**Classical Canonical Observables of the Nambu String**

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**Abstract**

As a prosecution of a preceeding paper, a canonical set of classical observables for the open Nambu string is determined. As a byproduct, a local involutory set of constants of motion is found. This result is obtained by determining a canonical transformation to a new set of variables, of which the constraints of the string are a subset. The Poincare' algebra is analyzed in terms of the new variables.

## 1. Introduction.

The present paper is the natural prosecution of a previous paper of the same authors[1], hereafter quoted as (I), on the study of the classical open Nambu string.

In (I) the many-time functional equations of motion of the open string have been studied, and their explicit solutions given. The aim of the present paper is to find a complete canonical set of observables, that is a set of canonical variables in strong involution with the constraints of the string. This allows in turn to find a new canonical basis, in which we may recognize two canonical subset: one formed by the constraints and their conjugated variables (gauge degrees of freedom) and the other corresponding to the physical degrees of freedom.

It is worth stressing that this is not merely equivalent to the determination of the Dirac brackets. The algebraic algorithm for the computation of the Dirac brackets[2] doesn't allow by itself the determination of the elementary canonical variables of the Dirac symplectic structure. On the contrary, the complete set of these new canonical variables, that we will give in Section 3, is such that, in terms of them, the Dirac brackets are simply the Poisson brackets restricted to the physical degrees of freedom (see for instance[3]).

Part of these new canonical variables are a generalization of the Del Giudice-Di Vecchia-Fubini oscillators (DDF)[4].

Of course, the canonical set we find is local, since it is determined in one of the charts discussed in (I). An equivalent canonical set could be given in another chart.

The discussion given in (I) is preliminary to the present work, as it is in particular the knowledge of the "many-times", that is the variables $B_{\pm}$, see Section 2, which make it possible to guess the structure of the generalized D.D.F. oscillators.

We analyse the Poincaré algebra in terms of these new variables. We follow reference [5] to exhibit the study the front form of the string dynamics.

The classical basis of the string quantum anomaly is found to be the non linear realization of the Thomas spin.

We analyse a possible set of action angle variables, but the classical Casimirs of the Poincare' algebra still depend on the angle variables.Since the whole approach is local, these variables are only locally defined.

Therefore we find a (infinite) canonical set of local constants of motion. Of course, half of them are in involution among themselves. An involutory global set of constants of motion of the string doesn't seem to exist, due to the essentially local character of the abelianization procedure. If an unvolutory global set of constants of motion does not exist, all the Dirac brackets associated to our local canonical basis only exist locally in the chosen chart. This would imply a kind of Gribov phenomenon for the string. In fact, the absence of a global Dirac brackets structure is due to the absence of a global gauge fixing, as shown in reference [6]. On the contrary, a non involutory global set of constants of motion has been found in reference [7]: all these constants can be (locally) decomposed in our basis, as it will be shown in a future paper.

The paper is organized as follows: in Section 2 some notation and some preliminary result quoted from (I) are given.

In Section 3 the new set of canonical variables is explicitly shown. In Section 4 the Poincaré algebra is given in terms of the new variables. Finally, the Appendix is devoted to the comparison of the present approach with the covariant one.

## 2. The Canonical Description of the Open Nambu String.

We summarize in this Section some results of (I), to which we refer for more details.

The action of the open Nambu string[8] is ($\hbar = c = 1$)

$$S = -N \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \sqrt{-h(\sigma,\tau)}, \qquad L = -N\sqrt{-h}, \tag{2.1}$$

where $N = \frac{1}{2\pi\alpha'}$, and

$$-h = -\det \parallel h_{\alpha\beta} \parallel = (\dot{x} \cdot x')^2 - \dot{x}^2 x'^2 \geq 0, \tag{2.2}$$

$$\| h_{\alpha\beta} \| = \| \partial_\alpha x^\mu \partial_\beta x_\mu \| = \begin{pmatrix} \dot{x}^2 & \dot{x}\cdot x' \\ \dot{x}\cdot x' & x'^2 \end{pmatrix}, \quad \alpha = 0,1, \quad \partial_0 = \frac{\partial}{\partial\tau}, \quad \partial_1 = \frac{\partial}{\partial\sigma}, \tag{2.3}$$

and where $-h \geq 0$ means that the surface swept by the string in the spacetime is everywhere time-like or null (i.e. it is a causal surface)[9]. The strip $0 < \sigma < \pi$ is mapped in the world-sheet spanned by the string in D-dimensional Minkowski space, (with signature $(+,-,...,-)$) by the string. The world-sheet is described be the coordinates $x^\mu(\sigma,\tau)$. $h_{\alpha\beta}(\sigma,\tau)$ is the induced metric.

We will not discuss here the boundary conditions, which has been carefully analyzed in (I).

The canonical momenta are given by

$$P^\mu(\sigma,\tau) = -\frac{\partial L}{\partial \dot{x}_\mu(\sigma,\tau)} = \frac{N}{\sqrt{-h}}((\dot{x}\cdot x')x'^\mu - x'^2\dot{x}^\mu); \tag{2.4}$$

They satisfy the identities

$$\begin{cases} \chi_1(\sigma,\tau) = P^2(\sigma,\tau) + N^2 x'^2(\sigma,\tau) = 0, \\ \chi_2(\sigma,\tau) = (P(\sigma,\tau)\cdot x'(\sigma,\tau)) = 0. \end{cases} \tag{2.5}$$

In order to define a Poisson structure in the phase-space, we use the following extension of the canonical variables from the interval $0 < \sigma < \pi$ to the whole real axis:

$$\begin{cases} x^\mu(\sigma,\tau) = x^\mu(-\sigma,\tau) = x^\mu(\sigma + 2n\pi,\tau), \\ P^\mu(\sigma,\tau) = P^\mu(-\sigma,\tau) = P^\mu(\sigma+2n\pi,\tau), \end{cases} \tag{2.6}$$

where $n$ is an integer.

Following reference [2] we introduce an even and odd delta function with period $2\pi$:

$$\Delta_\pm(\sigma,\sigma') = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(e^{in(\sigma-\sigma')} \pm e^{-in(\sigma+\sigma')}\right) =$$
$$= \sum_{n=-\infty}^{\infty}(\delta(\sigma-\sigma'+2n\pi) \pm \delta(\sigma+\sigma'+2n\pi)) \longrightarrow \delta(\sigma-\sigma'), \quad \text{for } \sigma,\sigma' \in (0,\pi), \tag{2.7}$$

with the following properties

$$\begin{cases} \Delta_+(\sigma,\sigma') = \Delta_+(-\sigma,\sigma') = \Delta_+(\sigma',\sigma) = \Delta_+(\sigma+2n\pi,\sigma'), \\ \Delta_-(\sigma,\sigma') = -\Delta_-(-\sigma,\sigma') = \Delta_-(\sigma',\sigma) = \Delta_-(\sigma+2n\pi,\sigma'), \\ \frac{\partial}{\partial\sigma}\Delta_\pm(\sigma,\sigma') = -\frac{\partial}{\partial\sigma'}\Delta_\mp(\sigma,\sigma'), \\ \int_{-\pi}^{\pi} d\sigma' f(\sigma')\Delta_\pm(\sigma',\sigma) = f(\sigma) \pm f(-\sigma). \end{cases} \tag{2.8}$$

To complete the Hamiltonian description we then introduce the following Poisson structure ($\eta^{\mu\nu} = (1;-1,-1,-1)$):

$$\{x^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = -\eta^{\mu\nu}\Delta_+(\sigma,\sigma') \quad \longrightarrow \quad -\eta^{\mu\nu}\delta(\sigma-\sigma'), \quad \text{for } \sigma,\sigma' \in (0,\pi). \tag{2.9}$$

The set of identity (2.5) are now constraints on the phase-space:

$$\begin{cases} P^2(\sigma) + N^2 x'^2(\sigma) \approx 0, \\ P(\sigma)\cdot x'(\sigma) \approx 0. \end{cases} \tag{2.10}$$

Let's define the following variables

$$A^\mu_\pm(\sigma,\tau) = A^\mu_\mp(-\sigma,\tau) = P^\mu(\sigma,\tau) \pm N x'^\mu(\sigma,\tau) = \frac{\partial}{\partial\sigma} B^\mu_\pm(\sigma,\tau), \tag{2.11}$$

$$B^\mu_\pm(\sigma,\tau) = -B^\mu_\mp(-\sigma,\tau). \tag{2.12}$$

With this definition, the constraints (2.10) becomes

$$\chi_\pm(\sigma) = \chi_\mp(-\sigma) = \chi_1(\sigma,\tau) \pm 2N\chi_2(\sigma,\tau) = A^2_\pm(\sigma) \approx 0, \tag{2.13}$$

with the following algebra:

$$\begin{cases} \{\chi_\pm(\sigma_1,\tau), \chi_\pm(\sigma_2,\tau)\} = \mp 2N\left(\chi_\pm(\sigma_1,\tau) + \chi_\pm(\sigma_2,\tau)\right)\cdot\left(\Delta'_+(\sigma_1,\sigma_2) + \Delta'_-(\sigma_1,\sigma_2)\right), \\ \{\chi_+(\sigma_1,\tau), \chi_-(\sigma_2,\tau)\} = -2N\left(\chi_+(\sigma_1,\tau) + \chi_-(\sigma_2,\tau)\right)\cdot\left(\Delta'_+(\sigma_1,\sigma_2) - \Delta'_-(\sigma_1,\sigma_2)\right). \end{cases} \tag{2.14}$$

Therefore the constraints are $1^{th}$-class, but they are in weak involution; this implies that the classical many-times equations of motion are not integrable as they stand. A set of $1^{th}$-class constraints in strong involution are needed.

