sometimes one makes use of the coherent states instead of the position operator eigenstates. In this case the basis in which one considers the wave functions is a basis of eigenfunctions of a non-hermitian operator

$$\psi(z) = \langle\psi|z\rangle \tag{1.26}$$

with

$$a|z\rangle = |z\rangle z \tag{1.27}$$

The wave functions of this type close an algebra, as $\langle z^*|\psi^*\rangle$ do. But this time the two types of eigenfunctions are not connected by any linear operation. In fact, the completeness relation is defined on the direct product of the two algebras

$$\int \frac{dz^* dz}{2\pi i}\exp(-z^* z)|z\rangle\langle z^*| = 1 \tag{1.28}$$

Therefore, in similar situations, we will not define the integration over the original algebra, but rather on the algebra obtained by the tensor product of the algebra times a copy. The copy corresponds to the complex conjugated functions of the previous example.

Chapter 2

Algebras

2.1 Algebras

We recall here some of the concepts introduced in [8], in order to define the integration rules over a generic algebra. We start by considering an algebra $\mathcal{A}$ given by $n+1$ basis elements x_i, with $i = 0, 1, \cdots n$ (we do not exclude the possibility of $n \to \infty$, or of a continuous index). We assume the multiplication rules

$$x_i x_j = f_{ijk} x_k \tag{2.1}$$

with the usual convention of sum over the repeated indices. For the future manipulations it is convenient to organize the basis elements x_i of the algebra in a ket

$$|x\rangle = \begin{pmatrix} x_0 \\ x_1 \\ \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \tag{2.2}$$

or in the corresponding bra

$$\langle x| = \begin{pmatrix} x_0, & x_1, & \cdots & x_n \end{pmatrix} \tag{2.3}$$

Important tools for the study of a generic algebra are the **right and left multiplication algebras**. We define the associated matrices by

$$R_i|x\rangle = |x\rangle x_i, \qquad \langle x|L_i = x_i\langle x| \tag{2.4}$$

For a generic element $a = \sum_i a_i x_i$ of the algebra we have $R_a = \sum_i a_i R_i$, and a similar equation for the left multiplication. In the following we will use also

$$L_i^T|x\rangle = x_i|x\rangle \tag{2.5}$$

The matrix elements of R_i and L_i are obtained from their definition

$$(R_i)_{jk} = f_{jik}, \quad (L_i)_{jk} = f_{ikj} \tag{2.6}$$

The algebra is completely characterized by the structure constants. The matrices R_i and L_i are just a convenient way of encoding their properties. For instance, in the case of associative algebras one has

$$x_i(x_j x_k) = (x_i x_j)x_k \tag{2.7}$$

implying the following relations (equivalent one with the other)

$$R_i R_j = f_{ijk} R_k, \quad L_i L_j = f_{ijk} L_k, \quad [R_i, L_j^T] = 0 \tag{2.8}$$

The first two say that R_i and L_i are linear representations of the algebra, called the regular representations. The third that the right and left multiplications commute for associative algebras. In this paper we will be interested in algebras with identity, and such that there exists a matrix C, satisfying

$$L_i = C R_i C^{-1} \tag{2.9}$$

We will call these algebras self-conjugated. In the case of associative algebras, the condition (2.9) says that the regular representations (see eq. (2.8)) spanned by L_i and R_i are equivalent. Therefore, the non existence of the matrix C boils down two the possibility that the associative algebra admits inequivalent regular representations. This happens, for instance, in the case of the bosonic algebra [8]. In all the examples we will consider here, the C matrix turns out to be symmetric

$$C^T = C \tag{2.10}$$

This condition of symmetry can be interpreted in terms of the opposite algebra $\mathcal{A}^D$, defined by

$$x_i^D x_j^D = f_{jik} x_k^D \tag{2.11}$$

The left and right multiplication in the dual algebra are related to those in $\mathcal{A}$ by

$$R_i^D = L_i^T, \quad L_i^D = R_i^T \tag{2.12}$$

Therefore, in the associative case, the matrices L_i^T are a representation of the dual algebra

$$L_i^T L_j^T |x\rangle = x_j x_i |x\rangle = f_{jik} L_k^T |x\rangle \tag{2.13}$$

We see that the property $C^T = C$ implies that the relation (2.9) holds also for the right and left multiplication in the opposite algebra

$$L_i^D = C R_i^D C^{-1} \tag{2.14}$$

In the case of associative algebras, the requirement of existence of an identity is not a strong one, because we can always extend the given algebra to another associative algebra with identity. In fact, let us call F the field over which the algebra is defined (usually F is the field of real or complex numbers). Then, the extension of $\mathcal{A}$ (call it $\mathcal{A}_1$), defined by the pairs

$$(\alpha, a) \in \mathcal{A}_1, \quad \alpha \in F, \quad a \in \mathcal{A} \tag{2.15}$$

with product rule

$$(\alpha, a)(\beta, b) = (\alpha\beta, \alpha a + \beta b + ab) \tag{2.16}$$

is an associative algebra with identity given by

$$I = (1, 0) \tag{2.17}$$

Of course, this is the same as adding to any element of $\mathcal{A}$ a term proportional to the identity, that is

$$\alpha I + a \tag{2.18}$$

and defining the multiplication by distributivity. An extension of this type exists also for many other algebras, but not for all. For instance, in the case of a Lie algebra one cannot add an identity with respect to the Lie product (since $I^2 = 0$). For self-conjugated algebras, L_i has an eigenket given by

$$L_i|Cx\rangle = CR_i|x\rangle = |Cx\rangle x_i, \qquad |Cx\rangle = C|x\rangle \tag{2.19}$$

as it follows from (2.9) and (2.4). Then, as explained in the Introduction, we define the integration for a self-conjugated algebra by the formula

$$\int_{(x)} |Cx\rangle\langle x| = 1 \tag{2.20}$$

where 1 is the identity in the space of the linear mappings on the algebra. In components the previous definition means

$$\int_{(x)} C_{ij} x_j x_k = C_{ij} f_{jkp} \int_{(x)} x_p = \delta_{ik} \tag{2.21}$$

This equation is meaningful only if it is possible to invert it in terms of $\int_{(x)} x_p$. This is indeed the case if $\mathcal{A}$ is an algebra with identity (say $x_0 = I$) [8], because by taking $x_k = I$ in eq. (2.21), we get

$$\int_{(x)} x_j = (C^{-1})_{j0} \tag{2.22}$$

We see now the reason for requiring the condition (2.9). In fact it ensures that the value (2.22) of the integral of an element of the basis of the algebra gives the solution to the equation (2.21). In fact we have

$$\int_{(x)} C_{ij} x_j x_k = C_{ij} f_{jkp} C^{-1}_{p0} = (CR_k C^{-1})_{i0} = (L_k)_{i0} = f_{k0i} = \delta_{ik} \tag{2.23}$$

as it follows from $x_k x_0 = x_k$. Notice that if C is symmetric we can write the integration also as

$$\int_{(x)} |x\rangle\langle Cx| = 1 \tag{2.24}$$

which is the form we would have obtained if we had started with the same assumptions but with the transposed version of eq. (2.4). We will define an arbitrary function on the algebra by

$$f(x) = \sum_i f_i x_i \equiv \langle x|f\rangle \tag{2.25}$$

and its conjugated as

$$f^*(x) = \sum_{ij} \bar{f}_i C_{ij} x_j = \langle f|Cx\rangle \tag{2.26}$$

where

$$|f\rangle = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \qquad \langle f| = (\, \bar{f}_0 \quad \bar{f}_1 \quad \cdots \quad \bar{f}_n \,) \tag{2.27}$$

where $\bar{f}_i$ is the complex conjugated of the coefficient f_i belonging to the field $\mathbb{C}$. Then a scalar product on the algebra is given by

$$\langle f|g\rangle = \int_{(x)} \langle f|Cx\rangle\langle x|g\rangle = \int_{(x)} f^*(x)g(x) = \sum_i \bar{f}_i g_i \tag{2.28}$$

2.2 Non existence of the C matrix

Consider now the case in which the C matrix does not exist. For associative algebras this happens when the left and right multiplications span inequivalent regular representations. In this case, let us take an isomorphic copy of $\mathcal{A}$, say $\mathcal{A}^*$

$$x_i^* x_j^* = f_{ijk} x_k^* \tag{2.29}$$

and

$$R_i|x^*\rangle = |x^*\rangle x_i^*, \qquad \langle x|L_i = x_i\langle x_i| \tag{2.30}$$

with $|x^*\rangle_i = x_i^*$. Define the integration over the direct product $\mathcal{A} \otimes \mathcal{A}^*$

$$\int_{(x,x^*)} |x^*\rangle\langle x| = 1 \tag{2.31}$$

or

$$\int_{(x,x^*)} x_i^* x_j = \delta_{ij} \tag{2.32}$$

giving rise to the scalar product

$$\langle f|g\rangle = \int_{(x,x^*)} \bar{f}(x^*)g(x) = \sum_i \bar{f}_i g_i \tag{2.33}$$

2.3 Algebras with involution

In some case, as for the toroidal algebras [9], the matrix C turns out to define a mapping which is an involution of the algebra. Let us consider the property of the involution on a given algebra $\mathcal{A}$. An involution is a linear mapping $^* : \mathcal{A} \to \mathcal{A}$, such that

$$(x^*)^* = x, \qquad (xy)^* = y^* x^*, \qquad x, y \in \mathcal{A} \tag{2.34}$$

Furthermore, if the definition field of the algebra is $\mathbb{C}$, the involution acts as the complex-conjugation on the field itself. Given a basis $\{x_i\}$ of the algebra, the involution can be expressed in terms of a matrix C such that

$$x_i^* = x_j C_{ji} \tag{2.35}$$

The eqs. (2.34) imply

$$(x_i^*)^* = x_j^* C_{ji}^* = x_k C_{kj} C_{ji}^* \tag{2.36}$$

from which

$$CC^* = 1 \tag{2.37}$$

From the product property applied to the equality

$$R_i |x\rangle = |x\rangle x_i \tag{2.38}$$

we get

$$(R_i |x\rangle)^* = \langle x^* | R_i^\dagger = \langle x | C R_i^\dagger = (|x\rangle x_i)^* = x_i^* \langle x^* | = x_i^* \langle x | C \tag{2.39}$$

and therefore

$$\langle x | C R_i^\dagger C^{-1} = x_j C_{ji} \langle x | = \langle x | L_j C_{ji} \tag{2.40}$$

that is

$$C R_i^\dagger C^{-1} = L_j C_{ji} \tag{2.41}$$

or also

$$C R_{x_i}^\dagger C^{-1} = L_{x_i^*} \tag{2.42}$$

If R_i and L_i are *-representations, that is

$$R_{x_i}^\dagger = R_{x_i^*} = R_{x_j} C_{ji} \tag{2.43}$$

we obtain

$$C R_{x_i}^\dagger C^{-1} = C R_{x_i^*} C^{-1} = L_{x_i^*} \tag{2.44}$$

Since the involution is non-singular, we get

$$C R_i C^{-1} = L_i \tag{2.45}$$

and comparing with the adjoint of eq. (2.44), we see that C is a unitary matrix which, from eq. (2.37), implies $C^T = C$. Therefore we have the theorem:

Given an associative algebra with involution, if the right and left multiplications are *-representations, then the algebra is self-conjugated.

In this case our integration is a *state* in the Connes terminology [4].

If the C matrix is an involution we can write the integration as

$$\int_{(x)} |x\rangle\langle x^*| = \int_{(x)} |x^*\rangle\langle x| = 1 \tag{2.46}$$

2.4 Derivations

We will discuss now the derivations on algebras with identity. Recall that a derivation is a linear mapping on the algebra satisfying

$$D(ab) = (Da)b + a(Db), \qquad a, b \in \mathcal{A} \tag{2.47}$$

We define the action of D on the basis elements in terms of its representative matrix, d,

$$Dx_i = d_{ij}x_j \tag{2.48}$$

If D is a derivation, then

$$S = \exp(\alpha D) \tag{2.49}$$

is an automorphism of the algebra. In fact, it is easily proved that

$$\exp(\alpha D)(ab) = (\exp(\alpha D)a)(\exp(\alpha D)b) \tag{2.50}$$

On the contrary, if $S(\alpha)$ is an automorphism depending on the continuous parameter α, then from (2.50), the following equation defines a derivation

$$D = \lim_{\alpha\to 0}\frac{S(\alpha) - 1}{\alpha} \tag{2.51}$$

In our formalism the automorphisms play a particular role. In fact, from eq. (2.50) we get

$$S(\alpha)(|x\rangle x_i) = (S(\alpha)|x\rangle)(S(\alpha)x_i) \tag{2.52}$$

and therefore

$$R_i(S(\alpha)|x\rangle) = S(\alpha)(R_i|x\rangle) = S(\alpha)(|x\rangle x_i) = (S(\alpha)|x\rangle)(S(\alpha)x_i) \tag{2.53}$$

meaning that $S(\alpha)|x\rangle$ is an eigenvector of R_i with eigenvalue $S(\alpha)x_i$. This equation shows that the basis $x_i' = S(\alpha)x_i$ satisfies an algebra with the same structure constants as those of the basis x_i. Therefore the matrices R_i and L_i constructed in the two basis, and as a consequence the C matrix, are identical. In other words, our formulation is invariant under automorphisms of the algebra (of course this is not true for a generic change of basis). The previous equation can be rewritten in terms of the matrix $s(\alpha)$ of the automorphism $S(\alpha)$, as

$$R_i\left(s(\alpha)|x\rangle\right) = \left(s(\alpha)|x\rangle\right) s_{ij}x_j = s_{ij}s(\alpha)R_j|x\rangle \tag{2.54}$$

or

$$s(\alpha)^{-1}R_i s(\alpha) = R_{S(\alpha)x} \tag{2.55}$$

If the algebra has an identity element, I, (say $x_0 = I$), then

$$Dx_0 = 0 \tag{2.56}$$

and therefore

$$Dx_0 = d_{0i}x_i = 0 \Longrightarrow d_{0i} = 0 \tag{2.57}$$

We will prove now some properties of the derivations. First of all, from the basic defining equation (2.47) we get

$$\begin{aligned} R_i d|x\rangle &= R_i D|x\rangle = D(R_i|x\rangle = D(|x\rangle x_i) \\ &= d|x\rangle x_i + |x\rangle Dx_i = dR_i|x\rangle + R_{Dx_i}|x\rangle \end{aligned} \tag{2.58}$$

or

$$[R_i, d\,] = R_{Dx_i} \tag{2.59}$$

which is nothing but the infinitesimal version of eq. (2.55). From the integration rules for a self-conjugated algebra with identity we get

$$\int_{(x)} Dx_i = d_{ij}\int_{(x)} x_j = d_{ij}(C^{-1})_{j0} \tag{2.60}$$

Showing that in order that the derivation D satisfies the integration by parts rule for any function, $f(x)$, on the algebra

$$\int_{(x)} D(f(x)) = 0 \tag{2.61}$$

the necessary and sufficient condition is

$$d_{ij}(C^{-1})_{j0} = 0 \tag{2.62}$$

implying that the d matrix must be singular and have $(C^{-1})_{j0}$ as a null eigenvector.

Next we show that, if a derivation satisfies the integration by part formula (2.61), then the matrix of related automorphism $S(\alpha) = \exp(\alpha D)$ obeys the equation

$$Cs(\alpha)C^{-1} = s^{T\,-1}(\alpha) \tag{2.63}$$

and it leaves invariant the measure of integration. The converse of this theorem is also true. Let us start assuming that D satisfies eq. (2.61), then

$$\begin{aligned} 0 &= \int_{(x)} D(C|x\rangle\langle x|) = \int_{(x)} Cd|x\rangle\langle x| + \int_{(x)} C|x\rangle\langle Dx| \\ &= CdC^{-1} + d^T \end{aligned} \tag{2.64}$$

that is

$$CdC^{-1} = -d^T \tag{2.65}$$

The previous expression can be exponentiated obtaining

$$C\exp(\alpha d)C^{-1} = \exp(-\alpha d^T) \tag{2.66}$$

from which the equation (2.63) follows, for $s(\alpha) = \exp(\alpha d)$. To show the invariance of the measure, let us consider the following identity

$$1 = \int_{(x)} s^{T-1}|Cx\rangle\langle x|s^T = \int_{(x)} Cs|x\rangle\langle x|s^T = \int_{(x)} C|Sx\rangle\langle Sx| = \int_{(x)} C|x'\rangle\langle x'| \tag{2.67}$$

where $x' = Sx$, and we have used eq. (2.63). For any automorphism of the algebra we have

$$\int_{(x')} |Cx'\rangle\langle x'| = 1 \tag{2.68}$$

since the numerical values of the matrices R_i and L_i, and consequently the C matrix, are left invariant. Comparing eqs. (2.67) and (2.68) we get

$$\int_{(x')} = \int_{(x)} \tag{2.69}$$

On the contrary, if the measure is invariant under an automorphism of the algebra, the chain of equalities

$$1 = \int_{(x')} |Cx'\rangle\langle x'| = \int_{(x)} |Cx'\rangle\langle x'| = \int_{(x)} Cs|x\rangle\langle x|s^T = CsC^{-1}s^T \tag{2.70}$$

implies eq. (2.63), together with its infinitesimal version eq. (2.65). From this (see the derivation in (2.64)), we get

$$0 = \int_{(x)} D(C_{ij}x_jx_k) \tag{2.71}$$

and by taking $x_k = I$,

$$\int_{(x)} Dx_i = 0 \tag{2.72}$$

for any basis element of the algebra. Therefore we have proven the following theorem:

If a derivation D satisfies the integration by part rule, eq. (2.61), the integration is invariant under the related automorphism $\exp(\alpha D)$*. On the contrary, if the integration is invariant under a continuous automorphism,* $\exp(\alpha D)$*, the related derivation, D, satisfies (2.61).*

This theorem generalizes the classical result about the Lebesgue integral relating the invariance under translations of the measure and the integration by parts formula.

2.4.1 Derivations on associative algebras

Next we will show that, always in the case of an associative self-conjugated algebra, $\mathcal{A}$, with identity, there exists a set of automorphisms such that the measure of integration is invariant. These are the so called **inner derivations**, that is derivations such that

$$D \in \mathcal{L}(\mathcal{A}) \tag{2.73}$$

where $\mathcal{L}(\mathcal{A})$ is the **Lie multiplication algebra** associated to $\mathcal{A}$. $\mathcal{L}(\mathcal{A})$ is defined in the following way: start with the linear space of left and right multiplications and define

$$\mathcal{M}_1 = \mathcal{M}_R + \mathcal{M}_{L^T} \tag{2.74}$$

that is the space generated by the vectors

$$R_a + L_b^T, \qquad a, b \in \mathcal{A} \tag{2.75}$$

Then

$$\mathcal{L}(\mathcal{A}) = \sum_{i=1}^{\infty} \mathcal{M}_i \tag{2.76}$$

where the spaces $\mathcal{M}_i$ are defined by induction

$$\mathcal{M}_{i+1} = [\mathcal{M}_1, \mathcal{M}_i] \tag{2.77}$$

Therefore $\mathcal{L}(\mathcal{A})$ is defined in terms of all the multiple commutators of the elements given in (2.75).

It is not difficult to prove that for a Lie algebra, $\mathcal{L}(\mathcal{A})$ coincides with the adjoint representation [11]. We will prove now an analogous result for associative algebras with identity. That is that $\mathcal{L}(\mathcal{A})$ coincides with the adjoint representation of the Lie algebra associated to $\mathcal{A}$ (the Lie algebra generated by $[a, b] = ab - ba$, for $a,\ b \in \mathcal{A}$). The proof can be found, for example, in ref. [11], but for completeness we will repeat it here. From the associativity conditions (2.8), and (2.13) one gets

$$[R_a + L_b^T, R_c + L_d^T] \in \mathcal{M}_1, \quad a, b, c, d \in \mathcal{A} \tag{2.78}$$

or

$$[\mathcal{M}_1, \mathcal{M}_1] \subset \mathcal{M}_1 \tag{2.79}$$

showing that

$$\mathcal{L}(\mathcal{A}) = \mathcal{M}_1 = \mathcal{M}_R + \mathcal{M}_{L^T} \tag{2.80}$$

Therefore the matrix associated to an inner derivation of an associative algebra must be of the form

$$d = R_a + L_b^T \tag{2.81}$$

We have now to require that this indeed a derivation, that is that eq. (2.59) holds. We start evaluating

$$[R_c, d] = [R_c, R_a + L_b^T] = R_{[c,a]} \tag{2.82}$$

where we have used the fact that the right multiplications form a representation of the algebra and that right and left multiplications commute. Then comparing with

$$R_{Dc} = R_{ca+cb} \tag{2.83}$$

we see that the two agree for $b = -a$. Then we get

$$Dx_i = x_i a - a x_i = -[a, x_i] = -(adj\ a)_{ij} x_j \tag{2.84}$$

This shows indeed that the inner derivations span the adjoint representation of the Lie algebra associated to $\mathcal{A}$.

We can now proof the following theorem:

For an associative self-conjugated algebra with identity, such that $C^T = C$, the measure of integration is invariant under the automorphisms generated by the inner derivations, or, equivalently, the inner derivations satisfy the rule of integration by parts.

In fact, this follows because the inner derivations satisfy eq. (2.65)

$$CdC^{-1} = C(R_a - L_a^T)C^{-1} = L_a - (C^{T^{-1}} L_a C^T)^T = L_a - R_a^T = -d^T \tag{2.85}$$

2.5 Integration over a subalgebra

Let us start with a self-conjugated algebra $\mathcal{A}$ with generators x_i, $i = 0, \cdots, n$. Let us further suppose that $\mathcal{A}$ has a self-conjugated sub-algebra $\mathcal{B}$ with generators y_α, with $\alpha = 0, \cdots, m$, $m < n$. As a vector space the algebra $\mathcal{A}$ can be decomposed as

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{C} \tag{2.86}$$

The vector space $\mathcal{C}$ is generated by vectors v_a, with $a = 1, \cdots n - m$. Since $\mathcal{B}$ is a subalgebra we have multiplication rules

$$\begin{aligned} y_\alpha y_\beta &= f_{\alpha\beta\gamma} y_\gamma \\ v_a y_\alpha &= f_{a\alpha\beta} y_\beta + f_{a\alpha b} v_b \\ v_a v_b &= f_{abc} v_c + f_{ab\alpha} y_\alpha \end{aligned} \tag{2.87}$$

By definition the integration is defined both in $\mathcal{A}$ and in $\mathcal{B}$. Our aim is to reconstruct the integration over $\mathcal{B}$ as an integration over $\mathcal{A}$ with a convenient measure. To this end, let us consider the matrix S which realizes the change of basis from x_i to (y_α, v_a), that is

$$y_\alpha = S_{\alpha i} x_i, \qquad v_a = S_{ai} x_i \tag{2.88}$$

This matrix is invertible by hypothesis, and we can reconstruct the original basis as

$$x_i = (S^{-1})_{i\alpha} y_\alpha + (S^{-1})_{ia} v_a \tag{2.89}$$

To reconstruct the integration over $\mathcal{B}$ in terms of an integration over $\mathcal{A}$, we will construct a function on the algebra

$$P = p_i x_i \tag{2.90}$$

such that

$$\int_{(\mathcal{A})} v_a P = 0, \qquad \int_{(\mathcal{A})} y_\alpha P = \int_{(\mathcal{B})} y_\alpha \tag{2.91}$$

These are equivalent to require

$$\int_{(\mathcal{A})} \mathcal{A}P = \int_{(\mathcal{A})} \mathcal{B}P = \int_{(\mathcal{B})} B \tag{2.92}$$

These are $n+1$ conditions over the $n+1$ unknown p_i. We will see immediately that there is one and only one solution to the problem. In fact, by using the matrix S we can make more explicit the previous equations by writing

$$0 = \int_{(\mathcal{A})} v_a P = S_{ai} p_j \int_{(\mathcal{A})} x_i x_j \tag{2.93}$$

and recalling that the general rule for integrating over a self-conjugated algebra $\mathcal{A}$ is

$$\int_{(\mathcal{A})} (C_\mathcal{A})_{ij} x_j x_k = \delta_{ik} \tag{2.94}$$

or

$$\int_{(\mathcal{A})} x_i x_j = (C_\mathcal{A}^{-1})_{ij} \tag{2.95}$$

with $C_\mathcal{A}$ the matrix realizing the equivalence between right and left multiplications of the algebra $\mathcal{A}$

$$L_i = C_\mathcal{A} R_i C_\mathcal{A}^{-1} \tag{2.96}$$

Therefore, we get

$$0 = S_{ai} p_j (C_\mathcal{A}^{-1})_{ij} \tag{2.97}$$

That is

$$(S C_\mathcal{A}^{-1})_{aj} p_j = 0 \tag{2.98}$$

and in analogous way

$$(S C_\mathcal{A}^{-1})_{\alpha j} p_j = \int_{(\mathcal{B})} y_\alpha \tag{2.99}$$

from which we obtain

$$(S C_\mathcal{A}^{-1})_{\alpha j} p_j = (C_\mathcal{B}^{-1})_{\alpha 0} \tag{2.100}$$

Since both S and C are invertible, the problem has a unique solution given by

$$p_i = (C_\mathcal{A} S^{-1})_{i\alpha} (C_\mathcal{B}^{-1})_{\alpha 0} \tag{2.101}$$

2.6 Change of variables

Consider again a self-conjugated algebra and the following linear change of variables

$$x'_i = S_{ij}x_j \tag{2.102}$$

The integration rules with respect to the new variables are

$$\int_{x'} x'_i x'_j = (C'^{-1})_{ij} \tag{2.103}$$

where C' satisfies

$$L'_i = C' R'_i C'^{-1} \tag{2.104}$$

and the right and left multiplications in the new basis are related to the ones in the old basis in the following manner. From

$$R'_i |x'\rangle = |x'\rangle x'_i \tag{2.105}$$

we get

$$R'_i S|x\rangle = S|x\rangle S_{ij}x_j = S_{ij} S R_j |x\rangle \tag{2.106}$$

or

$$R'_i = S_{ij} S R_j S^{-1} \tag{2.107}$$

In analogous way, from

$$\langle x'| L'_i = x'_i \langle x'| \tag{2.108}$$

we get

$$\langle x| S^T L_i = S_{ij} x_j \langle x| S^T = S_{ij} \langle x| L_j S^T \tag{2.109}$$

that is

$$L'_i = S_{ij} S^{T-1} L_j S^T \tag{2.110}$$

In the new basis the R- and L-representations are still equivalent

$$L'_i = C' R'_i C'^{-1} \tag{2.111}$$

implying

$$S^{T-1} L_i S^T = C' S R_i S^{-1} C'^{-1} \tag{2.112}$$

or

$$L_i = \left(S^T C' S\right) R_i \left(S^T C' S\right)^{-1} \tag{2.113}$$

Therefore we must have ($L_i = C R_i C^{-1}$)

$$C = S^T C' S A \tag{2.114}$$

with A invertible and

$$[R_i, A] = 0 \tag{2.115}$$

We get

$$(C'^{-1})_{ij} = \int_{x'} x'_i x'_j = \int_{x'} S_{il} S_{jm} x_l x_m \tag{2.116}$$

from which

$$\int_{x'} x_l x_m = (S^{-1} C'^{-1} S^{T^{-1}})_{lm} \tag{2.117}$$

and in particular

$$\int_{x'} x_i = (S^{-1} C'^{-1} S^{T^{-1}})_{i0} \tag{2.118}$$

The result can also be expressed in terms of the matrix A defined in eq. (2.114)

$$A = S^{-1} C'^{-1} S^{T^{-1}} C \tag{2.119}$$

obtaining

$$\int_{x'} x_i x_j = (A C^{-1})_{ij} \tag{2.120}$$

Chapter 3

Examples of Associative Self-Conjugated Algebras

3.1 The Grassmann algebra

We will discuss now the case of the Grassmann algebra $\mathcal{G}_1$, with generators $1, \theta$, such that $\theta^2 = 0$. The multiplication rules are

$$\theta^i \theta^j = \theta^{i+j}, \quad i, j, i+j = 0, 1 \tag{3.1}$$

and zero otherwise (see Table 1).

	1	θ
1	1	θ
θ	θ	0

Table 1: *Multiplication table for the Grassmann algebra* $\mathcal{G}_1$.