One possible solution of the constraint makes use of lightcone variables; of course other solutions are possible. In terms of the following lightcone variables

$$\begin{cases} A^+_\pm(\sigma,\tau) = \dfrac{1}{\sqrt{2}}\left(A^0_\pm(\sigma,\tau) + A^{D-1}_\pm(\sigma,\tau)\right), \\ A^-_\pm(\sigma,\tau) = \dfrac{1}{\sqrt{2}}\left(A^0_\pm(\sigma,\tau) - A^{D-1}_\pm(\sigma,\tau)\right), \end{cases} \tag{2.15}$$

with the Lorentz indices now running over $\mu = +, 1, 2, ..., D-2, -$, we may define the new constraints

$$\tilde{\chi}_\pm(\sigma,\tau) = \frac{\chi_\pm(\sigma,\tau)}{2A^+_\pm(\sigma,\tau)} = A^-_\pm(\sigma,\tau) - \frac{\vec{A}^2_\pm(\sigma,\tau)}{2A^+_\pm(\sigma,\tau)} \approx 0, \qquad \text{if} \qquad A^+_\pm(\sigma,\tau) \neq 0, \tag{2.16}$$

where $\vec{A}^2 = (A^1)^2 + ... + (A^{D-2})^2$.

In equation (2.16) it is assumed that $A^+_\pm$ don't vanish; otherwise, as fully discussed in (I), another chart must be used.

Thus our constraints are only weakly Poincaré invariant and are only locally defined, in those regions of the constraints manifold $\chi_\pm(\sigma,\tau) \approx 0$ where the denominators don't vanish. But they are now in strong involution:

$$\{\tilde{\chi}_\pm(\sigma,\tau), \tilde{\chi}_\pm(\sigma',\tau)\} = 0 \tag{2.17}$$

As a first step toward the new canonical basis, a new set of center-of-mass and relative coordinates is introduced. The center-of-mass coordinates of the string are

$$\begin{cases} X^\mu(\tau) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} d\sigma x^\mu(\sigma,\tau), \\ P^\mu = \dfrac{1}{2} \displaystyle\int_{-\pi}^{\pi} d\sigma P^\mu(\sigma,\tau), \end{cases} \tag{2.18}$$

where $P^\mu$ is the conserved generator of the space-time translations.
Let us introduce the following relative coordinates

$$\begin{cases} y^\mu(\sigma,\tau) = -x'^\mu(\sigma,\tau) = -y^\mu(-\sigma,\tau), \\ \mathcal{P}^\mu(\sigma,\tau) = \int_0^\sigma d\sigma' P^\mu(\sigma',\tau) - \dfrac{\sigma}{\pi} P^\mu = -\mathcal{P}^\mu(-\sigma,\tau) = \mathcal{P}^\mu(\sigma+2n\pi,\tau), \end{cases} \tag{2.19}$$

with the following properties

$$\begin{cases} \int_{-\pi}^{\pi} d\sigma y^\mu(\sigma,\tau) = \int_{-\pi}^{\pi} d\sigma \mathcal{P}^\mu(\sigma,\tau) = 0, \\ \\ \mathcal{P}^\mu(0) = \mathcal{P}^\mu(\pm\pi) = 0 \rightarrow \int_{-\pi}^{\pi} d\sigma \mathcal{P}'^\mu(\sigma,\tau) = 0. \end{cases} \tag{2.20}$$

It may be checked that the coordinates (2.18) and (2.19) constitute a basis of canonical variables

$$\begin{cases} \{X^\mu, P^\nu\} = -\eta^{\mu\nu}, \\ \\ \{y^\mu(\sigma,\tau), \mathcal{P}^\nu(\sigma',\tau)\} = -\eta^{\mu\nu}\Delta_-(\sigma,\sigma'), \end{cases} \tag{2.21}$$

with all the other Poisson brackets vanishing.

The inverse transformation is

$$\begin{cases} x^\mu(\sigma,\tau) = X^\mu(\tau) + \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 y^\mu(\sigma_2,\tau) - \int_0^\sigma d\sigma_2 y^\mu(\sigma_2,\tau), \\ P^\mu(\sigma,\tau) = \dfrac{1}{\pi} P^\mu + \mathcal{P}'^\mu(\sigma,\tau). \end{cases} \tag{2.22}$$

The variables $A^\mu_\pm(\sigma,\tau)$ and $B^\mu_\pm(\sigma,\tau)$ are expressed in terms of the new canonical set by

$$\begin{cases} A^\mu_\pm(\sigma,\tau) = \dfrac{1}{\pi} P^\mu + \mathcal{P}'^\mu(\sigma,\tau) \mp N y^\mu(\sigma,\tau) = \dfrac{\partial}{\partial\sigma} B^\mu_\pm(\sigma,\tau), \\ B^\mu_\pm(\sigma,\tau) = \dfrac{\sigma}{\pi} P^\mu + \mathcal{P}^\mu(\sigma,\tau) \pm N x^\mu(\sigma,\tau). \end{cases} \tag{2.23}$$

It is possible to express the original set $x^\mu(\sigma,\tau)$ and $P^\mu(\sigma,\tau)$ in term of these variables:

$$\begin{cases} x^\mu(\sigma,\tau) = \dfrac{1}{2N} \Big[ B^\mu_+(\sigma,\tau) - B^\mu_-(\sigma,\tau) \Big] \\ P^\mu(\sigma,\tau) = \dfrac{1}{2} \Big[ A^\mu_+(\sigma,\tau) + A^\mu_-(\sigma,\tau) \Big] \end{cases} \tag{2.24}$$

These are useful relations since the many-time functional equations of motion are most easily solved in terms of the variables $A^\mu_\pm(\sigma)$, $B^\mu_\pm(\sigma)$, as shown in (I), where the explicit solutions are given. Actually the $B^+_\pm(\sigma,\tau)$ turn out to be the generalization to an arbitrary gauge of the quantities $\tau \pm \sigma$ of the o.g. Moreover, the variables $A^\mu_\pm(\sigma,\tau)$ and $B^\mu_\pm(\sigma,\tau)$ are of interest, since, as it will be shown in the next Section, it is possible to give in terms of them a generalization of the Del Giudice-Di Vecchia-Fubini oscillators [4] to an arbitrary gauge.

In the orthonormal gauge, with the usual boundary conditions

$$x'^\mu(0,\tau) = x'^\mu(\pi,\tau) = 0, \tag{2.25}$$

to fix completely the gauge one usually adds the following gauge-fixing constraints (transverse light-cone gauge)

$$\begin{cases} \phi_1(\sigma,\tau) = x^+(\sigma,\tau) - q^+ - \dfrac{P^+\tau}{\pi N} \approx 0, \\ \phi_2(\sigma,\tau) = P^+(\sigma,\tau) - \dfrac{P^+}{\pi} \approx 0 \end{cases} \tag{2.26}$$

The solution is well known:

$$x^\mu(\sigma,\tau) = \frac{1}{2}\Big[Q^\mu(\tau+\sigma) + Q^\mu(\tau-\sigma)\Big], \tag{2.27}$$

where $Q^\mu(\tau)$ is the coordinate of the end point $\sigma = 0$ of the string, and is given by

$$Q^\mu(\tau) = x^\mu(0,\tau) = q^\mu + \frac{P^\mu}{\pi N}\tau + f^\mu(\tau), \tag{2.28}$$

where

$$f^\mu(\tau) = \frac{i}{\sqrt{\pi N}} \sum_{n\neq 0} \alpha^\mu_n e^{-in\tau} = f^\mu(\tau + 2N\pi), \tag{2.29}$$

and, from the constraints,

$$\left(\frac{P^\mu}{2\pi N} + \frac{d}{du} f^\mu(u)\right)^2 = 0, \qquad\qquad u = \tau \pm \sigma. \tag{2.30}$$

The quantities $q^\mu$ and $\alpha^\mu_n$ are constants of motion.

The DDF oscillators are given in this gauge by

$$\vec{A}_n = \sqrt{\frac{N}{2\pi}} \int_{-\pi}^{\pi} d\sigma \frac{d\vec{Q}(\sigma)}{d\sigma} e^{\frac{i\pi n}{P^+} Q^+(\sigma)}. \tag{2.31}$$

The usual o.g. case is completely defined by the additional requirement $f^+(u)$=0.

As shown in (I), in the o.g. case the variables $B^+_\pm(\sigma)$ reduce to

$$B^+_\pm(\sigma) \quad \xrightarrow{o.g.} \quad \pm Nq^+ \pm \frac{P^+}{\pi}(\tau \pm \sigma) = \pm NQ^+(\tau \pm \sigma). \tag{2.32}$$

As we will see in the following, this point is crucial as a guide to the construction of the generalization of the DDF oscillators to an arbitrary gauge.

## 3. The Observables and the Canonical Transformation.

Let us now look for a local set of observables with respect to the first class constraints $\tilde{\chi}_\pm$ in the $(\sigma, \tau)$ region where there are defined.

First of all we shall introduce the generalization to an arbitrary gauge of the Del Giudice-Di Vecchia-Fubini (DDF) oscillators [4], which commute with the Virasoro generators $L_n$ in the orthonormal gauge. They are the transverse part of the following objects:

$$\begin{cases} A^\mu_n(\tau) = \sqrt{|n|}\alpha^\mu_n = \frac{1}{\sqrt{4\pi N}} \int_{-\pi}^{\pi} d\sigma A^\mu_\pm(\sigma,\tau) \exp\left[\pm i\omega_n \frac{B^+_\pm(\sigma,\tau)}{2NP^+}\right], \qquad n = \pm 1, \pm 2, ..., \\ A^\mu_0 = \frac{1}{\sqrt{4\pi N}} \int_{-\pi}^{\pi} d\sigma A^\mu_\pm(\sigma,\tau) = \frac{P^\mu}{\sqrt{\pi N}}, \end{cases} \tag{3.1}$$

where $\omega_n = 2\pi N n$ and $P^+ \neq 0$. In the orthonormal gauge (2.26), remembering equations (2.32):

$$\frac{1}{N} B^\mu_\pm(\sigma,\tau) \to \pm Q^\mu(\tau \pm \sigma), \qquad A^\mu_\pm(\sigma,\tau) \to N \frac{dQ^\mu(\tau \pm \sigma)}{d(\tau \pm \sigma)},$$

we get:

$$\begin{aligned} A^\mu_n(\tau) \longrightarrow & \sqrt{\frac{N}{2\pi}} \int_{-\pi}^{\pi} d\sigma \frac{dQ^\mu(\tau+\sigma)}{d(\tau+\sigma)} \exp\left[+i\omega_n \frac{Q^+(\tau+\sigma)}{2P^+}\right] = \\ = & \sqrt{\frac{N}{2\pi}} \int_{-\pi}^{\pi} d\sigma \frac{dQ^\mu(\sigma)}{d\sigma} \exp\left[i\omega_n \frac{Q^+(\sigma)}{2P^+}\right] =^{DDF} A^\mu_n, \end{aligned} \tag{3.2}$$

owing to the $2\pi$-periodicity.

Moreover we have:

$$A^+_n(\tau) = \frac{1}{\sqrt{4\pi N}} \int_{-\pi}^{\pi} d\sigma \frac{\partial B^+_+(\sigma,\tau)}{\partial\sigma} \exp\left[i\omega_n \frac{B^+_+(\sigma,\tau)}{2NP^+}\right] = 0, \qquad n \neq 0, \tag{3.3}$$

owing to the fact that $B^+_\pm(\pi) - B^+_\pm(-\pi) = 2P^+$ (see equations (2.23)), so that $A^-_n = A^0_n - A^3_n$. This property, together with equations (2.8), is crucial to verify that:

$$\begin{cases} \left\{\vec{A}_n(\tau), \tilde{\chi}_\pm(\sigma,\tau)\right\} = 0, \\ \left\{A^-_n(\tau), \tilde{\chi}_\pm(\sigma,\tau)\right\} = -\frac{i\omega_n}{P^+} \exp\left[\pm i\omega n \frac{B^+_+(\sigma,\tau)}{2NP^+}\right] \cdot \tilde{\chi}_\pm(\sigma,\tau) \approx 0. \end{cases} \tag{3.4}$$

The $A_n^\mu$ satisfy the following Poisson brackets:

$$\{A_n^\mu, A_m^\nu\} = in\eta^{\mu\nu}\delta_{n,-m} + \frac{i\sqrt{\pi N}}{P^+}\left[m(\eta^{\mu 0}+\eta^{\mu 3})A_{n+m}^\nu - n(\eta^{\nu 0}+\eta^{\nu 3})A_{n+m}^\mu\right], \tag{3.5}$$

so that:

$$\begin{cases} \{A_n^a, A_m^b\} = -in\delta^{ab}\delta_{n,-m} \quad \Rightarrow \quad \text{for } n,m>0 \quad \{\alpha_n^a, \alpha_{-m}^b\} = -i\delta^{ab}\delta_{nm}, \\ \{A_n^a, A_m^-\} = -i\dfrac{\sqrt{\pi N}}{P^+} nA_{n+m}^a, \\ \{A_n^-, A_m^-\} = i\dfrac{\sqrt{\pi N}}{P^+} A_{n+m}^-(m-n). \end{cases} \tag{3.6}$$

Therefore the generalized DDF oscillators $\vec{A}_n(\tau)$ are strong observables, they also commute with the original constraints (2.13). Conversely, the $A_n^-(\tau)$ are only weak observables and at the classical level they are not independent quantities, because of the constraints $\tilde{\chi}_\pm \approx 0$. We have

$$A_n^-(\tau) = \sqrt{|n|}\alpha_n^- \equiv \frac{\sqrt{\pi N}}{P^+}U_n^- = \frac{\sqrt{\pi N}}{P^+}\left(\tilde{L}_n + \tilde{U}_n^-\right) \approx \frac{\sqrt{\pi N}}{P^+}\tilde{U}_n^-, \tag{3.7}$$

where

$$\begin{aligned} \tilde{L}_n(\tau) =& \frac{P^+}{2\pi N}\int_{-\pi}^{\pi} d\sigma \tilde{\chi}_\pm(\sigma,\tau)\exp\left[\pm i\omega_n \frac{B_\pm^+(\sigma,\tau)}{2NP^+}\right] \\ =& \frac{P^+}{\pi^2}\sum_{m=-\infty}^{+\infty} L_m(\tau)\int_{-\pi}^{\pi} d\sigma \frac{e^{\mp im\sigma}}{2A_\pm^+(\sigma,\tau)}\exp\left[\pm i\omega_n \frac{B_\pm^+(\sigma,\tau)}{2NP^+}\right] \approx 0, \end{aligned} \tag{3.8}$$

$$\tilde{U}_n^-(\tau) = \frac{P^+}{2\pi N}\int_{-\pi}^{\pi} d\sigma \frac{\vec{A}_\pm^2(\sigma,\tau)}{2A_\pm^+(\sigma,\tau)}\exp\left[\pm i\omega_n \frac{B_\pm^+(\sigma,\tau)}{2NP^+}\right]. \tag{3.9}$$

The $\tilde{L}_n(\tau)$ can be called the generalized Virasoro generators. In equation (3.8) we have used the standard definition of the Virasoro generators $L_n(\tau)$. In the o.g. (2.26), using equation (2.32), we get

$$\tilde{L}_n = L_n \exp\left[\frac{i\omega_n}{2P^+}\left(q^+ + \frac{P^+\tau}{\pi N}\right)\right].$$

The $U_n^-$, $\tilde{L}_n$, $\tilde{U}_n^-$ satisfy the Virasoro algebra

$$\{D_n D_m\} = i(m-n)D_{m+n}, \qquad D_n = U_n^-, \tilde{L}_n, \tilde{U}_n^-, \tag{3.10}$$

and

$$\begin{aligned} \left\{U_n^-, \tilde{L}_m\right\} =& 0, \\ \left\{\tilde{U}_n^-, \tilde{L}_m\right\} =& i(m-n)\tilde{L}_{m+n}, \\ \left\{U_n^-, \tilde{U}_m^-\right\} =& i(m-n)U_{n+m}^-. \end{aligned} \tag{3.11}$$

The inversion formula of equations (3.1) is

$$\begin{aligned}A^{\mu}_{\pm}(\sigma,\tau) =&\sqrt{4\pi N}\frac{A^{+}_{\pm}(\sigma,\tau)}{2P^{+}}\sum_{n=-\infty}^{+\infty} A^{\mu}_{n}(\tau)\exp\left[\mp i\omega_n \frac{B^{+}_{\pm}(\sigma,\tau)}{2NP^{+}}\right] = \\ =&\sqrt{4\pi N}\frac{A^{+}_{\pm}(\sigma,\tau)}{2P^{+}}\sum_{n=-\infty}^{+\infty} A^{\mu}_{\pm n}(\tau)\exp\left[-i\omega_n \frac{B^{+}_{\pm}(\sigma,\tau)}{2NP^{+}}\right],\end{aligned} \tag{3.12}$$

Using equation (3.12), we can put the $\tilde{U}^{-}_{n}(\tau)$ of equation (3.9) in the following form:

$$\begin{aligned}\tilde{U}^{-}_{n}(\tau) =& \frac{1}{2}\sum_{m=-\infty}^{+\infty}\vec{A}_{n-m}\cdot\vec{A}_{m} = \\ =& \frac{1}{2\pi N}\left[\sqrt{4\pi N}\vec{P}\cdot\vec{A}_{n} + 2\pi N\sum_{m>0}\vec{A}_{n+m}\cdot\vec{A}_{-m} + \pi N\sum_{m=1}^{n-1}\vec{A}_{n-m}\cdot A_{m}\right] = \\ =& \frac{1}{2\pi N}\left[\sqrt{2\omega_n}\vec{P}\cdot\vec{\alpha}_{n} + \sum_{m>0}\sqrt{\omega_m\omega_{n+m}}\vec{\alpha}_{n+m}\cdot\vec{\alpha}_{-m} + \frac{1}{2}\sum_{m=1}^{n-1}\sqrt{\omega_m\omega_{n-m}}\vec{\alpha}_{n-m}\cdot\vec{\alpha}_{m}\right].\end{aligned} \tag{3.13}$$

Then, using equations (3.1), (3.3), (3.7), (3.13), we get

$$\begin{aligned}\tilde{L}_{n}(\tau) =&\frac{P^{+}}{\sqrt{\pi N}}A^{-}_{n} - \tilde{U}^{-}_{n} = \\ =&A^{+}_{0}A^{-}_{n} - \frac{1}{2}\sum_{m=-\infty}^{+\infty}\vec{A}_{n-m}\cdot\vec{A}_{m} = \\ =&\frac{1}{2}\sum_{m=-\infty}^{+\infty}A^{\mu}_{m}A_{n-m,\mu}.\end{aligned} \tag{3.14}$$

While at the classical level $\tilde{U}^{-}_{n}$ is a dependent quantity, at the quantum level, with $D=4$, the $A^{-}_{n}$ become the independent longitudinal Brower modes of the no-ghost theorem in the covariant quantization approach[10–13]. See Appendix A for the relationship between the oscillators $A^{\mu}_{n}$ and the covariant oscillators $a^{\mu}_{n}$.