From the multiplication rules we get the structure constants

$$f_{ijk} = \delta_{i+j,k}, \quad i,\ j,\ k = 0, 1 \tag{3.2}$$

from which the explicit expressions for the matrices R_i and L_i follow

$$\begin{aligned}
(R_0)_{ij} &= f_{i0j} = \delta_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
(R_1)_{ij} &= f_{i1j} = \delta_{i+1,j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
(L_0)_{ij} &= f_{0ji} = \delta_{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
(L_1)_{ij} &= f_{1ji} = \delta_{i,j+1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{3.3}$$

Notice that R_1 and L_1 are nothing but the ordinary annihilation and creation Fermi operators with respect to the vacuum state $|0\rangle = (1,0)$. The C matrix exists and it is given by

$$(C)_{ij} = \delta_{i+j,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.4}$$

The ket and the bra eigenvectors of R_i are

$$|\theta\rangle = \begin{pmatrix} 1 \\ \theta \end{pmatrix}, \qquad \langle\theta| = (\theta, 1) \tag{3.5}$$

and the completeness reads

$$\int_{\mathcal{G}_1} |\theta\rangle\langle\theta| = \int_{\mathcal{G}_1} \begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.6}$$

or

$$\int_{\mathcal{G}_1} \theta^i \theta^{1-j} = \delta_{i,j} \tag{3.7}$$

which means

$$\int_{\mathcal{G}_1} 1 = 0, \qquad \int_{\mathcal{G}_1} \theta = 1 \tag{3.8}$$

The case of a Grassmann algebra $\mathcal{G}_n$, which consists of 2^n elements obtained by n anticommuting generators $\theta_1, \theta_2, \cdots, \theta_n$, the identity, 1, and by all their products, can be treated in a very similar way. In fact, this algebra can be obtained by taking a convenient tensor product of n Grassmann algebras $\mathcal{G}_1$, which means that the eigenvectors of the algebra of the left and right multiplications are obtained by tensor product of the eigenvectors of eq. (2.4). The integration rules extended by the tensor product give

$$\int_{\mathcal{G}_n} \theta_n \theta_{n-1} \cdots \theta_1 = 1 \tag{3.9}$$

and zero for all the other cases, which is equivalent to require for each copy of $\mathcal{G}_1$ the equations (3.8). It is worth to mention the case of the Grassmann algebra $\mathcal{G}_2$ because it can be obtained by tensor product of $\mathcal{G}_1$ times a copy $\mathcal{G}_1^*$. Then we can apply our second method of getting the integration rules and show that they lead to the same result with a convenient interpretation of the measure. The algebra $\mathcal{G}_2$ is generated by θ_1, θ_2. An involution of the algebra is given by the mapping

$$^* : \qquad \theta_1 \leftrightarrow \theta_2 \tag{3.10}$$

with the further rule that by taking the * of a product one has to exchange the order of the factors. It will be convenient to put $\theta_1 = \theta$, $\theta_2 = \theta^*$. This allows us to consider $\mathcal{G}_2$ as $\mathcal{G}_1 \otimes \mathcal{G}_1^* \equiv (\mathcal{G}_1, ^*)$. Then the ket and bra eigenvectors of left and right multiplication in $\mathcal{G}_1$ and $\mathcal{G}_1^*$ respectively are given by

$$\langle\theta| = (1, \theta) \qquad |\theta^*\rangle = \begin{pmatrix} 1 \\ \theta^* \end{pmatrix} \tag{3.11}$$

with

$$R_i|\theta^*\rangle = |\theta^*\rangle\theta^{*i}, \qquad \langle\theta|L_i = \theta^i\langle\theta| \tag{3.12}$$

The completeness relation reads

$$\int_{(\mathcal{G}_1,^*)} |\theta^*\rangle\langle\theta| = \int_{(\mathcal{G}_1,^*)} \begin{pmatrix} 1 & \theta \\ \theta^* & \theta^*\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.13}$$

This implies

$$\begin{aligned} \int_{(\mathcal{G}_1,^*)} 1 &= \int_{(\mathcal{G}_1,^*)} \theta^*\theta = 1 \\ \int_{(\mathcal{G}_1,^*)} \theta &= \int_{(\mathcal{G}_1,^*)} \theta^* = 0 \end{aligned} \tag{3.14}$$

These relations are equivalent to the integration over $\mathcal{G}_2$ if we do the following identification

$$\int_{(\mathcal{G}_1,^*)} = \int_{\mathcal{G}_2} \exp(\theta^*\theta) \tag{3.15}$$

The origin of this factor can be traced back to the fact that we have

$$\langle\theta|\theta^*\rangle = 1 + \theta\theta^* = \exp(-\theta^*\theta) \tag{3.16}$$

3.2 The Paragrassmann algebra

We will discuss now the case of a paragrassmann algebra of order p, $\mathcal{G}_1^p$, with generators 1, and θ, such that $\theta^{p+1} = 0$. The multiplication rules are defined by

$$\theta^i\theta^j = \theta^{i+j}, \quad i, j, i+j = 0, \cdots, p \tag{3.17}$$

and zero otherwise (see Table 2).

	1	θ	$\cdot$	θ^{p-1}	θ^p
1	1	θ	$\cdot$	θ^{p-1}	θ^p
θ	θ	θ^2	$\cdot$	θ^p	0
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
θ^{p-1}	θ^{p-1}	θ^p	0	0	0
θ^p	θ^p	0	0	0	0

Table 2: *Multiplication table for the paragrassmann algebra* $\mathcal{G}_1^p$.

From the multiplication rules we get the structure constants

$$f_{ijk} = \delta_{i+j,k}, \quad i,\ j,\ k = 0, 1, \cdots, p \tag{3.18}$$

from which we obtain the following expressions for the matrices R_i and L_i:

$$(R_i)_{jk} = \delta_{i+j,k}, \quad (L_i)_{jk} = \delta_{i+k,j}, \quad i, j, k = 0, 1 \cdots, p \tag{3.19}$$

The C matrix exists and it is given by

$$(C)_{ij} = \delta_{i+j,p} \tag{3.20}$$

In fact

$$(CR_iC^{-1})_{lq} = \delta_{l+m,p}\delta_{i+m,n}\delta_{n+q,p} = \delta_{i+p-l,p-q} = \delta_{i+q,l} = (L_i)_{lq} \tag{3.21}$$

The ket and the bra eigenvectors of L_i are given by

$$C|\theta\rangle = \begin{pmatrix} \theta^p \\ \theta^{p-1} \\ \cdot \\ 1 \end{pmatrix}, \quad \langle\theta| = (1, \theta \cdots, \theta^p) \tag{3.22}$$

and the completeness reads

$$\int_{\mathcal{G}_1^p} \theta^{p-i}\theta^j = \delta_{ij} \tag{3.23}$$

which means

$$\int_{\mathcal{G}_1^p} 1 = \int_{\mathcal{G}_1^p} \theta = \int_{\mathcal{G}_1^p} \theta^{p-1} = 0 \tag{3.24}$$

$$\int_{\mathcal{G}_1^p} \theta^p = 1 \tag{3.25}$$

in agreement with the results of ref. [6] (see also [12]).

3.2.1 Derivations on a Paragrassmann algebra

In order to define a derivation on an algebra it is enough to define it on the generators. The action upon the other elements will be obtained by using the distributivity law. We will be rather interested in what it is called a g-derivation (this is a particular case of a (s_1, s_2) derivation, see [13]). In our case we can define $p+1$ g-derivations as follows

$$D_i\theta = \theta^i \tag{3.26}$$

such that

$$D_i(\theta f(\theta)) = (D_i\theta)f(\theta) + g_i\theta(D_i f(\theta)) \tag{3.27}$$

These two equations determine uniquely the action of D_i upon the generic element of the algebra. In fact we have,

$$D_i\theta^2 = (1 + g_i)\theta \tag{3.28}$$

and then, by induction, we get easily

$$D_i\theta^n = (1 + g_i + g_i^2 + \cdots + g_i^{n-1})\theta^{n+i-1} = \frac{1 - g_i^n}{1 - g_i}\theta^{n+i-1} \tag{3.29}$$

In the case $i = 0$, we get a condition on g_0. In fact, if we take $n = p + 1$, we must have

$$D_0\theta^{p+1} = 0 \tag{3.30}$$

But using (3.29) we get

$$0 = D_0\theta^{p+1} = \frac{1 - g_0^{p+1}}{1 - g_0}\theta^p \tag{3.31}$$

implying that g_0 is a $(p + 1)$-root of unity

$$g_0^{p+1} = 1 \tag{3.32}$$

However, for $i \neq 0$ we get no restrictions. Take for instance $i = 1$, then

$$0 = D_1\theta^{p+1} = (1 + g_1 + g_1^2 + \cdots + g_1^p)\theta^{p+1} \tag{3.33}$$

is automatically satisfied. The same is true for the $i > 1$. If we introduce the linear transformation associated to a derivations, i.e.

$$Dx_i = d_{ij}x_j \tag{3.34}$$

in our case we get

$$D_i\theta^n = (1 + g_i + g_i^2 \cdots + g_i^{n-1})\theta^{n+i-1} = (d_i)_{nm}\theta^m \tag{3.35}$$

from which

$$(d_i)_{nm} = (1 + g_i + g_i^2 \cdots + g_i^{n-1})\delta_{n+i-1,m}, \qquad n \neq 0 \tag{3.36}$$

For $n = 0$, using the fact that $D_i\theta^0 = 0$, we have

$$(d_i)_{0m} = 0 \tag{3.37}$$

We will investigate now the integration by part rule. That is we want to determine which conditions imply on the derivations the rule

$$\int_\theta D_i f(\theta) = 0 \tag{3.38}$$

that is

$$\int_\theta D_i\theta^n = (d_i)_{nm}(C^{-1})_{m0} = (d_i)_{np} \tag{3.39}$$

In order to satisfy eq. (3.38) for any n we must have

$$(d_i)_{np} = (1+g_i+g_i^2+\cdots+g_i^{n-1})\delta_{n+i-1,p} = (1+g_i+g_i^2+\cdots+g_i^{p-i})\delta_{n+i-1,p} = 0 \tag{3.40}$$

This is automatically satisfied by D_0, due to eq. (3.32). For $i \neq 0$, we see that the only way that the eq. (3.38) holds, is to require that g_i is a $(p+1-i)$-root of unity, that is

$$g_i^{p+1-i} = 1 \tag{3.41}$$

It is interesting to notice that a particular set of D_i derivations can be generated by D_0. In fact, if we put

$$D_i = \theta^i D_0 \tag{3.42}$$

we get

$$D_i\theta^n = (1 + g_0 + g_0^2 + \cdots + g_0^{n-1})\theta^{n+i-1} \tag{3.43}$$

which coincides with eq. (3.29), with $g_i = g_0$. In this case only D_0 satisfies eq. (3.38). Notice that D_0 depends on the choice of g_0, except in the Grassmann case ($p = 1$).

3.3 The algebra of quaternions

The quaternionic algebra is defined by the multiplication rules

$$e_A e_B = -\delta_{AB} + \epsilon_{ABC} e_C, \quad A, B, C = 1, 2, 3 \tag{3.44}$$

where ϵ_{ABC} is the Ricci symbol in 3 dimensions. The quaternions can be realized in terms of the Pauli matrices $e_A = -i\sigma_A$. The automorphism group of the quaternionic algebra is $SO(3)$, but it is more useful to work in the so called split basis

$$u_0 = \frac{1}{2}(1 + ie_3), \qquad u_0^* = \frac{1}{2}(1 - ie_3)$$
$$u_+ = \frac{1}{2}(e_1 + ie_2), \qquad u_- = \frac{1}{2}(e_1 - ie_2) \tag{3.45}$$

In this basis the multiplication rules are given in Table 3.

	u_0	u_0^*	u_+	u_-
u_0	u_0	0	u_+	0
u_0^*	0	u_0^*	0	u_-
u_+	0	u_+	0	$-u_0$
u_-	u_-	0	$-u_0^*$	0

Table 3: *Multiplication table for the quaternionic algebra.*

The automorphism group of the split basis is $U(1)$, with u_0 and u_0^* invariant and u_+ and u_- with charges $+1$ and -1 respectively. We define, as usual, the vector

$$|u\rangle = \begin{pmatrix} u_0 \\ u_0^* \\ u_+ \\ u_- \end{pmatrix} \tag{3.46}$$

The matrices R_A and L_A satisfy the quaternionic algebra because this is an associative algebra. So R_+ and R_- satisfy the algebra of a Fermi oscillator (apart a sign). It is easy to get explicit expressions for the left and right multiplication matrices and check that the C matrix exists and that it is given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.47}$$

Therefore

$$\langle u| = (u_0, u_0^*, -u_-, -u_+) \tag{3.48}$$

The exterior product is given by

$$\begin{aligned} |u\rangle\langle u| &= \begin{pmatrix} u_0 \\ u_0^* \\ u_+ \\ u_- \end{pmatrix} (u_0, u_0^*, -u_-, -u_+) \\ &= \begin{pmatrix} u_0 & 0 & 0 & -u_+ \\ 0 & u_0^* & -u_- & 0 \\ 0 & u_+ & u_0 & 0 \\ u_- & 0 & 0 & u_0^* \end{pmatrix} \end{aligned} \tag{3.49}$$

According to our integration rules we get

$$\int_{(u)} u_0 = \int_{(u)} u_0^* = 1, \qquad \int_{(u)} u_+ = \int_{(u)} u_- = 0 \tag{3.50}$$

In terms of the original basis for the quaternions we get

$$\int_{(u)} 1 = 2, \qquad \int_{(u)} e_A = 0 \tag{3.51}$$

and we see that the integration coincides with taking the trace in the 2×2 representation of the quaternions (see next Section). That is, given an arbitrary functions $f(u)$ on the quaternions we get

$$\int_{(u)} f(u) = Tr[f(u)] \tag{3.52}$$

By considering the scalar product

$$\langle u'|u\rangle = u_0' u_0 + u_0^{*\prime} u_0 - u_-^{*}{}' u_+ - u_+' u_-^* \tag{3.53}$$

we see that

$$\langle u|u\rangle = 2 \tag{3.54}$$

and

$$\int_{(u)} \langle u'|u\rangle = u_0' + u_0^{*\prime} = 1 \tag{3.55}$$

Therefore $\langle u'|u\rangle$ behaves like a delta-function.

3.4 The algebra of matrices

Since an associative algebra admits always a matrix representation, it is interesting to consider the definition of the integral over the algebra $\mathcal{A}_N$ of the $N \times N$ matrices. These can be expanded in the following general way

$$A = \sum_{n,m=1}^{N} e^{(nm)} a_{nm} \tag{3.56}$$

where $e^{(nm)}$ are N^2 matrices defined by

$$e_{ij}^{(nm)} = \delta_i^n \delta_j^m, \quad i.j = 1, \cdots, N \tag{3.57}$$

These special matrices satisfy the algebra

$$e^{(nm)} e^{(pq)} = \delta_{mp} e^{(nq)} \tag{3.58}$$

Therefore the structure constants of the algebra are given by

$$f_{(nm)(pq)(rs)} = \delta_{mp} \delta_{nr} \delta_{qs} \tag{3.59}$$

from which

$$(R_{(pq)})_{(nm)(rs)} = \delta_{pm} \delta_{qs} \delta_{nr}, \quad (L_{(pq)})_{(nm)(rs)} = \delta_{qr} \delta_{pn} \delta_{ms} \tag{3.60}$$

The matrix C can be found by requiring that $|Cx\rangle$ is an eigenstate of L_{pq}, that is

$$(L_{(pq)})_{(nm)(rs)} [F(e)]^{(rs)} = [F(e)]^{(nm)} e^{(pq)} \tag{3.61}$$

where

$$F(e)^{(nm)} = C_{(nm)(rs)} e^{(rs)} \tag{3.62}$$

We get

$$[F(e)]^{(qm)} \delta_{pn} = [F(e)]^{(nm)} e^{(pq)} \tag{3.63}$$

By looking at the eq. (3.58), we see that this equation is satisfied by

$$[F(e)]^{(rs)} = e^{(sr)} \tag{3.64}$$

It follows

$$C_{(mn)(rs)} = \delta_{ms} \delta_{nr} \tag{3.65}$$

It is seen easily that C satisfies

$$C^T = C^* = C, \quad C^2 = 1 \tag{3.66}$$

Therefore the matrix algebra is a self-conjugated one. One easily checks that the right multiplications satisfy eq. (2.44), and therefore C is an involution. More precisely, since

$$e^{(mn)^*} = C_{(mn)(pq)} e^{(pq)} = e^{(nm)} \tag{3.67}$$

the involution is nothing but the hermitian conjugation

$$A^* = A^\dagger, \quad A \in \mathcal{A}_N \tag{3.68}$$

The integration rules give

$$(C^{-1})_{(rp)(qs)} = \delta_{rs}\delta_{pq} = \int_{(e)} e^{(rp)} e^{(qs)} = \delta_{pq} \int_{(e)} e^{(rs)} \tag{3.69}$$

We see that this is satisfied by

$$\int_{(e)} e^{(rs)} = \delta_{rs} \tag{3.70}$$

This result can be obtained also using directly eq. (2.22), noticing that the identity of the algebra is given by $I = \sum_n e^{(n,n)}$. Therefore

$$\int_{(e)} e^{(rs)} = \sum_n (C^{-1})_{(rs)(nn)} = \sum_n \delta_{ns}\delta_{nr} = \delta_{rs} \tag{3.71}$$

and, for a generic matrix

$$\int_{(e)} A = \sum_{m,n=1}^{N} a_{nm} \int_{(e)} e^{(nm)} = Tr(A) \tag{3.72}$$

Since the algebra of the matrices is associative, the inner derivations are given by

$$D_B A = [A, B] \tag{3.73}$$

Therefore

$$\int_{(e)} D_B A = \int_{(e)} [A, B] = 0 \tag{3.74}$$

and we see that the integration by parts formula corresponds to the cyclic property of the trace.