Since we already got the transverse oscillators observables $\vec{A}_{n}(\tau)=\sqrt{|n|}\vec{\alpha}_{n}$, we now only need to find 6 observables for the center-of-mass of the string. Three of them are $P^{+}$, $\vec{P}$. Their conjugate observable variables, $Z^{-}$, $\vec{Z}$, are:

$$\begin{cases} Z^{-}(\tau) = X^{-}(\tau) - \frac{1}{2P^{+}}\int_{-\pi}^{\pi}d\sigma\left\{\frac{x^{+}(\sigma,\tau)}{2}\left(\frac{\vec{A}^{2}_{-}(\sigma,\tau)}{2A^{+}_{-}(\sigma,\tau)}+\frac{\vec{A}^{2}_{+}(\sigma,\tau)}{2A^{+}_{+}(\sigma,\tau)}\right)+\right. \\ \qquad\left. -\frac{\mathcal{P}^{+}(\sigma,\tau)}{2N}\left(\frac{\vec{A}^{2}_{-}(\sigma,\tau)}{2A^{+}_{-}(\sigma,\tau)}-\frac{\vec{A}^{2}_{+}(\sigma,\tau)}{2A^{+}_{+}(\sigma,\tau)}\right)\right\}, \\ \vec{Z}(\tau) = \vec{X}^{+}(\tau) - \frac{1}{2P^{+}}\int_{-\pi}^{\pi}d\sigma\left[x^{+}(\sigma,\tau)\vec{P}(\sigma,\tau) - \vec{y}(\sigma,\tau)\mathcal{P}^{+}(\sigma,\tau)\right]. \end{cases} \tag{3.15}$$

The so defined quantities are effectively observables, since it can be checked that:

$$\left\{Z^-(\tau), \tilde{\chi}_\pm(\sigma,\tau)\right\} = \left\{\vec{Z}(\tau), \tilde{\chi}_\pm(\sigma,\tau)\right\} = 0. \tag{3.16}$$

Besides, the non vanishing Poisson brackets among these 6 observables are

$$\begin{cases} \{Z^-, P^+\} = -1, & \\ \left\{Z^a, P^b\right\} = \delta^{ab}, & a, b = 1, 2. \end{cases} \tag{3.17}$$

We can now define a canonical transformation from the variables $x^\mu(\sigma,\tau)$, $P^\mu(\sigma,\tau)$ to a new canonical basis adapted to the multitemporal approach of the previous Section. This new base should be an appropriate starting point of departure toward the construction of Dirac brackets [2] associated to gauge-fixing constraints like those of equations (2.26). This kind of canonical transformation for a system with first class constraints [3] generates new canonical variables divided into two sets. In one set, half of the canonical variables are functions of the first class constraints, hence vanishing on the manifold defined by the constraints in the phase-space, while the other half constitute a possible choice of the gauge degrees of freedom of the theory. In the second set we have those observables which have vanishing Poisson brackets with the chosen gauge degrees of freedom. The gauge sector of the new variables is composed by

$$\begin{cases} Y^-(\sigma,\tau) = \frac{1}{2N}\left(\tilde{\chi}_-(\sigma,\tau) - \tilde{\chi}_+(\sigma,\tau)\right), \\ \mathcal{P}^+(\sigma,\tau), \end{cases} \tag{3.18}$$

$$\begin{cases} x^+(\sigma,\tau), \\ \Pi^-(\sigma,\tau) = \frac{1}{2}\left(\tilde{\chi}_-(\sigma,\tau) + \tilde{\chi}_+(\sigma,\tau)\right). \end{cases} \tag{3.19}$$

If we separate $\Pi^-(\sigma,\tau)$ in its center-of-mass and relative part:

$$\Pi^-(\sigma,\tau) = \frac{\Xi^-_{tot}(\tau)}{\pi} + \Xi'^-_{rel}(\sigma,\tau), \tag{3.20}$$

where

$$\begin{cases} \Xi_{tot} = \frac{1}{2}\int_{-\pi}^{\pi} d\sigma \Pi^-(\sigma), \\ \Xi^-_{rel} = \int_0^\sigma d\sigma' \Pi^-(\sigma') - \frac{\sigma}{\pi}\Xi^-_{tot}. \end{cases} \tag{3.21}$$

we may replace $x^+(\sigma,\tau)$, $\Pi^-(\sigma,\tau)$ with the new gauge variables

$$\begin{cases} X^+(\tau), \\ \Xi^-_{tot}(\tau), \end{cases} \qquad \begin{cases} y^+(\sigma,\tau), \\ \Xi^-_{rel}(\sigma,\tau). \end{cases} \tag{3.22}$$

The constraints $\tilde{\chi}_\pm \approx 0$ are equivalent to $Y^-(\sigma,\tau) \approx 0$, $\Pi^-(\sigma,\tau) \approx 0$ (or $\Xi^-_{tot}(\tau) \approx 0$, $\Xi^-_{rel}(\sigma,\tau) \approx 0$).

The sector of the observables is composed by the $\vec{\alpha}_n$, $P^+$, $Z^-$, $\vec{P}$, $\vec{Z}$. It is only a matter of calculation to verify that the non-vanishing Poisson brackets are

$$
\begin{aligned}
&\left\{Y^-(\sigma,\tau), \mathcal{P}^+(\sigma',\tau)\right\} = -\Delta_-(\sigma,\sigma'),\\
&\left\{x^+(\sigma,\tau), \Pi^-(\sigma',\tau)\right\} = -\Delta_+(\sigma,\sigma'),\\
&\left\{X^+(\tau), \Xi^-_{tot}(\tau)\right\} = -1,\\
&\left\{y^+(\sigma,\tau), \Xi^-_{rel}(\sigma',\tau)\right\} = -\Delta_-(\sigma,\sigma'),\\
&\left\{\alpha^a_n, \alpha^b_{-m}\right\} = -i\delta^{ab}\delta_{nm}, \qquad\qquad a,b=1,2,\\
&\left\{Z^-, P^+\right\} = -1,\\
&\left\{Z^a, P^b\right\} = \delta_{ab}, \qquad\qquad a,b=1,2.
\end{aligned}
\tag{3.23}
$$

With some calculation, it is possible to find the inverse canonical transformation, which is defined by the following expressions:

$$
\begin{aligned}
&x^+(\sigma,\tau);\\
&P^-(\sigma,\tau) = \Pi^-(\sigma,\tau) + \frac{P^+(\sigma,\tau)}{2P^{+2}}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right) +\\
&\qquad + \frac{\pi N}{P^+}\sum_{n=1}^{\infty}\Bigg\{\Bigg(\frac{A^+_+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}\right] + \frac{A^+_-(\sigma,\tau)}{2P^+}\cdot\\
&\qquad\cdot\exp\left[i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}\right]\Bigg)\tilde{U}^-_n + \left(\frac{A^+_+(\sigma,\tau)}{2P^+}\exp\left[+i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}\right]\right. +\\
&\qquad \left. + \frac{A^+_-(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}\right]\right)\tilde{U}^-_{-n}\Bigg\};\\
&x^-(\sigma,\tau) = Z^- + \frac{x^+(\sigma,\tau)}{2P^{+2}}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right) +\\
&\qquad + \frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma_1\int_0^{\sigma_1}d\sigma_2 Y^-(\sigma_2,\tau) - \int_0^{\sigma}d\sigma_2 Y^-(\sigma_2,\tau) +\\
&\qquad + \frac{i}{2P^+}\sum_{n=1}^{\infty}\frac{1}{n}\Bigg[\left(\exp\left[-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}\right] + \exp\left[i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}\right]\right)\tilde{U}^-_n(\tau) +\\
&\qquad - \left(\exp\left[-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}\right] + \exp\left[i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}\right]\right)\tilde{U}^-_{-n}(\tau)\Bigg];\\
&P^+(\sigma,\tau) = \frac{P^+}{\pi} + \mathcal{P}'^+(\sigma,\tau);\\
&\vec{x}(\sigma,\tau) = \vec{Z}(\tau) + \frac{\vec{P}}{P^+}x^+(\sigma,\tau) +
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
&+\frac{i}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{\omega_n}}\left[\left(\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\exp[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_n+\\
&-\left(\exp\left[-i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]+\exp\left[i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_{-n}\Bigg]; \qquad (3.24)\\
\vec{P}(\sigma,\tau)=&\frac{\vec{P}}{P^+}P^+(\sigma,\tau)+\sum_{n=1}^{\infty}\sqrt{\frac{\omega_n}{2}}\left[\left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\right.\right.\\
&\left.+\frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_n+\\
&\left.+\left(\frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]+\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]\right)\vec{\alpha}_{-n}\right],
\end{aligned}
$$

where $\tilde{U}_n^-$ is given by equation (3.13) and $A_\pm^+$, $B_\pm^+$ are expressed in terms of $P^+$, $\mathcal{P}^+(\sigma,\tau)$, $x^+(\sigma,\tau)$ in equations (2.23).

It is also possible to write the results of equations (3.24) in a more compact form through the use of the generalized DDF oscillators; we obtain:

$$
\begin{aligned}
x^\mu(\sigma,\tau)=&Z^\mu(\tau)+\frac{P^\mu}{P^+}x^+(\sigma,\tau)+\\
&+\frac{i}{2\sqrt{\pi N}}\sum_{n\neq 0}\frac{A_n^\mu(\tau)}{n}\left(\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)+\\
&+\left(\eta^{\mu 0}+\eta^{\mu 3}\right)\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma_1\int_0^{\sigma_1}d\sigma_2 Y^-(\sigma_2,\tau)-\int_0^{\sigma}d\sigma_2 Y^-(\sigma_2,\tau)+\right. \qquad (3.25)\\
&\left.-\frac{i}{2P^+}\sum_{n\neq 0}\frac{\tilde{L}_n(\tau)}{n}\left(\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
P^\mu(\sigma,\tau)=&\frac{P^\mu}{P^+}P^+(\sigma,\tau)+\sqrt{\pi N}\sum_{n\neq 0}A_n^\mu(\tau)\left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\right.\\
&\left.+\frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)+\\
&+\left(\eta^{\mu 0}+\eta^{\mu 3}\right)\left[\Pi^-(\sigma,\tau)-\frac{P^+(\sigma,\tau)}{P^+}\Xi^-_{tot}(\tau)-\frac{\pi N}{P^+}\sum_{n\neq 0}\tilde{L}_n(\tau)\cdot\right. \qquad (3.26)\\
&\left.\cdot\left(\frac{A_+^+(\sigma,\tau)}{2P^+}\exp\left[-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}\right]+\frac{A_-^+(\sigma,\tau)}{2P^+}\exp\left[i\omega_n\frac{B_-^+(\sigma,\tau)}{2NP^+}\right]\right)\right],
\end{aligned}
$$

where $P^-$ is given by

$$
\begin{aligned}
P^- =&\frac{1}{2P^+}\left(\vec{P}^2+\sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right)+\Xi_{tot-}(\tau)= \\
=&\frac{1}{2P^+}\left(\vec{P}^2+2\pi N\sum_{n=1}^{\infty}\vec{A}_n\cdot\vec{A}_{-n}\right)+\Xi_{tot-}(\tau),
\end{aligned}
\tag{3.27}
$$

and

$$
Z^+ = 0. \tag{3.28}
$$

The canonical transformation (3.23) and its inverse (3.24) could have been obtained also starting from another set of constraints, similar to $\tilde{\chi}_\pm$ of equation (2.16), but with $A^-_\pm$ in place of $A^+_\pm$. These constraints were called $\tilde{\chi}^{(-)}_\pm$ in (I). This is to show that in the overlap region where $A^+_\pm \neq 0$, $A^-_\pm \neq 0$ we can go from the basis for $\tilde{\chi}_\pm$ to the basis for $\tilde{\chi}^{(-)}_\pm$ with a canonical transformation.

## 4. The Poincaré Generators.

Let us start the discussion of the Poincare' algebra in $D = 4$ space time dimensions. We will discuss the general case in the following of this Section.

The invariance of the action (2.1) under the Poincaré transformations implies the conservation of the following ten quantities:

$$
\begin{aligned}
P^\mu =&\frac{1}{2}\int_{-\pi}^{\pi} d\sigma P^\mu(\sigma,\tau) \\
J^{\mu\nu} =&\frac{1}{2}\int_{-\pi}^{\pi} d\sigma \left[x^\mu(\sigma,\tau)P^\nu(\sigma,\tau)-x^\nu(\sigma,\tau)P^\mu(\sigma,\tau)\right] \\
=&L^{\mu\nu}+S^{\mu\nu}= \\
=&X^\mu(\tau)P^\nu - X^\nu(\tau)P^\mu + \frac{1}{2}\int_{-\pi}^{\pi} d\sigma\Big[Y^\mu(\sigma,\tau)\mathcal{P}^\nu(\sigma,\tau)-Y^\nu(\sigma,\tau)\mathcal{P}^\mu(\sigma,\tau)\Big]
\end{aligned}
\tag{4.1}
$$

which satisfy the usual Poincaré algebra.

Due to the constraints, the components $S^{+-}$ and $S^{a-}$ of $S^{\mu\nu}$ are not independent from the others, so that there are only 3 degrees of freedom for the spin.

In the light-cone basis (see Appendix B of reference[14]) these constants of motion take the form

$$
\begin{aligned}
P^- =& \frac{1}{2}(P^0 - P^3), \qquad P^+ = P^0 + P^3, \qquad P^a, \qquad a = 1, 2;\\
E^a =& J^{-a} = -\frac{1}{2}(\epsilon^{ab}J^b + N^a) = E^a_L + E^a_S,\\
&\text{where}: \quad E^a_L = L^{-a} = -\frac{1}{2}(\epsilon^{ab}L^b + N^a_L), \qquad E^a_S = S^{-a} = -\frac{1}{2}(\epsilon^{ab}S^b + N^a_S);\\
F^a =& J^{+a} = -\frac{1}{2}(\epsilon^{ab}J^b - N^a) = F^a_L + F^a_S,\\
&\text{where}: \quad F^a_L = L^{+a} = (\epsilon^{ab}L^b - N^a_L), \qquad F^a_S = S^{+a} = (\epsilon^{ab}S^b - N^a_S);\\
J^3 =& J^{12} = L^3 + S^3, \qquad J^a = \epsilon^{ab}(E^b - \frac{1}{2}F^b);\\
N^3 =& J^{+-} = J^{30} = N^3_L + N^3_S, \qquad N^a = -(E^a + \frac{1}{2}F^a)
\end{aligned}
\tag{4.2}
$$

and the Poincaré algebra is:

$$
\begin{array}{lll}
\{J^3, E^a\} = \epsilon^{ab}E^b, & \{N^3, P^+\} = P^+, & \{J^3, P^a\} = \epsilon^{ab}P^b,\\
\{J^3, F^a\} = \epsilon^{ab}F^b, & \{E^a, P^+\} = -P^a, & \{E^a, P^b\} = -\delta^{ab}P^-,\\
\{N^3, E^a\} = -E^a, & \{N^3, P^-\} = -P^-, & \{F^a, P^b\} = -\delta^{ab}P^+,\\
\{N^3, F^a\} = F^a, & \{F^a, P^-\} = -P^a, & \{E^a, F^b\} = \epsilon^{ab}J^3 - \delta^{ab}N^3.
\end{array}
\tag{4.3}
$$

After a long calculation the generators (4.2) expressed in terms of the new canonical basis are found to be:

$$
\begin{aligned}
P^- =& \frac{1}{2P^+}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right) + \Xi^-_{tot}(\tau) \approx \frac{1}{2P^+}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right)\\
P^+&\\
\vec{P}&\\
N^3 =& J^{+-} = X^+\Xi^-_{tot} - Z^-P^+ + \frac{1}{2}\int_{-\pi}^{\pi}d\sigma\left[Y^+(\sigma,\tau)\Xi^-_{rel}(\sigma,\tau) - Y^-(\sigma,\tau)\mathcal{P}^+(\sigma,\tau)\right] \approx -Z^-P^+\\
J^3 =& J^{12} = \epsilon^{ab}\left(Z^aP^b + i\sum_{n=1}^{\infty}\alpha^a_n\cdot\alpha^b_{-n}\right)\\
F^a =& J^{+a} = -Z^aP^+\\
E^a =& J^{-a} = Z^-P^a - \frac{Z^a}{2P^+}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right) - \frac{2\pi iN}{P^+}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\omega_n}}\left(\alpha^a_n\tilde{U}^-_{-n} - \alpha^a_{-n}\tilde{U}^-_n\right) -\\
&- \left[Z^a + \frac{X^+}{P^+}P^a + \sum_{n=1}^{\infty}\frac{i}{\sqrt{2\omega_n}}\left(\frac{\alpha^a_n}{\pi}\int_{-\pi}^{\pi}d\sigma e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}} - \frac{\alpha^a_{-n}}{\pi}\int_{-\pi}^{\pi}d\sigma e^{-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}}\right)\right]\Xi^-_{tot} -\\
&- \frac{P^a}{2P^+}\int_{-\pi}^{\pi}d\sigma\left[Y^+(\sigma,\tau)\Xi^-_{rel}(\sigma,\tau) - Y^-(\sigma,\tau)\mathcal{P}^+(\sigma,\tau)\right] +
\end{aligned}
\tag{4.4}
$$

$$+ i\sum_{n=1}^{\infty}\Bigg(\alpha_n^a\int_{-\pi}^{\pi}d\sigma\Xi^-_{rel}(\sigma,\tau)\frac{\partial}{\partial\sigma}e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}} - \alpha^a_{-n}\int_{-\pi}^{\pi}d\sigma\Xi^-_{rel}(\sigma,\tau)\frac{\partial}{\partial\sigma}e^{-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}}$$

$$+ \sum_{n=1}^{\infty}\frac{iN}{\sqrt{2\omega_n}}\left(\alpha_n^a\int_{-\pi}^{\pi}d\sigma Y^-(\sigma,\tau)e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}} + \alpha^a_{-n}\int_{-\pi}^{\pi}d\sigma Y^-(\sigma,\tau)e^{-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}}\right) \approx$$

$$\approx Z^-P^a - \frac{Z^a}{2P^+}\left(\vec{P}^2 + \sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\right) - \frac{2\pi iN}{P^+}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\omega_n}}\alpha^a_n\tilde{U}^-_{-n} - \alpha^a_{-n}\tilde{U}^-_n.$$

The last term in $E^a$, trilinear in the $\vec{\alpha}_n$, coincides with the analogous term in $J^{-a}$, which engenders the conditions $D = 26$ and $\alpha_0 = 1$ in the non-covariant quantization of the string 10. So we find the usual results, but now we are able to disentangle their inner reason.