3.4.1 The subalgebra $\mathcal{A}_{N-1}$

Consider the algebra $\mathcal{A}_N$ of the $N \times N$ matrices, and its subalgebra $\mathcal{A}_{N-1}$. We have the decomposition

$$\mathcal{A}_N = \mathcal{A}_{N-1} \oplus \mathcal{C} \tag{3.75}$$

with

$$\mathcal{C} = \sum_{i=1}^{N-1} \left(e^{(i,N)} \oplus e^{(N,i)} \right) \oplus e^{(N,N)} \tag{3.76}$$

and

$$\mathcal{A}_{N-1} = \sum_{i,j=1}^{N-1} \oplus e^{(i,j)} \tag{3.77}$$

Let us put

$$P = \sum_{i,j=1}^{N} p_{ij} e^{(i,j)} \tag{3.78}$$

then we require

$$\int_{\mathcal{A}_N} \mathcal{C} P = 0 \tag{3.79}$$

which implies

$$\int_{\mathcal{A}_N} e^{(i,N)} P = p_{Ni} = 0 \tag{3.80}$$

and analogously

$$p_{iN} = p_{NN} = 0 \tag{3.81}$$

The other condition

$$\int_{\mathcal{A}_N} \mathcal{A}_{N-1} P = \int_{\mathcal{A}_{N-1}} \mathcal{A}_{N-1} \tag{3.82}$$

gives

$$\int_{\mathcal{A}_N} e^{(i,j)} P = p_{ji} = \delta_{ij} \tag{3.83}$$

Therefore

$$P = \begin{pmatrix} 1_{N-1} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.84}$$

where 1_{N-1} is the identity matrix in $N-1$ dimensions. This result can be checked immediately by computing the product $A_N P$ with A a generic matrix of $\mathcal{A}_N$

$$A_N P = \begin{pmatrix} A_{N-1} & B \\ C & D \end{pmatrix} \begin{pmatrix} 1_{N-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{N-1} & 0 \\ C & 0 \end{pmatrix} \tag{3.85}$$

implying

$$\int_{\mathcal{A}_N} A_N P = Tr(A_N P) = Tr(A_{N-1}) = \int_{\mathcal{A}_{N-1}} A_{N-1} \tag{3.86}$$

3.4.2 Paragrassmann algebras as subalgebras of $\mathcal{A}_{p+1}$

A paragrassmann algebra of order p can be seen as a subalgebra of the matrix algebra A_{p+1}. In fact, ince a paragrassmann is associative it has a matrix representation (the regular one) in terms the $(p+1)\times(p+1)$ right multiplication matrices, R_i (see eq. (3.19). These are given by

$$(R_i)_{jk} = \delta_{i+j,k} \tag{3.87}$$

Defining

$$R_\theta \equiv R_1 \tag{3.88}$$

we can write, in terms of the matrices defined in eq. (3.57)

$$R_\theta = \sum_{i=1}^{p} e^{(i,i+1)} \tag{3.89}$$

and

$$R_\theta^k = \sum_{i=1}^{p+1-k} e^{(i,i+k)} \tag{3.90}$$

Therefore, the most general function on the paragrassmann algebra (as a subalgebra of the matrices $(p+1) \times (p+1)$) is given by

$$f(R_\theta) = \sum_{i=1}^{p+1} a_i R_\theta^{p+1-i} = \sum_{i=1}^{p+1} a_i \sum_{j=1}^{i} e^{(j,p+1+j-i)} \tag{3.91}$$

In order to construct the matrix P defined in Section (2.5), let us consider a generic matrix $B \in \mathcal{A}_{p+1}$. We can always decompose it as (see later)

$$B = f(R_\theta) + \tilde{B} \tag{3.92}$$

In order to construct this decomposition, let us consider the most general $(p+1) \times (p+1)$ matrix. We can write

$$B = \sum_{i,j=1}^{p+1} b_{ij} e^{(ij)} = \sum_{i=1}^{p+1} \sum_{j=1}^{p} b_{ij} e^{(ij)} + \sum_{i=1}^{p+1} b_{i,p+1} e^{(i,p+1)} \tag{3.93}$$

By adding and subtracting

$$\sum_{i=2}^{p+1} b_{i,p+1} \sum_{j=1}^{i-1} e^{(j,p+1+j-i)} \tag{3.94}$$

we get the decomposition (3.92) with

$$f(R_\theta) = \sum_{i=1}^{p+1} b_{i,p+1} R_\theta^{p+1-i} \tag{3.95}$$

and

$$\tilde{B} = \sum_{i=1}^{p+1} \sum_{j=1}^{p} b_{ij} e^{(ij)} - \sum_{i=2}^{p+1} b_{i,p+1} \sum_{j=1}^{i-1} e^{(j,p+1+j-i)} \tag{3.96}$$

Now, we can check that the matrix P such that

$$\int_\theta f(\theta) = \int_{A_{p+1}} BP = \int_{A_{p+1}} f(R_\theta) P \tag{3.97}$$

is given by

$$P = e^{(p+1,1)} \tag{3.98}$$

In fact, we have

$$\tilde{B} e^{(p+1,1)} = 0 \tag{3.99}$$

implying

$$B e^{(p+1,1)} = f(R_\theta) e^{(p+1,1)} \tag{3.100}$$

Furthermore

$$Tr[R_\theta^k e^{(p+1,1)}] = \sum_{i=1}^{p+1-k} Tr[e^{(i,i+k)} e^{(p+1,1)}] = Tr[e^{(p+1-k,1)}] = \delta_{p,k} \qquad (3.101)$$

and therefore

$$\int_\theta \theta^k = Tr[R_\theta^k e^{(p+1,1)}] = \delta_{p,k} \qquad (3.102)$$

showing that $e^{(p+1,1)}$ is the matrix P we were looking for.

We notice that the matrices $\tilde{B}$ and $f(R_\theta)$ appearing in the decomposition (3.92) can be written more explicitly as

$$\tilde{B} = \begin{pmatrix} \tilde{b}_{1,1} & \tilde{b}_{1,2} & \cdots & \tilde{b}_{1,p} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \tilde{b}_{p,1} & \tilde{b}_{p,2} & \cdots & \tilde{b}_{p,p} & 0 \\ \tilde{b}_{p+1,1} & \tilde{b}_{p+1,2} & \cdots & \tilde{b}_{p+1,p} & 0 \end{pmatrix} \qquad (3.103)$$

and

$$f(R_\theta) = \begin{pmatrix} a_{p+1} & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ 0 & a_{p+1} & a_p & \cdots & a_3 & a_2 \\ 0 & 0 & a_{p+1} & \cdots & a_4 & a_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_p & a_{p-1} \\ 0 & 0 & 0 & \cdots & a_{p+1} & a_p \\ 0 & 0 & 0 & \cdots & 0 & a_{p+1} \end{pmatrix} \qquad (3.104)$$

The $p \times (p+1)$ parameters appearing in $\tilde{B}$ and the $p+1$ parameters in $f(R_\theta)$ can be easily expressed in terms of the $(p+1) \times (p+1)$ parameters defining the matrix B.

In the particular case of a Grassmann algebra we have

$$R_\theta = e^{(1,2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_+, \quad P = e^{(2,1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_- \qquad (3.105)$$

The decomposition in eq. (3.92), for a 2×2 matrix

$$B = a + b\sigma_3 + c\sigma_+ + d\sigma_- \qquad (3.106)$$

is given by

$$\tilde{B} = b(1 + \sigma_3) + d\sigma_-, \qquad f(R_\theta) = f(\sigma_+) = a - b + c\sigma_+ \qquad (3.107)$$

and the integration is

$$\int_{(\theta)} f(\theta) = Tr[f(\sigma_+)\sigma_-] \qquad (3.108)$$

from which

$$\int_{(\theta)} 1 = Tr[\sigma_-] = 0, \qquad \int_{(\theta)} \theta = Tr[\sigma_+\sigma_-] = 1 \qquad (3.109)$$

3.5 Projective group algebras

Let us start defining a projective group algebra. We consider an arbitrary projective linear representation, $a \to x(a)$, $a \in G$, $x(a) \in \mathcal{A}(G)$, of a given group G. The representation $\mathcal{A}(G)$ defines in a natural way an associative algebra with identity (it is closed under multiplication and it defines a generally complex vector space). This algebra will be denoted by $\mathcal{A}(G)$. The elements of the algebra are given by the combinations

$$\sum_{a \in G} f(a)x(a) \tag{3.110}$$

For a group with an infinite number of elements, there is no a unique definition of such an algebra. The one defined in eq. (3.110) corresponds to consider a formal linear combination of a finite number of elements of G. This is very convenient because we will not be concerned here with topological problems. Other definitions correspond to take complex functions on G such that

$$\sum_{a \in G} |f(a)| < \infty \tag{3.111}$$

Or, in the case of compact groups, the sum is defined in terms of the Haar invariant measure. In the following we will not need to be more precise about this point. The basic product rule of the algebra follows from the group property

$$x(a)x(b) = e^{i\alpha(a,b)}x(ab) \tag{3.112}$$

where $\alpha(a,b)$ is called a cocycle. This is constrained, by the requirement of associativity of the representation, to satisfy

$$\alpha(a,b) + \alpha(ab,c) = \alpha(b,c) + \alpha(a,bc) \tag{3.113}$$

Changing the element $x(a)$ of the algebra by a phase factor $e^{i\phi(a)}$, that is, defining

$$x'(a) = e^{-i\phi(a)}x(a) \tag{3.114}$$

we get

$$x'(a)x'(b) = e^{i(\alpha(a,b)-\phi(ab)+\phi(a)+\phi(b))}x'(ab) \tag{3.115}$$

This is equivalent to change the cocycle to

$$\alpha'(a,b) = \alpha(a,b) - [\phi(ab) - \phi(a) - \phi(b)] \tag{3.116}$$

In particular, if $\alpha(a,b)$ is of the form $\phi(ab) - \phi(a) - \phi(b)$, it can be transformed to zero, and therefore the corresponding projective representation is isomorphic to a vector one. For this reason the combination

$$\alpha(a,b) = \phi(ab) - \phi(a) - \phi(b) \tag{3.117}$$

is called a trivial cocycle. Let us now discuss some properties of the cocycles. We start from the relation (e is the identity element of G)

$$x(e)x(e) = e^{i\alpha(e,e)}x(e) \tag{3.118}$$

By the transformation $x'(e) = e^{-i\alpha(e,e)}x(e)$, we get

$$x'(e)x'(e) = x'(e) \tag{3.119}$$

Therefore we can assume

$$\alpha(e,e) = 0 \tag{3.120}$$

Then, from

$$x(e)x(a) = e^{i\alpha(e,a)}x(a) \tag{3.121}$$

multiplying by $x(e)$ to the left, we get

$$x(e)x(a) = e^{i\alpha(e,a)}x(e)x(a) \tag{3.122}$$

implying

$$\alpha(e,a) = \alpha(a,e) = 0 \tag{3.123}$$

where the second relation is obtained in analogous way. Now, taking $c = b^{-1}$ in eq. (3.113), we get

$$\alpha(a,b) + \alpha(ab,b^{-1}) = \alpha(b,b^{-1}) \tag{3.124}$$

Again, putting $a = b^{-1}$

$$\alpha(b^{-1},b) = \alpha(b,b^{-1}) \tag{3.125}$$

We can go farther by considering

$$x(a)x(a^{-1}) = e^{i\alpha(a,a^{-1})}x(e) \tag{3.126}$$

and defining

$$x'(a) = e^{-i\alpha(a,a^{-1})/2}x(a) \tag{3.127}$$

from which

$$x'(a)x'(a^{-1}) = e^{-i\alpha(a,a^{-1})}x(a)x(a^{-1}) = x(e) = x'(e) \tag{3.128}$$

Therefore we can transform $\alpha(a,a^{-1})$ to zero without changing the definition of $x(e)$,

$$\alpha(a,a^{-1}) = 0 \tag{3.129}$$

As a consequence, equation (3.124) becomes

$$\alpha(a,b) + \alpha(ab,b^{-1}) = 0 \tag{3.130}$$

We can get another relation using $x(a^{-1}) = x(a)^{-1}$

$$\begin{aligned} x(a^{-1})x(b^{-1}) &= e^{i\alpha(a^{-1},b^{-1})}x(a^{-1}b^{-1}) = x(a)^{-1}x(b)^{-1} \\ &= (x(b)x(a))^{-1} = e^{-i\alpha(b,a)}x(a^{-1}b^{-1}) \end{aligned} \tag{3.131}$$

from which

$$\alpha(a^{-1}, b^{-1}) = -\alpha(b, a) \tag{3.132}$$

and together with eq. (3.130) we get

$$\alpha(ab, b^{-1}) = \alpha(b^{-1}, a^{-1}) \tag{3.133}$$

The last two relations will be useful in the following. From the product rule

$$x(a)x(b) = e^{i\alpha(a,b)}x(ab) = \sum_{c\in G} f_{abc}x(c) \tag{3.134}$$

we get the structure constants of the algebra

$$f_{abc} = \delta_{ab,c}e^{i\alpha(a,b)} \tag{3.135}$$

The delta function is defined according to the nature of the sum over the group elements.

To define the integration over $\mathcal{A}(G)$, we start as usual by introducing a ket with elements given by $x(a)$, that is $|x\rangle_a = x(a)$, and the corresponding transposed bra $\langle x|$. From the algebra product, we get immediately

$$(R(a))_{bc} = f_{bac} = \delta_{ba,c}e^{i\alpha(b,a)}, \quad (L(a))_{bc} = f_{acb} = \delta_{ac,b}e^{i\alpha(a,c)} \tag{3.136}$$

We show now that also these algebras are self-conjugated. Let us look for eigenkets of $L(a)$

$$L(a)|Cx\rangle = |Cx\rangle x(a) \tag{3.137}$$

giving

$$\delta_{ac,b}(Cx)_c e^{i\alpha(a,c)} = e^{i\alpha(a,a^{-1}b)}(Cx)_{a^{-1}b} = (Cx)_b x(a) \tag{3.138}$$

By putting

$$(Cx)_a = k_a x(a^{-1}) \tag{3.139}$$

we obtain

$$k_{a^{-1}b}x(b^{-1}a)e^{i\alpha(a,a^{-1}b)} = k_b e^{i\alpha(b^{-1},a)}x(b^{-1}a) \tag{3.140}$$

Then, from eqs. (3.133) and (3.132)

$$k_{a^{-1}b} = k_b \tag{3.141}$$

Therefore $k_a = k_e$, and assuming $k_e = 1$, it follows

$$(Cx)_a = x(a^{-1}) = x(a)^{-1} \tag{3.142}$$

giving

$$C_{a,b} = \delta_{ab,e} \tag{3.143}$$

This shows also that

$$C^T = C \tag{3.144}$$

at least in the cases of discrete and compact groups. The mapping $C : \mathcal{A} \to \mathcal{A}$ is an involution of the algebra. In fact, by defining

$$x(a)^* = x(b)C_{b,a} = x(a^{-1}) = x(a)^{-1} \tag{3.145}$$

we have $(x(a)^*)^* = x(a)$, and $x(b)^* x(a)^* = (x(a)x(b))^*$. In refs. [8, 10] we have shown that for a self-conjugated algebra, it is possible to define an integration rule in terms of the matrix C

$$\int_{(x)} x(a) = C^{-1}_{e,a} = \delta_{e,a} \tag{3.146}$$

From this definition and the eq. (3.144) it follows [8, 10]

$$\int_{(x)} |x\rangle\langle xC| = \int_{(x)} |xC\rangle\langle x| = 1 \tag{3.147}$$

Therefore we are allowed to expand a function on the group ($|f\rangle_a = f(a)$) as

$$f(a) = \int_{(x)} x(a^{-1})\langle x|f\rangle \tag{3.148}$$

with $\langle x|f\rangle = \sum_{b\in G} x(b)f(b)$. It is also possible to define a scalar product among functions on the group. Defining, $\langle f|_a = \bar{f}(a)$, where $\bar{f}(a)$ is the complex conjugated of $f(a)$, we put

$$\langle f|g\rangle = \int_{(x)} \langle f|xC\rangle\langle x|g\rangle = \int_{(x)} \bar{f}(x(a)^{-1})g(x) = \sum_{a\in G} \bar{f}(a)g(a) \tag{3.149}$$

It is important to stress that this definition depends only on the algebraic properties of $\mathcal{A}(G)$ and not on the specific representation chosen for this construction.

3.5.1 What is the meaning of the algebraic integration?

As we have said in the previous Section, the integration formula we have obtained is independent on the group representation we started with. In fact, it is based only on the structure of right and left multiplications, that is on the abstract algebraic product. This independence on the representation suggests that in some way we are "summing" over all the representations. To understand this point, we will study in this Section vector representations. To do that, let us introduce a label λ for the vector representation we are actually using to define $\mathcal{A}(G)$. Then a generic function on $\mathcal{A}(G)_\lambda$

$$\hat{f}(\lambda) = \sum_{a\in G} f(a)x_\lambda(a) \tag{3.150}$$

can be thought as the Fourier transform of the function $f : G \to \mathbb{C}$. Using the algebraic integration we can invert this expression (see eq. (3.148))

$$f(a) = \int_{(x_\lambda)} \hat{f}(\lambda)x_\lambda(a^{-1}) \tag{3.151}$$

But it is a well known result of the harmonic analysis over the groups, that in many cases it is possible to invert the Fourier transform, by an appropriate sum over the representations. This is true in particular for finite and compact groups. Therefore the algebraic integration should be the same thing as summing or integrating over the labels λ specifying the representation. In order to show that this is the case, let us recall a few facts about the Fourier transform over the groups [16]. First of all, given the group G, one defines the set $\hat{G}$ of the equivalence classes of the irreducible representations of G. Then, at each point λ in $\hat{G}$ we choose a unitary representation x_λ belonging to the class λ, and define the Fourier transform of the function $f : G \to \mathbb{C}$, by the eq. (3.150). In the case of compact groups, instead of the sum over the group element one has to integrate over the group by means of the invariant Haar measure. For finite groups, the inversion formula is given by

$$f(a) = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\hat{f}(\lambda) x_\lambda(a^{-1})] \tag{3.152}$$

where n_G is the order of the group and d_λ the dimension of the representation λ. Therefore, we get the identification

$$\int_{(x)} \{\cdots\} = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\{\cdots\}] \tag{3.153}$$

A more interesting way of deriving this relation, is to take in (3.150), $f(a) = \delta_{e,a}$, obtaining for its Fourier transform, $\hat{\delta} = x_\lambda(e) = 1_\lambda$, where the last symbol means the identity in the representation λ. By inserting this result into (3.152) we get the identity

$$\delta_{e,a} = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\hat{x}_\lambda(a^{-1})] \tag{3.154}$$

which, compared with eq. (3.146), gives (3.153). This shows explicitly that the algebraic integration for vector representations of G is nothing but the sum over the representations of G.

An analogous relation is obtained in the case of compact groups. This can also be obtained by a limiting procedure from finite groups, if we insert $1/n_G$, the volume of the group, in the definition of the Fourier transform. That is one defines

$$\hat{f}(\lambda) = \frac{1}{n_G} \sum_{a \in G} f(a) x_\lambda(a) \tag{3.155}$$

from which

$$f(a) = \sum_{\lambda \in \hat{G}} d_\lambda tr[\hat{f}(\lambda) x_\lambda(a^{-1})] \tag{3.156}$$

Then one can go to the limit by substituting the sum over the group elements with the Haar measure

$$\hat{f}(\lambda) = \int_G d\mu(a) f(a) x_\lambda(a) \tag{3.157}$$

The inversion formula (3.156) remains unchanged. We see that in these cases the algebraic integration sums over the elements of the space $\hat{G}$, and therefore it can be thought as the dual of the sum over the group elements (or the Haar integration for compact groups). By using the Fourier transform (3.150) and its inversion (3.151), one can easily establish the Plancherel fromula. In fact by multiplying together two Fourier transforms, one gets

$$\hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{a\in G}\left(\sum_{b\in G} f_1(b)f_2(b^{-1}a)\right) x_\lambda(a) \tag{3.158}$$

from which

$$\int_{(x)} \hat{f}_1(\lambda)\hat{f}_2(\lambda)x_\lambda(a^{-1}) = \sum_{b\in G} f_1(b)f_2(b^{-1}a) \tag{3.159}$$

and taking $a = e$ we obtain

$$\int_{(x)} \hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{b\in G} f_1(b)f_2(b^{-1}) \tag{3.160}$$

This formula can be further specialized, by taking $f_2 \equiv f$ and for f_1 the involuted of f. That is

$$\hat{f}^*(\lambda) = \sum_{a\in G} \bar{f}(a)x_\lambda(a^{-1}) \tag{3.161}$$

where use has been made of eq. (3.145). Then, from eq. (3.160) we get the Plancherel formula

$$\int_{(x)} \hat{f}^*(\lambda)\hat{f}(\lambda) = \sum_{a\in G} \bar{f}(a)f(a) \tag{3.162}$$

Let us also notice that eq. (3.158) says that the Fourier transform of the convolution of two functions on the group is the product of the Fourier transforms.

We will consider now projective representations. In this case, the product of two Fourier transforms is given by

$$\hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{a\in G} h(a)x_\lambda(a) \tag{3.163}$$

with

$$h(a) = \sum_{b\in G} f_1(b)f_2(b^{-1}a)e^{i\alpha(b,b^{-1}a)} \tag{3.164}$$

Therefore, for projective representations, the convolution product is deformed due to the presence of the phase factor. However, the Plancherel formula still holds. In fact, since in

$$h(e) = \sum_{b\in G} f_1(b)f_2(b^{-1}) \tag{3.165}$$

using eq. (2.20), the phase factor disappears, the previous derivation from eq. (3.160) to eq. (3.162) is still valid. Notice that eq. (3.163) tells us that the Fourier transform of the deformed convolution product of two functions on the group, is equal to the product of the Fourier transforms.

	$G = R^D$ $\hat{G} = R^D$	$G = Z^D$ $\hat{G} = T^D$	$G = T^D$ $\hat{G} = Z^D$	$G = Z_N^D$ $\hat{G} = Z_N^D$
$\vec{a}$	$-\infty \leq a_i \leq +\infty$	$a_i = \frac{2\pi m_i}{L}$ $m_i \in Z$	$0 \leq a_i \leq L$	$a_i = k_i,$ $0 \leq k_i \leq n-1$
$\vec{q}$	$-\infty \leq q_i \leq +\infty$	$0 \leq q_i \leq L$	$q_i = \frac{2\pi m_i}{L}$ $m_i \in Z$	$q_i = \frac{2\pi \ell_i}{N}$ $0 \leq \ell_i \leq n-1$

Table 1: *Parameterization of the abelian group G and of its dual $\hat{G}$, for $G = \mathbb{R}^D$, Z^D, T^D, Z_n^D.*

3.5.2 The case of abelian groups

In this Section we consider the case of abelian groups, and we compare the Fourier analysis made in the framework of the ATI with the more conventional one made in terms of the characters. A fundamental property of the abelian groups is that the set $\hat{G}$ of their vector unitary irreducible representations (VUIR), is itself an abelian group, the dual of G (in the sense of Pontryagin [16]). Since the VUIR's are one-dimensional, they are given by the characters of the group. We will denote the characters of G by $\chi_\lambda(a)$, where $a \in G$, and λ denotes the representation of G. For what we said before, the parameters λ can be thought as the elements of the dual group. The parameterization of the group element a and of the representation label λ are given in Table 1, for the most important abelian groups and for their dual groups, where we have used the notation $a = \vec{a}$ and $\lambda = \vec{q}$.

The characters are given by

$$\chi_\lambda(a) \equiv \chi_{\vec{q}}\,(\vec{a}) = e^{-i\vec{q}\cdot\vec{a}} \tag{3.166}$$

and satisfy the relation (here we use the additive notation for the group operation)

$$\chi_\lambda(a+b) = \chi_\lambda(a)\chi_\lambda(b) \tag{3.167}$$

and the dual

$$\chi_{\lambda_1+\lambda_2}(a) = \chi_{\lambda_1}(a)\chi_{\lambda_2}(a) \tag{3.168}$$

That is they define vector representations of the abelian group G and of its dual, $\hat{G}$. Also we can easily check that the operators

$$D_{\vec{q}}\chi_{\vec{q}}\left(\vec{a}\right) = -i\vec{a}\chi_{\vec{q}}\left(\vec{a}\right) \tag{3.169}$$

are derivations on the algebra (3.167) of the characters for any G in Table 1.

We can use the characters to define the Fourier transform of the function $f(g) : G \to \mathbb{C}$

$$\tilde{f}(\lambda) = \sum_{a\in G} f(a)\chi_\lambda(a) \tag{3.170}$$

If we evaluate the Fourier transform of the deformed convolution of eq. (3.164), we get

$$\tilde{h}(\lambda) = \sum_{a\in G} h(a)\chi_\lambda(a) = \sum_{a,b\in G} f(a)\chi_\lambda(a)e^{i\alpha(a,b)}g(b)\chi_\lambda(b) \tag{3.171}$$

In the case of vector representations the Fourier transform of the convolution is the product of the Fourier transforms. In the case of projective representations, the result, using the derivation introduced before, can be written in terms of the Moyal product (we omit here the vector signs)

$$\tilde{h}(\lambda) = \tilde{f}(\lambda) \star \tilde{g}(\lambda) = e^{-i\alpha(D_{\lambda'},D_{\lambda''})}\tilde{f}(\lambda')\tilde{g}(\lambda'')\Big|_{\lambda'=\lambda''=\lambda} \tag{3.172}$$

Therefore, the Moyal product arises in a very natural way from the projective group algebra. On the other hand, we have shown in the previous Section, that the use of the Fourier analysis in terms of the projective representations avoids the Moyal product. The projective representations of abelian groups allow a derivation on the algebra, analogous to the one in eq. (3.169), with very special features. In fact we check easily that

$$\vec{D}x_\lambda(\vec{a}) = -i\vec{a}x_\lambda(\vec{a}) \tag{3.173}$$

is a derivation, and furthermore

$$\int_{(x_\lambda)} \vec{D}x_\lambda(\vec{a}) = 0 \tag{3.174}$$

From this it follows, by linearity, that the integral of $\vec{D}$ applied to any function on the algebra is zero

$$\int_{(x_\lambda)} \vec{D}\left(\sum_{a\in G} f(\vec{a})x_\lambda(\vec{a})\right) = 0 \tag{3.175}$$

This relation is very important because, as we have shown in [10], the automorphisms generated by $\vec{D}$, that is $\exp(\vec{\alpha}\cdot\vec{D})$, leave invariant the integration measure of the ATI (see also later on). Notice that this derivation generalizes the derivative with respect to the parameter $\vec{q}$, although this has no meaning in the present case. In the case of nonabelian groups, a derivation sharing the previous properties can be defined only if there exists a mapping $\sigma : G \to \mathbb{C}$, such that

$$\sigma(ab) = \sigma(a) + \sigma(b), \qquad a, b \in G \tag{3.176}$$

since in this case, defining

$$Dx(a) = \sigma(a)x(a) \tag{3.177}$$

we get

$$\begin{aligned} D(x(a)x(b)) &= \sigma(ab)x(a)x(b) = (\sigma(a)+\sigma(b))x(a)x(b) \\ &= (Dx(a))x(b) + x(a)(Dx(b)) \end{aligned} \tag{3.178}$$

Having defined derivations and integrals one has all the elements for the harmonic analysis on the projective representations of an abelian group.