Let us remark that the expressions for $N^3$ and $F^a$ confirm equations (3.15). Moreover, if we remember that $Z^+ = 0$, $A^+_n = 0$ (see equations (3.28) and (3.3)) for $n \neq 0$, equations (4.4) imply

$$J^{\mu\nu} = Z^\mu(\tau)P^\nu - Z^\nu(\tau)P^\mu + i\sum_{n\neq 0}\frac{A^\mu_n(\tau)A^\nu_{-n}(\tau)}{n} + G^{\mu\nu}, \tag{4.5}$$

where

$$\begin{aligned}
G^{12} =&0\\
G^{30} =&X^+(\tau)\Xi_{tot}(\tau) + \frac{1}{2}\int_{-\pi}^{\pi}d\sigma\left[Y^+(\sigma,\tau)\Xi^-_{rel}(\sigma,\tau) - Y^-(\sigma,\tau)\mathcal{P}^+(\sigma,\tau)\right]\\
G^{a0} =&-\frac{P^a}{P^+}x^+(\tau)\Xi^-_{tot}(\tau) + \frac{i\sqrt{\pi N}}{P^+}\sum_{n\neq 0}\frac{A^a_n(\tau)\tilde{L}_{-n}(\tau)}{n}+\\
&+\frac{i\Xi^-_{tot}(\tau)}{2\sqrt{\pi N}}\frac{1}{n}\frac{A^a_n(\tau)}{\pi}\int_{-\pi}^{\pi}d\sigma e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}}+\\
&-i\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[A^a_n(\tau)\int_{-\pi}^{\pi}d\sigma\Xi^-_{rel}(\sigma,\tau)\frac{\partial}{\partial\sigma}e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}} - A^a_{-n}(\tau)\int_{-\pi}^{\pi}d\sigma\Xi^-_{rel}(\sigma,\tau)\frac{\partial}{\partial\sigma}e^{-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}}\right]+\\
&-\frac{i}{2}\sqrt{\frac{N}{\pi}}\sum_{n=1}^{\infty}\frac{1}{n}\left[A^a_n(\tau)\int_{-\pi}^{\pi}d\sigma Y^-(\sigma,\tau)e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}} + A^a_{-n}(\tau)\int_{-\pi}^{\pi}d\sigma Y^-(\sigma,\tau)e^{-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}}\right]\\
G^a =&\frac{1}{2}\epsilon^{ajk}G^{jk} = \qquad\qquad (4.6)\\
=&\epsilon^{ab}\Bigg\{-\frac{P^b}{P^+}X^+(\tau)\Xi^-_{tot}(\tau) + \frac{i\sqrt{\pi N}}{P^+}\frac{A^b_n(\tau)\tilde{L}_{-n}(\tau)}{n}+\\
&-\frac{i\Xi^-_{tot}(\tau)}{2\sqrt{\pi N}}\frac{1}{n}\frac{A^b_n(\tau)}{\pi}\int_{-\pi}^{\pi}d\sigma e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}}+\\
&+i\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left[A^b_n(\tau)\int_{-\pi}^{\pi}d\sigma\Xi^-_{rel}(\sigma,\tau)\frac{\partial}{\partial\sigma}e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}} - A^b_{-n}(\tau)\int_{-\pi}^{\pi}d\sigma\Xi^-_{rel}(\sigma,\tau)\frac{\partial}{\partial\sigma}e^{-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}}\right]+
\end{aligned}$$

$$+\frac{i}{2}\sqrt{\frac{N}{\pi}}\sum_{n=1}^{\infty}\frac{1}{n}\left[A^b_n(\tau)\int_{-\pi}^{\pi}d\sigma Y^-(\sigma,\tau)e^{-i\omega_n\frac{B^+_+(\sigma,\tau)}{2NP^+}}+A^b_{-n}(\tau)\int_{-\pi}^{\pi}d\sigma Y^-(\sigma,\tau)e^{-i\omega_n\frac{B^+_-(\sigma,\tau)}{2NP^+}}\right]\Bigg\}$$

From equations (4.4) we get the mass Poincaré Casimir

$$P^2=\sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}+2P^+\Xi^-_{tot}(\tau)\approx\sum_{n=1}^{\infty}\omega_n\vec{\alpha}_n\cdot\vec{\alpha}_{-n}\geq 0, \tag{4.7}$$

so that $P^2=0$ if the oscillatory modes $\vec{\alpha}_n$ are not excited.

Following reference [14], the Pauli-Lubansky 4-vector has the form

$$\begin{aligned}W^\mu=&\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma}=(\vec{P}\cdot\vec{S};P^0\vec{S}+\vec{P}\times\vec{N}_S)=\\=&\Big((\frac{1}{2}P^+-P^-)S^3+\epsilon^{ab}P^a(E^b_S-\frac{1}{2}F^b_S);\epsilon^{ab}(P^+E^b_S-P^-F^b_S+P^bN^3_S),\\&(\frac{1}{2}P^++P^-)S^3-\epsilon^{ab}P^a(E^b_S+\frac{1}{2}F^b_S)\Big).\end{aligned}\tag{4.8}$$

When $P^2>0$ we have the other Poincaré Casimir

$$W^2=-\frac{1}{2}P^2S_{\mu\nu}S^{\mu\nu}-P^\mu S_{\mu\nu}S^{\nu\rho}P_\rho=-P^2\vec{S}^2(P), \tag{4.9}$$

where $\vec{S}(P)$ is the Thomas spin. To find $\vec{S}(P)$, let us introduce the boost $L^\mu{}_\nu(P,\overset{\circ}{P})$ [14] from the rest frame $\overset{\circ}{P}{}^\mu=\eta\sqrt{P^2}(1;\vec{0})$, $\eta\equiv\mathrm{sign}P^0$, to the general frame $P^\mu=L^\mu{}_\nu(P,\overset{\circ}{P})\overset{\circ}{P}{}^\nu$. Then we define the rest frame Pauli-Lubansky vector:

$$\begin{aligned}W^\mu(P)=&\Big(0;\eta\sqrt{P^2}\vec{S}(P)\Big)\qquad\Longrightarrow\qquad W^\mu=\Big(\vec{P}\cdot\vec{S}(P);\eta\sqrt{P^2}\vec{S}(P)+\frac{\vec{P}\cdot\vec{S}(P)}{P^0+\eta\sqrt{P^2}}\vec{P}\Big)\\W_\mu=&W_\nu L^\nu{}_\mu(P,\overset{\circ}{P}).\end{aligned}\tag{4.10}$$

Following reference[15], let us define a new spin vector $\vec{\mathcal{S}}$:

$$\begin{aligned}\mathcal{S}^3=&\frac{W^+}{P^+}=S^3-\epsilon^{ab}\frac{P^a}{P^+}F^b_S=\frac{\eta\sqrt{P^2}}{P^+}S^3(P)+\frac{P^++\eta\sqrt{P^2}}{P^+(\frac{1}{2}P^++P^-+\eta\sqrt{P^2})}\vec{P}\cdot\vec{S}(P),\\\mathcal{S}^a=&\frac{1}{\eta\sqrt{P^2}}\Big(W^a-\frac{P^a}{P^+}W^+\Big)=\qquad\qquad(a=1,2)\\=&\frac{1}{\eta\sqrt{P^2}}\Big[\epsilon^{ab}\Big(P^+E^b_S-P^-F^b_S+P^bN^3_S\Big)-F^a\Big(S^3-\epsilon^{cd}\frac{P^c}{P^+}F^d_S\Big)\Big]=\\=&S^a(P)-\frac{P^a}{P^+}\Big(S^3(P)+\frac{\vec{P}\cdot\vec{S}(P)}{\frac{1}{2}P^++P^-P\eta\sqrt{P^2}}\Big).\end{aligned}\tag{4.11}$$

We have $\vec{\mathcal{S}}^2 = \vec{S}^2(P)$, and $\vec{\mathcal{S}} = \vec{S}(P)$ in the rest frame where $P^1 = P^2 = P^3 = 0$.

From the algebra of the Pauli-Lubansky vector

$$\Big\{W^\mu, W^\nu\Big\} = \epsilon^{\mu\nu\rho\sigma} P_\rho W_\sigma, \qquad \Big\{W^\mu, P^\nu\Big\} = 0,$$

we get

$$\Big\{\mathcal{S}^i, \mathcal{S}^j\Big\} = \epsilon^{ijk}\mathcal{S}^k, \qquad i,j,k = 1,2,3; \tag{4.12}$$

while from equation (4.11) we get

$$E^a_S = \frac{P^-}{P^+} N^3_S - \frac{\epsilon^{ab}}{P^+}\Big(\eta\sqrt{P^2}\mathcal{S}^b + P^b \mathcal{S}^3\Big), \tag{4.13}$$

that is, $E^a_S$ may be written in terms of $\vec{\mathcal{S}}$, $F^a_S$, $N^3_S$.

As we are using the front form of the dynamics[16], only three of the Poincaré generators are dynamical, *i.e.* $P^-$, $E^a$. The other seven, $P^+$, $\vec{P}$, $N^3$, $N^a$, $J^3$ are kinematical as they constitute the stability group of the null plane $x^+ = 0$ (and act transitively on the half space $P^+ > 0$). $P^-$ translates this plane keeping it parallel to itself, while the $E^a$ rotate it around the light-cone to which it is tangent. As shown in reference[17], instead of the three dynamical Hamiltonians $P^-$, $E^a$ one may consider as basic the $U(2)$ dynamical algebra generated by $M = \eta\sqrt{P^2}$, $\vec{\zeta}$, with $\Big\{M, \vec{\zeta}\Big\} = 0$. $\zeta^3$ is also contained in the kinematical generator $J^3$ and is a Casimir of the stability group called null-plane helicity [17].