Let us start considering $G = R^D$. In the case of vector representations we have

$$x_{\vec{q}}\,(\vec{a}) = e^{-i\vec{q}\cdot\vec{a}} \tag{3.179}$$

with $\vec{a} \in G$, and $\vec{q} \in \hat{G} = R^D$ labels the representation. The Fourier transform is

$$\hat{f}(\vec{q}) = \int d^D\vec{a} f(\vec{a}) e^{-i\vec{q}\cdot\vec{a}} \tag{3.180}$$

Here the Haar measure for G coincides with the ordinary Lebesgue measure. Also, since $\hat{G} = R^D$, we can invert the Fourier transform by using the Haar measure on the dual group, that is, again the Lebesgue measure. In the projective case, eq. (3.179) still holds true, if we assume $\vec{q}$ as a vector operator satisfying the commutation relations

$$[q_i, q_j] = i\eta_{ij} \tag{3.181}$$

with η_{ij} numbers which can be related to the cocycle, by using the Baker-Campbell-Hausdorff formula

$$e^{-i\vec{q}\cdot\vec{a}} e^{-i\vec{q}\cdot\vec{b}} = e^{-i\eta_{ij}a_i b_j/2} e^{-i\vec{q}\cdot(\vec{a}+\vec{b})} \tag{3.182}$$

giving

$$\alpha(\vec{a}, \vec{b}) = -\frac{1}{2}\eta_{ij} a_i b_j \tag{3.183}$$

The inversion of the Fourier transform can now be obtained by the ATI in the form

$$f(\vec{a}) = \int_{(\vec{q})} \hat{f}(\vec{q}) x_{\vec{q}}\,(-\vec{a}) \tag{3.184}$$

where the dependence on the representation is expressed in terms of $\vec{q}$, thought now they are not coordinates on $\hat{G}$. We recall that in this case, eq. (3.146) gives

$$\int_{(\vec{q})} x_{\vec{q}}\,(\vec{a}) = \delta^D(\vec{a}) \tag{3.185}$$

Therefore, the relation between the integral in ATI and the Lebesgue integral in $\hat{G}$, in the vector case is

$$\int_{(\vec{q})} = \int \frac{d^D\vec{q}}{(2\pi)^D} \tag{3.186}$$

In the projective case the right hand side of this relation has no meaning, whereas the left hand side is still well defined. Also, we cannot maintain the interpretation of the q_i's as coordinates on the dual space $\hat{G}$. However, we can define elements of $\mathcal{A}(G)$ having the properties of the q_i's (in particular satisfying eq. (3.181)), by using the Fourier analysis. That is we define

$$q_i = \int d^D\vec{a} \left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right) x_{\vec{q}}\,(\vec{a}) \tag{3.187}$$

which is an element of $\mathcal{A}(G)$ obtained by Fourier transforming a distribution over G, which is a honestly defined space. From this definition we can easily evaluate the product

$$q_i x_{\vec{q}}\,(\vec{a}) = \int d^D\vec{b} \left(-i\frac{\partial}{\partial b_i}\delta^D(\vec{b})\right) x_{\vec{q}}\,(\vec{b}) x_{\vec{q}}\,(\vec{a}) \tag{3.188}$$

Using the algebra and integrating by parts, one gets the result

$$q_i x_{\vec{q}}\,(\vec{a}) = i\nabla_i x_{\vec{q}}\,(\vec{a}) \tag{3.189}$$

where

$$\nabla_i = \frac{\partial}{\partial a_i} + i\alpha_{ij}a_j \tag{3.190}$$

where $\alpha_{ij} = \alpha(\vec{e}_{(i)}, \vec{e}_{(j)})$, with $\vec{e}_{(i)}$ an orthonormal basis in $\mathbb{R}^D$. In a completely analogous way one finds

$$x_{\vec{q}}\,(\vec{a}) q_i = i\overline{\nabla}_i x_{\vec{q}}\,(\vec{a}) \tag{3.191}$$

where

$$\overline{\nabla}_i = \frac{\partial}{\partial a_i} - i\alpha_{ij}a_j \tag{3.192}$$

Then, we evaluate the commutator

$$[q_i, \hat{f}(\vec{q})] = \int d^D\vec{a} \left[-i\left(\overline{\nabla}_i - \nabla_i\right) f(\vec{a})\right] x_{\vec{q}}(\vec{a}) \tag{3.193}$$

where we have done an integration by parts. We get

$$[q_i, \hat{f}(\vec{q})] = -2i\alpha_{ij} D_{q_j} \hat{f}(\vec{q}) \tag{3.194}$$

where D_{q_j} is the derivation (3.173), with q_j a reminder for the direction along wich the derivation acts upon. In particular, from

$$D_{q_j} q_i = \int d^D\vec{a} \left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right)(-ia_j)x_{\vec{q}}(\vec{a}) = \delta_{ij} \tag{3.195}$$

we get

$$[q_i, q_j] = -2i\alpha_{ij} \tag{3.196}$$

in agreement with eq. (3.181), after the identification $\alpha_{ij} = -\eta_{ij}/2$.

The automorphisms induced by the derivations (3.173) are easily evaluated

$$S(\vec{\alpha})x_{\vec{q}}\,(\vec{a}) = e^{\vec{\alpha}\cdot D_{\vec{q}}}x_{\vec{q}}\,(\vec{a}) = e^{-i\vec{\alpha}\cdot\vec{a}}x_{\vec{q}}\,(\vec{a}) = x_{\vec{q}+\vec{\alpha}}(\vec{a}) \tag{3.197}$$

where the last equality follows from

$$\int d^D\vec{a}\left(-i\frac{\partial}{\partial a_i}\delta^D(\vec{a})\right)e^{\vec{\alpha}\cdot D_{\vec{q}}}x_{\vec{q}}\,(\vec{a}) = q_i + \alpha_i \tag{3.198}$$

Meaning that in the vector case, $S(\vec{\alpha})$ induces translations in $\hat{G}$. Since $D_{\vec{q}}$ satisfies the eq. (3.175), it follows from [10] that the automorphism $S(\vec{\alpha})$ leaves invariant the algebraic integration measure

$$\int_{(\vec{q})} = \int_{(\vec{q}+\vec{\alpha})} \tag{3.199}$$

This shows that it is possible to construct a calculus completely analogous to the one that we have on $\hat{G}$ in the vector case, just using the Fourier analysis following by the algebraic definition of the integral. We can push this analysis a little bit further by looking at the following expression

$$\int_{(\vec{q})}\hat{f}(\vec{q})\;q_i\;x_{\vec{q}}\,(-\vec{a}) = -i\left(\frac{\partial}{\partial a_i} + i\alpha_{ij}a_j\right)f(\vec{a}) \tag{3.200}$$

where we have used eq. (3.189). In the case $D = 2$ this equation has a physical interpretation in terms of a particle of charge e, in a constant magnetic field B. In fact, the commutators among canonical momenta are

$$[\pi_i, \pi_j] = ieB\epsilon_{ij} \tag{3.201}$$

where ϵ_{ij} is the 2-dimensional Ricci tensor. Therefore, identifying π_i with q_i, we get $\alpha_{ij} = -eB\epsilon_{ij}/2$. The corresponding vector potential is given by

$$A_i(\vec{a}) = -\frac{1}{2}\epsilon_{ij}Ba_j = \frac{1}{e}\alpha_{ij}a_j \tag{3.202}$$

Then, eq. (3.200) tells us that the operation $\hat{f}(\vec{q}) \to \hat{f}(\vec{q})q_i$, corresponds to take the covariant derivative

$$-i\frac{\partial}{\partial a_i} + eA_i(\vec{a}) \tag{3.203}$$

of the inverse Fourier transform of $\hat{f}(\vec{q})$. An interesting remark is that a translation in $\vec{q}$ generated by $\exp(\vec{\alpha}\cdot\vec{D})$, gives rise to a phase transformation on $f(\vec{a})$. First of all, by using the invariance of the integration measure we can check that

$$\hat{f}(\vec{q}+\vec{\alpha}) = e^{\vec{\alpha}\cdot\vec{D}}\hat{f}(\vec{q}) \tag{3.204}$$

In fact

$$\int_{(\vec{q})} \hat{f}(\vec{q}+\vec{\alpha})x_{\vec{q}}\,(-\vec{a}) = \int_{(\vec{q}-\vec{\alpha})} \hat{f}(\vec{q})x_{\vec{q}-\vec{\alpha}}\,(-\vec{a}) = e^{-i\vec{\alpha}\cdot\vec{a}}f(\vec{a}) \tag{3.205}$$

Then, we have

$$\int_{(\vec{q})} \left(e^{\vec{\alpha}\cdot\vec{D}}\hat{f}(\vec{q})\right) x_{\vec{q}}\,(-\vec{a}) = \int_{(\vec{q})} \hat{f}(\vec{q}) \left(e^{-\vec{\alpha}\cdot\vec{D}}x_{\vec{q}}(-\vec{a})\right) = e^{-i\vec{\alpha}\cdot\vec{a}}f(\vec{a}) \tag{3.206}$$

where we have made use of eq. (3.197). This shows eq. (3.204), and at the same time our assertion. From eq. (3.200), this is equivalent to a gauge transformation on the gauge potential $\mathcal{A}_i = \alpha_{ij}a_j$, $\mathcal{A}_i \to \mathcal{A}_i - \partial_i\Lambda$, with $\Lambda = \vec{\alpha}\cdot\vec{a}$. Therefore, we see here explicitly the content of a projective representation in the basis of the functions on the group. One starts assigning the two-form α_{ij}. Given that, one makes a choice for the vector potential. For instance in the previous analysis we have chosen $\alpha_{ij}a_j$. Any possible projective representation corresponds to a different choice of the gauge. In the dual Fourier basis this corresponds to assign a fixed set of operators q_i, with commutation relations determined by the two-form. All the possible projective representations are obtained by translating the operators q_i's. Of course, this is equivalent to say that the projective representations are the central extension of the vector ones, and that they are determined by the cocycles. But the previous analysis shows that the projective representations generate noncommutative spaces, and that the algebraic integration, allowing us to define a Fourier analysis, gives the possibility of establishing the calculus rules.

Consider now the case $G = Z^D$. Let us introduce an orthonormal basis on the square lattice defined by Z^D, $\vec{e}_{(i)}$, $i = 1,\cdots,D$. Then, any element of the algebra can be reconstructed in terms of a product of the elements

$$U_i = x(\vec{e}_{(i)}) \tag{3.207}$$

corresponding to a translation along the direction i by one lattice site. In general we will have

$$x(\vec{m}) = e^{i\theta(\vec{m})}U_1^{m_1}\cdots U_D^{m_D}, \qquad \vec{m} = \sum_i m_i\vec{e}_{(i)} \tag{3.208}$$

with θ a calculable phase factor. The quantities U_i play the same role of $\vec{q}$ of the previous example. The Fourier transform is defined by

$$\hat{f}(\vec{U}) = \sum_{\vec{m}\in Z^D} f(\vec{m})x_{\vec{U}}(\vec{m}) \tag{3.209}$$

where the dependence on the representation is expressed in terms of $\vec{U}$, denoting the collections of the U_i's. The inverse Fourier transform is defined by

$$f(\vec{m}) = \int_{\vec{U}} \hat{f}(\vec{U})x_{\vec{U}}(-\vec{m}) \tag{3.210}$$

where the integration rule is

$$\int_{(\vec{U})} x_{\vec{U}}(\vec{m}) = \delta_{\vec{m},\vec{0}} \tag{3.211}$$

Therefore, the Fourier transform of U_i is simply $\delta_{\vec{m},\vec{e}_{(i)}}$. The algebraic integration for the vector case is

$$\int_{(\vec{U})} \to \int_0^L \frac{d^D\vec{q}}{L^D} \tag{3.212}$$

Since the set $\vec{U}$ is within the generators of the algebra, to establish the rules of the calculus is a very simple matter. Eq. (3.207) is the definition of the set $\vec{U}$, analogous to eq. (3.187). In place of eq. (3.194) we get

$$U_i\hat{f}(\vec{U})U_i^{-1} = e^{-2\alpha_{ij}D_j}\hat{f}(\vec{U}) \tag{3.213}$$

Here D_j is the j-th component of the derivation $\vec{D}$ which acts upon U_i as

$$D_iU_j = -i\delta_{ij}U_j \tag{3.214}$$

By choosing $\hat{f}(\vec{U}) = U_k$ we have

$$U_iU_kU_i^{-1}U_k^{-1} = e^{2i\alpha_{ik}} \tag{3.215}$$

which is the analogue of the commutator among the q_i's. The automorphisms generated by $\vec{D}$ are

$$S(\vec{\phi})x_{\vec{U}}(\vec{m}) = e^{\vec{\phi}\cdot\vec{D}}x_{\vec{U}}(\vec{m}) = e^{-i\vec{\phi}\cdot\vec{m}}x_{\vec{U}}(\vec{m}) \tag{3.216}$$

From which we see that

$$U_i \to S(\vec{\phi})U_i = e^{-i\phi_i}U_i \tag{3.217}$$

This transformation corresponds to a trivial cocycle. As in the case $G = \mathbb{R}^D$ it gives rise to a phase transformation on the group functions

$$\int_{(\vec{U})} \left(e^{\vec{\alpha}\cdot\vec{D}}\hat{f}(\vec{U})\right) x_{\vec{U}}(-\vec{m}) = \int_{\vec{U})} \hat{f}(\vec{U}) \left(e^{-\vec{\alpha}\cdot\vec{D}}x_{\vec{U}}(-\vec{m})\right) = e^{i\vec{\phi}\cdot\vec{m}}f(\vec{m}) \tag{3.218}$$

Of course, all these relations could be obtained by putting $U_i = \exp(-iq_i)$, with q_i defined as in the case $G = R^D$.

Finally, in the case $G = Z_n^D$, the situation is very much alike Z^D, that is the algebra can be reconstructed in terms of a product of elements

$$U_i = x(\vec{e}_{(i)}) \tag{3.219}$$

satisfying

$$U_i^n = 1 \tag{3.220}$$

Therefore we will not repeat the previous analysis but we will consider only the case $D = 2$, where U_1 and U_2 can be expressed as [17]

$$(U_1)_{a,b} = \delta_{a,b-1} + \delta_{a,n}\delta_{b,1}, \quad (U_2)_{a,b} = e^{\frac{2\pi i}{n}(a-1)}\delta_{a,b}, \quad a,b = 1,\cdots,n \tag{3.221}$$

The elements of the algebra are reconstructed as

$$x_{\vec{U}}(\vec{m}) = e^{i\frac{\pi}{n}m_1 m_2} U_1^{m_1} U_2^{m_2} \tag{3.222}$$

The cocycle is now

$$\alpha(\vec{m}_1, \vec{m}_2) = -\frac{2\pi}{n}\epsilon_{ij} m_{1i} m_{2j} \tag{3.223}$$

In this case we can compare the algebraic integration rule

$$\int_{\vec{U}} x_{\vec{U}}(\vec{m}) = \delta_{\vec{m},\vec{0}} \tag{3.224}$$

with

$$Tr[x_{\vec{U}}(\vec{m})] = n\delta_{\vec{m},\vec{0}} \tag{3.225}$$

A generic element of the algebra is a $n \times n$ matrix

$$A = \sum_{m_1,m_2=0}^{n-1} c_{m_1 m_2} x_{\vec{U}}(\vec{m}) \tag{3.226}$$

and therefore

$$\int_{\vec{U}} A = \frac{1}{n} Tr[A] \tag{3.227}$$

In ref. [10] we have shown that the algebraic integration over the algebra of the $n \times n$ matrices $\mathcal{A}_n$ is given by

$$\int_{\mathcal{A}_n} A = Tr[A] \tag{3.228}$$

implying

$$\int_{\vec{U}} A = \frac{1}{n} \int_{\mathcal{A}_n} A \tag{3.229}$$

3.5.3 The example of the algebra on the circle

A particular example of a group algebra is the algebra on the circle defined by

$$z^n z^m = z^{n+m}, \quad -\infty \leq n, m \leq +\infty \tag{3.230}$$

with z restricted to the unit circle.

$$z^* = z^{-1} \tag{3.231}$$

This is a group algebra over $\mathbb{Z}$. Defining the ket

$$|z\rangle = \begin{pmatrix} \cdot \\ z^{-i} \\ \cdot \\ 1 \\ z \\ \cdot \\ z^i \\ \cdot \end{pmatrix} \tag{3.232}$$

the R_i and L_i matrices are given by

$$(R_i)_{jk} = \delta_{i+j,k}, \quad (L_i)_{jk} = \delta_{i+k,j} \tag{3.233}$$

and from our previous construction, the matrix C is given by

$$(C)_{ij} = (C^{-1})_{ij} = \delta_{i+j,0} \tag{3.234}$$

or. more explicitly by

$$C = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 1 & \cdot \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{3.235}$$

showing that

$$C: \quad z^i \to z^{-i} \tag{3.236}$$

In fact

$$(C^{-1}L_iC)_{lp} = \delta_{l,-m}\delta_{i+n,m}\delta_{n,-p} = \delta_{i-p,-l} = \delta_{i+l,p} = (R_i)_{lp} \tag{3.237}$$