By using equation (4.5) we get

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P^\nu \Big[i\frac{A_{n\rho}A_{-n\sigma}}{n} + G_{\rho\sigma}\Big]. \tag{4.14}$$

If we remember equations (4.11) we then get

$$\begin{aligned}
\mathcal{S}^3 =& i\frac{A^1_n A^2_{-n}}{n} + P^1(G^{20} + G^1) - P^2(G^{10} - G^2),\\
\mathcal{S}^a =& \frac{1}{\eta\sqrt{P^2}}\Big\{ -i\frac{P^+ \epsilon ab A^b_n A^-_{-n} + P^a A^1_n A^2_{-n}}{n} + \\
& + P^0 G^a + \epsilon^{ajk} P^j G^{k0} - \frac{P^a}{P^+}\Big[P^1\left(G^{20} + G^1\right) - P^2\left(G^{10} - G^2\right)\Big]\Big\}.
\end{aligned} \tag{4.15}$$

By looking at equations (4.4) and by using equations (3.7) to connect $A^-_n$ and $\tilde{U}^-_n$, we see that, modulo the constraints, $\vec{\mathcal{S}}$ is the spin with respect to the observable center of mass $(Z^-, \vec{Z})$. In the rest frame $\vec{\zeta} = \vec{S}(P)$. Since the string is described by a numerable set of bidimensional oscillators all lying in the same plane, to build a tridimensional spin like $\vec{\mathcal{S}}$ (or $\vec{S}(P)$), needed for $W^2 = -P^2\vec{S}^2(P) = -P^2\vec{\mathcal{S}}^2$, we necessarily must have $\mathcal{S}^1$, $\mathcal{S}^2$ realized in a non-linear way: $\mathcal{S}^3$ is bilinear in the $\vec{\alpha}_n$, while $\mathcal{S}^a$ contains trilinear terms.

It can be checked that this non-linearity is at the basis of the string model's quantum anomaly: $\left\{\mathcal{S}^1, \mathcal{S}^2\right\} = i\mathcal{S}^3$ is not preserved after quantization, due to ordering problems.

When $P^2 > 0$ and $D > 4$ the previous construction $\vec{\mathcal{S}}$ does not work because $W^\mu$ is not defined any more. The Lorentz group is now $O(D-1,1)$ and equations (4.1)-(4.3) are still valid with $3 \to D-1$. The Thomas spin $\vec{S}(P)$ and $\vec{\mathcal{S}}$ are replaced by the generators $S^{ij}(P)$ and $\mathcal{S}^{ij}$, $i,j = 1,2,...,D-1$, of the little group $SO(D-1)$ of $P^2 > 0$: the generators $\mathcal{S}^{ab}, a,b = 1,2,...,D-2$ of the $SO(D-2)$ built with the oscillators $\alpha^a_n(\tau)$, (see $\mathcal{S}^3$ in equation (4.15)) are bilinear in them; the remaining generators $\mathcal{S}^{a,D-1}$ will contain trilinear terms. The form of $\mathcal{S}^{a,D-1}$ may be extracted from equation (4.13) in $D$ dimension given the previous $\mathcal{S}^{a,b}$

$$E^a_S = \frac{P^-}{P^+} F^a_S - \frac{P^a}{P^+} N^{D-1}_S + \frac{1}{P^+}(\eta\sqrt{P^2}\mathcal{S}^{a,D-1} - \mathcal{S}^{a,b}P^b). \tag{4.16}$$

See reference[18] for related problems and for the Lorentz covariance properties of the transverse oscillators. When $D = 26$ for some non trivial reason the algebra of this non-linearly realized SO(25) is preserved by quantization. Another problem with quantization is that $P^2 > 0$, $P^0 > 0$ requires $P^+ > 0$, and as it is shown in reference [19], there is no self-adjoint operator satisfying all the properties of $Z^-$, so that the longitudinal observable position coordinate $Z^-$ has no quantum counterpart.

When $P^2 = 0$ a similar analysis may be performed (now $W^\mu \propto P^\mu\Sigma$, where $\Sigma = W^0/P^0$ is the helicity). We only notice that when $P^+ > 0$ the reference frame is $\mathring{P}^\mu = \omega(1;0,0,1)$: here $\mathring{P}^- = \vec{\mathring{P}} = 0$. But these are the conditions for some of the Patrascioiu modes [20]: they are the massless longitudinal modes when all the oscillators are at rest.

As a final remark let us introduce the action angle variables for the oscillators $\vec{\alpha}_n$ to see how $P^2$, $W^2 = -P^2\vec{\mathcal{S}}^2$, $\mathcal{S}^3$ depend from them when $P^2 > 0$ and the constrains are put equal to zero. Let us first go to a circular basis

$$b_{n(\pm)} = \frac{1}{\sqrt{2}}(\alpha^1_n \pm i\alpha^2_n), \qquad b_{-n(\pm)} = (b_{n(\pm)})^*, \qquad n > 0,$$
$$\left\{b_{n(\pm)}, b_{-m(\pm)}\right\} = -i\delta_{mn}, \tag{4.17}$$

so that we get for the occupation numbers

$$N_n = \sum_{a=1}^{2} N^a_n = \sum_{a=1}^{2}(\alpha^a_n\alpha^a_{-n}) = N_{n(+)} + N_{n(-)} = b_{n(+)}b_{-n(+)} + b_{n(-)}b_{-n(-)}. \tag{4.18}$$

In terms of the circular oscillators we have

$$\begin{aligned} P^2 =& \sum_{n=1}^{\infty}\omega_n(N^1_n + N^2_n) = \sum_{n=1}^{\infty}\omega_n[N_{n(+)} + N_{n(-)}], \\ \mathcal{S}^3 =& i\epsilon^{ab}\sum_{n=1}^{\infty}\alpha^a_n\alpha^b_{-n} = \sum_{n=1}^{\infty}[N_{n(-)} - N_{n(+)}]. \end{aligned} \tag{4.19}$$

The action-angle variable for the circular oscillators are

$$
\begin{aligned}
&b_{n(\pm)} = -ie^{-i\phi_{n(\pm)}}\sqrt{I_{n(\pm)}} = (b_{-n(\pm)})^*, \\
&I_{n(\pm)} = N_{n(\pm)}, \\
&\sin\phi_{n(\pm)} = -\frac{b_{n(\pm)} + b_{-n(\pm)}}{2\sqrt{I_{n(\pm)}}}, \qquad\qquad n > 0, \\
&\cos\phi_{n(\pm)} = \frac{i}{2}\frac{b_{n(\pm)} - b_{-n(\pm)}}{\sqrt{I_{n(\pm)}}}, \qquad\qquad n > 0, \\
&\left\{\phi_{n(\pm)}, I_{m(\pm)}\right\} = \delta_{n,m}.
\end{aligned}
\tag{4.20}
$$

In terms of them we get

$$
\begin{aligned}
P^2 &= \sum_{n=1}^{\infty}\omega_n[I_{n(+)} + I_{n(-)}], \\
\mathcal{S}^3 &= \sum_{n=1}^{\infty}[I_{n(-)} - I_{n(+)}], \\
W^2 &= -P^2\vec{\mathcal{S}}^2 = f(I_{n(\pm)}, \phi_{n(\pm)}).
\end{aligned}
\tag{4.21}
$$

Therefore $W^2$ still depends upon the angles even if $\left\{W^2, \mathcal{S}^3\right\} = \left\{W^2, P^2\right\} = 0$, due to $\omega_{n+m} = \omega_n + \omega_m$. This is due to the already quoted fact that in the front form dynamic we have three Hamiltonians ($P^2$, $\mathcal{S}^1$, $\mathcal{S}^2$) and not only one. Oppositely to the case of a non-relativistic completely integrable system[21], the variables (4.20) do not allow to express $P^2$, $W^2$, $\mathcal{S}^3$ in terms of a denumerable set of canonical variables in involution. Here this means that by quantizing the variables (4.20) we would not get the reducible Poincaré representation associated to the string model decomposed according to its irreducible components. In $D = 26$ equations (4.21) should be replaced by a complete set of commuting Casimirs of $O(D-1,1)$. See reference[22] for the difficulties in counting the spin levels lying on each mass level, not being able to decompose the Poincaré representation in a general way.

In conclusion, we have seen that it is possible to find a set of (infinite) constants of motion, which are in involution among themselves, but which are only locally defined, since we found them in a given chart of the phase- space manifold.

In reference [7], an (infinite) set of constants of motion has been found. These last are not in involution, contrary to the set found here (half of the new canonical variables). These new constants can be decomposed in our canonical basis, as it will be shown in a future paper.

## Appendix A.

We want now to compare our results with the covariant approach of reference [2]. Let us perform a Fourier expansion of $x^\mu(\sigma,\tau)$, $P^\mu(\sigma,\tau)$ satisfying equations (2.6):

$$\begin{aligned}
x^\mu(\sigma,\tau) =& X^\mu(\tau) + 2\sum_{n=1}^{\infty} q_n^\mu(\tau)\cos n\sigma = \\
=& X^\mu(\tau) + i\sqrt{2}\sum_{n=1}^{\infty}\frac{C_n^\mu(\tau) - C_n^{*\mu}(\tau)}{\sqrt{\omega_n}}\cos n\sigma = \\
=& X^\mu(\tau) + i\sqrt{2}\sum_{n\neq 0}\frac{C_n^\mu(\tau)}{\omega_n}\cos n\sigma \qquad (A.1)\\
P^\mu(\sigma,\tau) =& \frac{P^\mu}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\Pi_n^\mu(\tau)\cos n\sigma = \\
=& \frac{P^\mu}{\pi} + \frac{1}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\sqrt{\omega_n}\big(C_n^\mu(\tau) + C_n^{*\mu}(\tau)\big)\cos n\sigma = \\
=& \frac{P^\mu}{\pi} + \frac{1}{\pi\sqrt{2}}\sum_{n\neq 0} a_n^\mu(\tau)\cos n\sigma
\end{aligned}$$

where $\omega_n = 2\pi N n$, and $X^\mu(\tau)$ and $P^\mu(\tau)$ are given by equations (2.18), and the other quantities are defined as follow:

$$\begin{aligned}
q_n^\mu(\tau) =& \frac{i}{\sqrt{2\omega_n}}\Big[C_n^\mu(\tau) - C_n^{*\mu}(\tau)\Big] = \frac{i}{\sqrt{2}\omega_n}\big(a_n^\mu(\tau) - a_{-n}^\mu(\tau)\big) \quad && n>0\\
\Pi^\mu(\tau) =& \sqrt{\frac{\omega_n}{2}}\big[C_n^\mu(\tau) + C_n^{*\mu}(\tau)\big] = \frac{1}{\sqrt{2}}\big(a_n^\mu(\tau) + a_{-n}^\mu(\tau)\big) && n>0\\
C_n^\mu(\tau) =& \frac{-i}{\sqrt{2}}\big(\sqrt{\omega_n}q_n^\mu(\tau) + \frac{i\Pi_n^\mu(\tau)}{\sqrt{\omega_n}}\big) && n>0 \qquad (A.2)\\
C_n^{*\mu}(\tau) =& \big(C_n^\mu(\tau)\big)^* && n>0\\
a_n^\mu(\tau) =& \sqrt{\omega_n}C_n^\mu(\tau) = \frac{-i}{\sqrt{2}}\big(\omega_n q_n^\mu(\tau) + i\Pi_n^\mu(\tau)\big) && n>0\\
a_{-n}^\mu(\tau) =& \sqrt{\omega_n}C_n^{*\mu}(\tau) = \big(a_n^\mu(\tau)\big)^* && n>0
\end{aligned}$$