In this case the C matrix is nothing but the complex conjugation ($z \to z^* = z^{-1}$). The completeness relation reads now

$$\int_{(z)} z^i z^{-j} = \delta_{ij} \tag{3.238}$$

from which

$$\int_{(z)} z^k = \delta_{k0} \tag{3.239}$$

Our algebraic definition of integral can be interpreted as an integral along a circle C around the origin. In fact we have

$$\int_{(z)} = \frac{1}{2\pi i}\int_C \frac{dz}{z} \tag{3.240}$$

Chapter 4

Examples of Associative Non Self-Conjugated Algebras

4.1 The algebra of the bosonic oscillator

We will start trying to reproduce the integration rules in the bosonic case. It is convenient to work in the coherent state basis. The coherent states are defined by the relation

$$a|z\rangle = |z\rangle z \tag{4.1}$$

where a is the annihilation operator, $[a, a^\dagger] = 1$. The representative of a state at fixed occupation number in the coherent state basis is

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}} \tag{4.2}$$

So we will consider as elements of the algebra the quantities

$$x_i = \frac{z^i}{\sqrt{i!}}, \quad i = 0, 1, \cdots, \infty \tag{4.3}$$

We introduce the ket

$$\begin{pmatrix} 1 \\ z \\ z^2/\sqrt{2!} \\ \cdot \\ \cdot \end{pmatrix} \tag{4.4}$$

The algebra is defined by the multiplication rules

$$x_i x_j = \frac{z^{i+j}}{\sqrt{i!\,j!}} = x_{i+j}\sqrt{\frac{(i+j)!}{i!\,j!}} \tag{4.5}$$

from which

$$f_{ijk} = \delta_{i+j,k}\sqrt{\frac{k!}{i!\,j!}} \tag{4.6}$$

It follows

$$(R_i)_{jk} = \delta_{i+j,k}\sqrt{\frac{k!}{i!\,j!}} \tag{4.7}$$

and

$$(L_i)_{jk} = \delta_{i+k,j}\sqrt{\frac{j!}{i!\,k!}} \tag{4.8}$$

In particular we get

$$(R_1)_{jk} = \sqrt{k}\,\delta_{j+1,k}, \qquad (L_1)_{jk} = \sqrt{k+1}\,\delta_{j-1,k} \tag{4.9}$$

Therefore R_1 and L_1 are nothing but the representative, in the occupation number basis, of the annihilation and creation operators respectively. It follows that the C matrix cannot exist, because $[R_1, L_1] = 1$, and a unitary transformation cannot change this commutation relation into $[L_1, R_1] = -1$. For an explicit proof consider, for instance, R_1

$$R_1 = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \tag{4.10}$$

If the matrix C would exist it would be possible to find states $\langle z|$ such that

$$\langle z|R_1 = z\langle z| \tag{4.11}$$

with

$$\langle z| = (f_0(z), f_1(z), \cdots) \tag{4.12}$$

This would mean

$$\langle z|R_1 = (0, f_0(z), \sqrt{2}f_1(z), \cdots) = (zf_0(z), zf_1(z), zf_2(z), \cdots) \tag{4.13}$$

which implies

$$f_0(z) = f_1(z) = f_2(z) = \cdots = 0 \tag{4.14}$$

Now, having shown that no C matrix exists, we will consider the complex conjugated algebra with generators constructed in terms of z^*, where z^* is the complex conjugate of z. Correspondingly we introduce the vectors

$$R_i|z^*\rangle = |z^*\rangle z_i^*, \qquad \langle z|L_i = z_i\langle z| \tag{4.15}$$

where, for instance

$$\langle z| = (1, \frac{z}{\sqrt{1!}}, \frac{z^2}{\sqrt{2!}}, \cdots) \tag{4.16}$$

and the integration rules give

$$\int_{(z,z^*)} \frac{z^i z^{*j}}{\sqrt{i!j!}} = \delta_{i,j} \tag{4.17}$$

We see that they are equivalent to the gaussian integration

$$\int_{(z,z^*)} = \int \frac{dz^* dz}{2\pi i} \exp(-|z|^2) \tag{4.18}$$

4.2 The q-oscillator

A generalization of the bosonic oscillator is the q-bosonic oscillator [14]. We will use the definition given in [15]

$$b\bar{b} - q\bar{b}b = 1 \tag{4.19}$$

with q real and positive. We assume as elements of the algebra $\mathcal{A}$, the quantities

$$x_i = \frac{z^i}{\sqrt{i_q!}} \tag{4.20}$$

where z is a complex number,

$$i_q = \frac{q^i - 1}{q - 1} \tag{4.21}$$

and

$$i_q! = i_q (i-1)_q \cdots 1 \tag{4.22}$$

The structure constants are

$$f_{ijk} = \delta_{i+j,k} \sqrt{\frac{k_q!}{i_q! j_q!}} \tag{4.23}$$

and therefore

$$(R_i)_{jk} = \delta_{i+j,k} \sqrt{\frac{k_q!}{i_q! j_q!}}, \qquad (L_i)_{jk} = \delta_{i+k,j} \sqrt{\frac{j_q!}{i_q! k_q!}} \tag{4.24}$$

In particular

$$(R_1)_{jk} = \delta_{j+1,k} \sqrt{k_q}, \qquad (L_1)_{jk} = \delta_{j-1,k} \sqrt{(k+1)_q} \tag{4.25}$$

We see that R_1 and L_1 satisfy the q-bosonic algebra

$$R_1 L_1 - q L_1 R_1 = 1 \tag{4.26}$$

For q real and positive, no C matrix exists, so, according to our rules

$$\int_{(z,z^*)_q} \frac{z^i z^{*j}}{i_q! j_q!} = \delta_{ij} \tag{4.27}$$

This integration can be expressed in terms of the so called q-integral (see ref. [18]), by using the representation of $n_q!$ as a q-integral

$$n_q! = \int_0^{1/(1-q)} d_q t \; e_{1/q}^{-qt} \, t^n \tag{4.28}$$

where the q-exponential is defined by

$$e_q^t = \sum_{n=0}^{\infty} \frac{z^n}{n_q!} \tag{4.29}$$

and the q-integral through

$$\int_0^a d_q t f(t) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n \tag{4.30}$$

Then the two integrations are related by $(z = |z| \exp(i\phi))$

$$\int_{(z,z^*)_q} = \int \frac{d\phi}{2\pi} \int d_q(|z|^2)\ e_{1/q}^{-q|z|^2} \tag{4.31}$$

The Jackson integral is the inverse of the q-derivative

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} \tag{4.32}$$

In fact, for

$$F(a) = \int_0^a d_q t f(t) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n \tag{4.33}$$

one has

$$(D_q F)(a) = f(a) \tag{4.34}$$

as can be checked using

$$F(qa) = qa(1-q) \sum_{n=0}^{\infty} f(aq^{n+1}) q^n = a(1-q) \sum_{n=1}^{\infty} f(aq^n) q^n = F(a) - a(1-q) f(a) \tag{4.35}$$

Chapter 5

Examples of Nonassociative Self-Conjugated Algebras

5.1 Jordan algebras of symmetric bilinear forms

The Clifford algebra C_N is defined in terms of $N+1$ generators

$$e_i = (e_0, e_a), \quad a = 1, \cdots, N, \quad i = 0, 1, \cdots, N+1 \tag{5.1}$$

such that

$$[e_a, e_b]_+ = 2\delta_{ab} \tag{5.2}$$

and e_0 is the identity. We will introduce the associated Jordan algebra as the algebra composed by the elements e_i, with product rules (see ref. [19])

$$e_i \star e_j = \frac{1}{2}[e_i, e_j]_+ \tag{5.3}$$

In the following we will omit the $\star$ in the product of the elements of the algebra. Therefore, the product rules are

$$e_a e_b = \delta_{ab} e_0 \tag{5.4}$$

$$e_0 e_a = e_a e_0 = e_a \tag{5.5}$$

$$e_0 e_0 = e_0 \tag{5.6}$$

This algebra is a non-associative one. The structure constants are

$$f_{abc} = 0 \tag{5.7}$$

$$f_{ab0} = f_{a0b} = f_{0ab} = \delta_{ab} \tag{5.8}$$

$$f_{a00} = f_{0a0} = f_{00a} = 0 \tag{5.9}$$

$$f_{000} = 1 \tag{5.10}$$

Correspondingly we get the following expressions for the matrices of the right multiplications ($(R_i)_{jk} = f_{jik}$)

$$(R_a)_{bc} = 0, \qquad (R_0)_{bc} = \delta_{bc} \tag{5.11}$$

$$(R_a)_{00} = 0, \qquad (R_0)_{00} = 1 \tag{5.12}$$

$$(R_a)_{0b} = \delta_{ab}, \qquad (R_0)_{0a} = 0 \tag{5.13}$$

$$(R_a)_{b0} = \delta_{ab}, \qquad (R_0)_{a0} = 0 \tag{5.14}$$

Notice that

$$R_i^T = R_i \tag{5.15}$$

and that the algebra is commutative

$$e_i e_j = e_j e_i \tag{5.16}$$

By using the vectors

$$|e\rangle = \begin{pmatrix} e_0 \\ e_1 \\ \cdot \\ \cdot \\ e_N \end{pmatrix} \tag{5.17}$$

we have from the very definition

$$R_i|e\rangle = |e\rangle e_i \tag{5.18}$$

By taking the transposed of this equation we get

$$\langle e|R_i^T = \langle e|e_i \tag{5.19}$$

and using the symmetry of R_i and the commutativity of the algebra we get

$$\langle e|R_i = e_i\langle e| \tag{5.20}$$

showing that

$$L_i = R_i \tag{5.21}$$

Therefore

$$C = 1 \tag{5.22}$$

Then the integration rules follow from

$$\int_{C_N} e_i e_j = \delta_{ij} \tag{5.23}$$

that is

$$\int_{C_N} e_i = \delta_{i0} \tag{5.24}$$

Therefore

$$\int_{C_N} e_a = 0, \qquad \int_{C_N} e_0 = 1 \tag{5.25}$$

If the algebra C_N is realized by matrices od dimension d, then

$$\int_{C_N} [\cdots] = \frac{1}{d} Tr[\cdots] \tag{5.26}$$

Consider now the most general derivation on C_N. Let us put

$$De_i = d_{ij} e_j \tag{5.27}$$

The conditions for D being a derivation are

$$D(e_0^2) = 2De_0 = De_0 \to De_0 = 0 \tag{5.28}$$

implying

$$d_{0i} = 0 \tag{5.29}$$

Then

$$D(e_a e_b) = \delta_{ab} De_0 = 0 = e_a d_{bi} e_i + d_{ai} e_i e_b = (e_b d_{a0} + e_a d_{b0}) + (d_{ab} + d_{ba}) \tag{5.30}$$

giving rise to

$$d_{ab} = -d_{ba}, \qquad d_{a0} = 0 \tag{5.31}$$

That is

$$De_0 = 0, \qquad De_a = d_{ab} e_b \tag{5.32}$$

with d an antisymmetric matrix. The corresponding automorphism

$$S = \exp(D) \tag{5.33}$$

corresponds to an orthogonal transformation of the elements e_a leaving invariant e_0. From this derivation we have found the the most general continuous automorphism of C_N is an element of the group $SO(N)$. The complete group of automorphisms is given by $O(N)$ since the algebra is invariant under the parity operation

$$e_a \to -e_a, \quad e_0 \to e_0 \tag{5.34}$$

From our integration rules follow also that for the most general derivation on C_N one has

$$\int_{C_N} Df(e) = 0 \tag{5.35}$$

where $f(e)$ is an arbitrary (linear) function on C_N. Therefore the integration is invariant under $SO(N)$ transformations.

The previous algebra has the following interpretation. Given a vector space with a basis e_a equipped with a metrics

$$\langle v_1 | v_2 \rangle = g_{ab} v_1^a v_2^b \tag{5.36}$$

we define an algebra by enlarging the elements e_a to $e_i = (e_0, e_a)$ where e_0 is the identity and product rule

$$e_a e_b = g_{ab} e_0 \tag{5.37}$$

or, for generic elements

$$v_1 v_2 = \langle v_1 | v_2 \rangle e_0 \tag{5.38}$$

The structure constants and the matrix elements of the right multiplications are given by

$$f_{abc} = 0 \tag{5.39}$$
$$f_{a0b} = g_{ab} \tag{5.40}$$
$$f_{ab0} = f_{0ab} = \delta_{ab} \tag{5.41}$$
$$f_{a00} = f_{0a0} = f_{00a} = 0 \tag{5.42}$$
$$f_{000} = 1 \tag{5.43}$$

and

$$(R_a)_{bc} = 0, \qquad (R_0)_{bc} = \delta_{bc} \tag{5.44}$$
$$(R_a)_{00} = 0, \qquad (R_0)_{00} = 1 \tag{5.45}$$
$$(R_a)_{0b} = \delta_{ab}, \qquad (R_0)_{0a} = 0 \tag{5.46}$$
$$(R_a)_{b0} = g_{ab}, \qquad (R_0)_{a0} = 0 \tag{5.47}$$

We see that the R_i matrices are given by

$$R_0 = 1, \quad R_a = \begin{pmatrix} 0 & v_a \\ w_a & 0 \end{pmatrix} \tag{5.48}$$

where the vectors v_a and w_a are defined by

$$(v_a)_b = \delta_{ab}, \qquad (w_a)_b = g_{ab} \tag{5.49}$$

These vectors have the following property

$$w_a = g v_a \tag{5.50}$$

where g is the matrix with elements g_{ab}. The matrices R_a are not any more symmetric, but since the algebra is abelian we have

$$L_a = R_a^T \tag{5.51}$$

Therefore the C matrix is defined by the relation

$$C R_a = R_a^T C \tag{5.52}$$

Writing

$$C = \begin{pmatrix} c & z \\ z & A \end{pmatrix} \tag{5.53}$$

where a is a number, z a vector and A a matrix. One gets the conditions

$$z \otimes v_a = v_a \otimes z, \quad A w_a = a v_a \tag{5.54}$$

implying

$$z = 0, \quad A = a g^{-1} \tag{5.55}$$

By choosing $a = 1$, we get

$$C = \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \tag{5.56}$$

Then we obtain the integration rules

$$\int_{(e)} e_a = 0, \quad \int_{(e)} e_0 = 1 \tag{5.57}$$

From this it is easy to extend the treatment to a complex vector space equipped with a Kähler metrics. A basis will be given by $e_\alpha = (e_a, e_a^*)$, with an algebra

$$e_a e_b^* = \eta_{a\bar{b}} = e_b^* e_a = \eta_{\bar{b}a}, \quad e_a e_b = e_a^* e_b^* = 0 \tag{5.58}$$

We require also that the conjugation, $e_a \to e_a^*$ is an involution of the algebra, implying

$$\eta_{a\bar{b}}^* = \eta_{b\bar{a}} \tag{5.59}$$

Notice that in the case the metrics reduce to a Kronecker delta, the algebra is nothing but the one obtained by taking as a product the anticommutators of Fermi creation and annihilation operators. Since we have ($\alpha = (a, \bar{a})$)

$$e_\alpha e_\beta = \begin{pmatrix} 0 & \eta_{a\bar{b}} \\ \eta_{\bar{a}b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \eta_{a\bar{b}} \\ \eta_{\bar{b}a}^* & 0 \end{pmatrix} = g_{\alpha\beta} \tag{5.60}$$

where we have made use of eq. (5.59) and of the commutativity of the algebra, if follows

$$g = g^\dagger \tag{5.61}$$

Then we check easily that

$$C = \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \tag{5.62}$$

with

$$g^{-1} = \begin{pmatrix} 0 & \eta^{-1} \\ (\eta^\dagger)^{-1} & 0 \end{pmatrix} \tag{5.63}$$

a hermitian matrix. In particular

$$C|e\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \eta^{-1} \\ 0 & (\eta^\dagger)^{-1} & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_a \\ e_a^* \end{pmatrix} = \begin{pmatrix} e_0 \\ \eta^{-1} e_a^* \\ (\eta^\dagger)^{-1} e_a \end{pmatrix} \tag{5.64}$$

In particular for $\eta = 1$, C is the matrix representation of the involution. In general, one has to introduce the matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^\dagger \end{pmatrix} \tag{5.65}$$

Then, the involution is given by the product

$$IC = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.66}$$

The integration rules are

$$\int_{(e)} e_a = 0, \quad \int_{(e)} e_a^* = 0, \quad \int_{(e)} e_0 = 1 \tag{5.67}$$

5.2 The algebra of octonions

We will discuss now how to integrate over the octonionic algebra (see [20]). This algebra (said also a Cayley algebra) is defined in terms of the multiplication table of its seven imaginary units e_A

$$e_A e_B = -\delta_{AB} + a_{ABC} e_C, \quad A, B, C = 1, \cdots, 7 \tag{5.68}$$

where a_{ABC} is completely antisymmetric and equal to $+1$ for $(ABC) = (1, 2, 3)$, (2,4,6), (4,3,5), (3,6,7), (6,5,1), (5,7,2) and (7,1,4). The automorphism group of the algebra is G_2. We define also in this case the split basis as

$$\begin{aligned} u_0 &= \frac{1}{2}(1 + ie_7), \qquad u_0^* = \frac{1}{2}(1 - ie_7) \\ u_i &= \frac{1}{2}(e_i + ie_{i+3}), \qquad u_i^* = \frac{1}{2}(e_i - ie_{i+3}) \end{aligned} \tag{5.69}$$

where $i = 1, 2, 3$. In this basis the multiplication rules are given in Table 4.

	u_0	u_0^*	u_j	u_j^*
u_0	u_0	0	u_j	0
u_0^*	0	u_0^*	0	u_j^*
u_i	0	u_i	$\epsilon_{ijk} u_k^*$	$-\delta_{ij} u_0$
u_i^*	u_i^*	0	$-\delta_{ij} u_0^*$	$\epsilon_{ijk} u_k$

Table 4: *Multiplication table for the octonionic algebra.*

This algebra is non-associative and in the split basis it has an automorphism group $SU(3)$. The non-associativity can be checked by taking, for instance,

$$u_i(u_j u_k^*) = u_i(-\delta_{jk}u_0) = 0 \tag{5.70}$$

and comparing with

$$(u_i u_j)u_k^* = \epsilon_{ijm}u_m^* u_k^* = -\epsilon_{ijk}\epsilon_{kmn}u_n \tag{5.71}$$

We introduce the ket

$$|u\rangle = \begin{pmatrix} u_0 \\ u_0^* \\ u_i \\ u_i^* \end{pmatrix} \tag{5.72}$$

and one can easily evaluate the matrices R and L corresponding to right and left multiplication. We will not give here the explicit expressions, but one can easily see some properties. For instance, one can evaluate the anticommutator $[R_i, R_j^*]_+$, by using the following relation

$$[R_i, R_j^*]_+|u\rangle = R_i|u\rangle u_j^* + R_j^*|u\rangle u_i = (|u\rangle u_i)u_j^* + (|u\rangle u_j^*)u_i \tag{5.73}$$

The algebra of the anticommutators of R_i, R_i^* turns out to be the algebra of three Fermi oscillators (apart from the sign)

$$[R_i, R_j^*]_+ = -\delta_{ij}, \quad [R_i, R_j]_+ = 0, \quad [R_i^*, R_j^*]_+ = 0 \tag{5.74}$$

The matrices R_0 and R_0^* define orthogonal projectors

$$R_0^2 = R_0, \quad (R_0^*)^2 = R_0^*, R_0 R_0^* = R_0^* R_0 = 0 \tag{5.75}$$

Further properties are

$$R_0 + R_0^* = 1 \tag{5.76}$$

and

$$R_i^* = -R_i^T \tag{5.77}$$

Similar properties hold for the left multiplication matrices. One can also show that there is a matrix C connecting left and right multiplication matrices. This is given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1_3 \\ 0 & 0 & -1_3 & 0 \end{pmatrix} \tag{5.78}$$

where 1_3 is the 3×3 identity matrix. We have $C^T = C$. It follows that the eigenbras of the matrices of type R, are

$$\langle u| = (u_0, u_0^*, -u_i^*, -u_i) \tag{5.79}$$

For getting the integration rules we need the external product

$$
\begin{aligned}
|u\rangle\langle u| &= \begin{pmatrix} u_0 \\ u_0^* \\ u_i \\ u_i^* \end{pmatrix} \left(u_0, u_0^*, -u_j^*, -u_j\right) \\
&= \begin{pmatrix} u_0 & 0 & 0 & -u_j \\ 0 & u_0^* & -u_j^* & 0 \\ 0 & u_i & \delta_{ij} u_0 & -\epsilon_{ijk} u_k^* \\ u_i^* & 0 & -\epsilon_{ijk} u_k & \delta_{ij} u_0^* \end{pmatrix}
\end{aligned} \tag{5.80}
$$

According to our rules we get

$$
\int_{(u)} u_0 = \int_{(u)} u_0^* = 1, \qquad \int_{(u)} u_i = \int_{(u)} u_i^* = 0 \tag{5.81}
$$

Other interesting properties are

$$
\langle u|u\rangle = u_0 + u_0^* + 3u_0^* + 3u_0 = 4 \tag{5.82}
$$

and using

$$
\langle u'|u\rangle = u_0' u_0 + u_0^{*\prime} u_0 - u_i^{*\prime} u_i - u_i' u_i^* \tag{5.83}
$$

we get

$$
\int_{(u)} \langle u'|u\rangle = u_0' + u_0^{*\prime} = 1 \tag{5.84}
$$

Showing that $\langle u'|u\rangle$ behaves like a delta-function.

Appendix A

A.1 Jordan algebras

A Jordan algebra, $\mathcal{A}$, is a nonassociative algebra which enjoys the following two properties
1) **Commutativity**: $xy = yx$
2) **Jordan identity**: $(xy)x^2 = x(yx^2)$
where $x, y \in \mathcal{A}$. For example, one can obtain a Jordan algebra from an associative one, by defining the Jordan product (which will be denoted in this particular example by $\star$) as

$$x \star y = \frac{1}{2}(xy + yx) \tag{A.1}$$

Noticing that

$$x \star y = y \star x, \qquad x \star x = x^2 \tag{A.2}$$

one can verify immediately that this is a Jordan algebra. In dealing with nonassociative algebra it turns out convenient to define the **associator**

$$(a, b, c) = (ab)c - a(bc) \tag{A.3}$$

which represents a measure of the nonassociativity of the algebra. By using the product rules we get in a basis (x_i)

$$(x_i x_j)x_k = f_{ijl} f_{lkm} x_m = (L_i^T R_k)_{jm} x_m = (L_{x_i x_j}^T)_{km} x_m = (R_j R_k)_{im} x_m \tag{A.4}$$

$$x_i(x_j x_k) = f_{jkl} f_{ilm} x_m = (R_k L_i^T)_{jm} x_m = (R_{x_j x_k})_{im} x_m = (L_j^T L_i^T)_{im} x_m \tag{A.5}$$

More symbolically

$$\begin{aligned} (ab)c &= (L_a^T R_c)b = L_{ab}^T c = (R_b R_c)a \\ a(bc) &= (R_c L_a^T)b = R_{bc} a = (L_b^T L_a^T)c \end{aligned} \tag{A.6}$$

In particular for associative algebras we get the equivalent identities for the right and left multiplications

$$[L_a^T, R_b] = 0, \quad L_a L_b = L_{ab}, \quad R_a R_b = R_{ab} \tag{A.7}$$

Notice that the commutativity condition gives

$$ab = ba \longrightarrow R_b a = L_b^T a \tag{A.8}$$

that is

$$R_a = L_a^T \tag{A.9}$$

In terms of the associator the Jordan identity can be written as

$$x(yx^2) = (xy)x^2 \longrightarrow (x, y, x^2) = 0 \tag{A.10}$$

By sending $x \to x + \lambda z$ ($\lambda \in F$, where F is the field over wich the algebra is defined) we get

$$\begin{aligned} 0 &= (x + \lambda z, y, (x + \lambda z)^2) = (x, y, x^2) + \lambda[(z, y, x^2) + 2(x, y, xz)] \\ &+ \lambda^2[(x, y, z^2) + 2(z, y, xz)] \end{aligned} \tag{A.11}$$

This is satisfied if

$$2(x, y, xz) + (z, y, x^2) = 0 \tag{A.12}$$

Again we send $x \to x + \lambda w$ in this identity obtaining from the coefficients of λ and λ^2 (the constant term is identically zero) the following relations

$$\begin{aligned} \lambda &: \quad (w, y, xz) + (x, y, wz) + (z, y, xw) = 0 \\ \lambda^2 &: \quad 2(w, y, wz) + (z, y, w^2) = 0 \end{aligned} \tag{A.13}$$

The coefficient of λ^2 is zero for eq. (A.12), therefore we get the further condition

$$(x, y, wz) + (w, y, xz) + (z, y, xw) = 0 \tag{A.14}$$

Using the commutativity condition (A.9) and eq. (A.6), we can rewrite the associator in the equivalent forms

$$(a, b, c) = [R_a, R_c]b = (R_{ab} - R_b R_a)c = (R_b R_c - R_{bc})a \tag{A.15}$$

Therefore we can write the condition (A.14) in two equivalent ways

$$[R_x, R_{wz}] + [R_w, R_{xz}] + [R_z, R_{xw}] = 0 \tag{A.16}$$

and

$$(R_{xy} - R_y R_x)wz + (R_y R_{xz} - R_{y(xz)})w + (R_{zy} - R_y R_z)xw = 0 \tag{A.17}$$

from which

$$R_z(R_{xy} - R_y R_x) + R_y R_{xz} - R_{y(xz)} + R_x(R_{zy} - R_y R_z) = 0 \tag{A.18}$$

exchanging $x \leftrightarrow y$

$$R_z(R_{yx} - R_x R_y) + R_x R_{yz} - R_{x(yz)} + R_y(R_{zx} - R_x R_z) = 0 \tag{A.19}$$

and subtracting these two expressions

$$R_z(R_{[x,y]} + [R_x, R_y]) + R_y R_{[x,z]} + R_x R_{[z,y]} - [R_x, R_y]R_z - R_{y(xz)} + R_{x(yz)} = 0 \tag{A.20}$$

and using again commutativity

$$[R_z, [R_x, R_y]] = R_{(xz)y} - R_{x(zy)} = R_{(x,z,y)} = R_{[R_x,R_y]z} \tag{A.21}$$

or

$$[R_z, [R_x, R_y]] = R_{[R_x,R_y]z} \tag{A.22}$$

recalling that the condition for the linear mepping D to be a derivation is

$$[R_i, d] = R_{Dx_i} = R_{dx_i} \tag{A.23}$$

(where as for the right and left multiplications we use the convention $dx_i = d_{ij}x_j$), we see that the expression

$$D_{(x,y)} = [R_x, R_y] \tag{A.24}$$

is a derivation for any pair $x, y \in \mathcal{A}$. Therefore a basis for the derivations is

$$D_{ij} = [R_i, R_j] \tag{A.25}$$

Bibliography

[1] F.A.Berezin and M.S.Marinov, JETP Lett. **21** (1975) 321, *ibidem* Ann. of Phys. **104** (1977) 336.

[2] R.Casalbuoni, Il Nuovo Cimento, **33A** (1976) 115 and *ibidem* 389.

[3] F.A.Berezin, *The method of second quantization*, Academic Press (1966).

[4] A.Connes, *Noncommutative geometry*, Academic Press (1994).

[5] V.G.Drinfeld, *Quantum Groups*, in Proceedings of the International Congress of Mathematicians, Berkeley 1986, pp. 798-820, AMS, Providence, RI.

[6] J.L.Martin, Proc. Roy. Soc. **251A** (1959) 543.

[7] A.Messiah, *Quantum Mechanics*, North Holland, Amsterdam (1962).

[8] R.Casalbuoni, Int. J. Mod. Phys. **A12** (1997) 5803, `physics/9702019`.

[9] R. Casalbuoni, Int. Journ. of Mod. Phys., **14** (1999) 129, `math-ph/9804020`.

[10] R. Casalbuoni, Int. Journ. of Mod. Phys. **A13** (1998) 5459, `physics/9803024`

[11] R.D.Schafer, *An introduction to nonassociative algebras*, Academic Press (1966).

[12] A.P.Isaev, in Proceedings of 'II International Workshop on Classical and Quantum Integrable Systems' (Dubna, 8-12 July, 1996), q-alg/9609030.

[13] N.Jacobson, *Lie Algebras*, Interscience Publish., N.-Y.-London (1962).

[14] L.Biedenharn, J. Phys. **A22** (1989) 4873; A.Macfarlane, J.Phys. **A22** (1989) 4581.

[15] L.Baulieu and E.G.Floratos, Phys. Lett. **B258**(1991) 171.

[16] A.A. Kirillov, *Éléments de la Théorie des Représentations*, Éditions MIR, Moscou (1974); *ibidem Representation Theory and Noncommutative Harmonic Analysis I*. Springer-Verlag (1994).

[17] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. **B305** (1988) 545; D. Fairlie, P. Fletcher and C. Zachos, J. Math. Phys. **31** (1990) 1088; J. Hoppe, Int. J. Mod. Phys. **A4** (1989) 5235.

[18] T.H.Koornwinder, *Representations of Lie groups and quantum groups*, eds. V.Baldoni and M.A.Picardello, Pitman Research Notes in Mathematical Series 311, Longman Scientific & Technical (1994), pp. 46-128; see also M.Chaichian, A.P.Demichev and P.P.Kulish, HIP 1997-02/Th, q-alg/9702023.

[19] N.Jacobson, *Exceptional Lie Algebras*, M. Dekker. Inc., N.-Y. (1971).

[20] R.Casalbuoni, G.Domokos and S. Kövesi-Domokos, Il Nuovo Cimento, **31A** (1976) 423.