Remembering equations (2.19), we get:

$$\begin{aligned}
q_n^\mu(\tau) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma x^\mu(\sigma,\tau)\cos n\sigma = \frac{1}{2\pi n}\int_{-\pi}^{\pi} d\sigma Y^\mu(\sigma,\tau)\sin n\sigma,\\
\Pi_n^\mu(\tau) =& \int_{-\pi}^{\pi} d\sigma P^\mu(\sigma,\tau)\cos n\sigma = \frac{1}{n}\int_{-\pi}^{\pi} d\sigma \mathcal{P}(\sigma,\tau)\sin n\sigma, \qquad (A.3)
\end{aligned}$$

so that

$$
\begin{aligned}
Y^\mu(\sigma,\tau) =&2\sum_{n=1}^{\infty} n q_n^\mu(\tau)\sin n\sigma = i\sqrt{2}\sum_{n=1}^{\infty}\frac{n}{\sqrt{\omega_n}}\big(C_n^\mu(\tau)-C_n^{*\mu}(\tau)\big)\sin n\sigma = \\
=&\frac{1}{\pi N\sqrt{2}}\sum_{n\neq 0} a_n^\mu(\tau)\sin n\sigma, \\
\mathcal{P}^\mu(\sigma,\tau) =&\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Pi_n^\mu(\tau)}{n}\sin n\sigma = \frac{1}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sqrt{\omega_n}}{n}\big(C_n^\mu(\tau)+C_n^{*\mu}(\tau)\big)\sin n\sigma = \\
=&\frac{1}{\pi\sqrt{2}}\sum_{n\neq 0}\frac{a_n^\mu(\tau)}{n}\sin n\sigma, \\
A_\pm^\mu(\sigma,\tau) =&\frac{P^\mu}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\big(\Pi_n^\mu(\tau)\cos n\sigma \mp \omega_n q_n^\mu(\tau)\sin n\sigma\big) = \\
=&\frac{P^\mu}{\pi}+\frac{1}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\sqrt{\omega_n}\big(C_n^\mu(\tau)e^{\mp in\sigma}+C_n^{*\mu}(\tau)e^{\pm in\sigma}\big) = \\
=&\frac{P^\mu}{\pi}+\frac{1}{\pi\sqrt{2}}\sum_{n\neq 0} a_n^\mu(\tau)e^{\mp in\sigma}, \\
B_\pm^\mu(\sigma,\tau) =&\frac{\sigma}{\pi}P^\mu \pm N X^\mu(\tau)+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\big(\Pi_n^\mu(\tau)\sin n\sigma \pm \omega_n q_n^\mu(\tau)\cos n\sigma\big) = \\
=&\frac{\sigma}{\pi}P^\mu \pm N X^\mu(\tau) \pm \frac{i}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sqrt{\omega_n}}{n}\big(C_n^\mu(\tau)e^{\mp in\sigma}-C_n^{*\mu}(\tau)e^{\pm in\sigma}\big) = \\
=&\frac{\sigma}{\pi}P^\mu \pm N X^\mu(\tau) \pm \frac{i}{\pi\sqrt{2}}\sum_{n\neq 0}\frac{a_n^\mu(\tau)}{n}e^{\mp in\sigma}.
\end{aligned}
\tag{A.4}
$$

We have the following non vanishing Poisson brackets:

$$
\begin{aligned}
&\big\{q_n^\mu,\Pi_m^\nu\big\} = -\eta^{\mu\nu}\delta_{mn}, && n,m>0, \\
&\big\{C_n^\mu,C_m^{*\nu}\big\} = i\eta^{\mu\nu}\delta_{mn}, && n,m>0, \\
&\big\{A_n^\mu,a_{-m}^\nu\big\} = i\omega_n\delta_{mn}, && n,m=0,\pm1,\pm2,...,
\end{aligned}
\tag{A.5}
$$

where we have introduced

$$
a_0^\mu = \sqrt{2}P^\mu. \tag{A.6}
$$

From equation (3.1) we may express the $A_n^\mu$'s in terms of the $a_n^\mu$'s:

$$
\begin{aligned}
A_0^\mu =&\frac{a_0^\mu}{\sqrt{2\pi N}}, \\
A_n^\mu(\tau) =&\frac{1}{\sqrt{2\pi N}}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma \sum_{m=-\infty}^{\infty} a_m^\mu(\tau)e^{-im\sigma}.
\end{aligned}
\tag{A.7}
$$

$$
\cdot \exp\Big[\frac{i\omega_n}{2NP^+}\left(\frac{\sigma P^+}{\pi}+NX^+(\tau)+\frac{i}{\pi\sqrt{2}}\sum_{r\neq 0}\frac{a_r^+(\tau)}{r}e^{-ir\sigma}\right)\Big]
$$

$$
=\frac{1}{\sqrt{2\pi N}}e^{i\frac{\omega_n}{\sqrt{2}a_0^+}X^+(\tau)}\sum_{m=-\infty}^{\infty}a_m^\mu(\tau)\prod_{r\neq 0}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma\exp\big[i(n-m)\sigma-\frac{n}{a_0^+}\frac{a_r^+(\tau)}{r}e^{-ir\sigma}\big].
$$

The Virasoro generators are

$$
L_n(\tau)=\frac{1}{4N}\int_{-\pi}^{\pi}d\sigma e^{\pm in\sigma}\chi_\pm(\sigma,\tau)=\frac{1}{2}\sum_{m=-\infty}^{\infty}a_m^\mu\cdot a_{n-m,\mu}\approx 0, \tag{A.8}
$$

that is:

$$
\begin{aligned}
L_0 =&P^2+\frac{1}{2}\sum_{m\neq 0}a_m^\mu\cdot a_{-m\mu}=P^2+\sum_{m=1}^{\infty}\omega_m C_m^\mu C^*_{m\mu},\\
L_n =&\sqrt{2}P_\mu\cdot a_n^\mu+\sum_{m=1}^{\infty}a_{n+m}^\mu\cdot a_{-m\mu}+\frac{1}{2}\sum_{m=1}^{n-1}a_{m-n}^\mu\cdot a_{m\mu},
\end{aligned} \tag{A.9}
$$

so that equations (3.14) and (3.8) imply:

$$
\begin{aligned}
\tilde{L}_n(\tau)=&\frac{1}{2}\sum_{m=-\infty}^{\infty}A_m^\mu\cdot A_{n-m\mu}=\\
=&\frac{1}{2}a_0^+e^{i\frac{\omega_n}{\sqrt{2}a_0^+}X^+(\tau)}\cdot\\
&\cdot\sum_{m=-\infty}^{\infty}a_m^\mu\cdot a_{n-m\mu}\cdot\prod_{r\neq 0}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma e^{i(n-m)\sigma}\cdot\frac{\exp\Big[-i\frac{n}{a_0^+}\frac{a_r^+(\tau)}{r}e^{-ir\sigma}\Big]}{\sum_{-\infty}^{\infty}a_s^+(\tau)e^{-is\sigma}}\approx 0.
\end{aligned} \tag{A.10}
$$

The Lorentz generators $J^{\mu\nu}$ (equations (4.1)) have the following expression in terms of the $a_n^\mu$:

$$
\begin{aligned}
J^{\mu\nu}=&X^\mu(\tau)P^\nu-X^\nu(\tau)P^\mu+i\sum_{n\neq 0}\frac{1}{\omega_n}a_n^\mu(\tau)a_{-n}^\nu(\tau)=\\
=&Z^\mu(\tau)P^\nu-Z^\nu(\tau)P^\mu+i\sum_{n\neq 0}\frac{a_n^\mu(\tau)a_{-n}^\nu(\tau)}{\omega_n}+\\
&+\frac{i}{\sqrt{\pi N}}\sum_{n\neq 0}\frac{A_n^\mu(\tau)P^\nu-A_n^\nu(\tau)P^\mu}{n}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma e^{-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}}+\\
&-\big[(\eta^{\mu 0}+\eta^{\mu 3})P^\nu-(\eta^{\nu 0}+\eta^{\nu 3})P^\mu\big]\cdot\Big\{\frac{X^+(\tau)}{P^+}\Xi^-_{tot}(\tau)+\frac{i}{P^+}\sum_{n\neq 0}\frac{1}{n}\frac{\tilde{L}_n(\tau)}{2\pi}\int_{-\pi}^{\pi}d\sigma e^{-i\omega_n\frac{B_+^+(\sigma,\tau)}{2NP^+}}\Big\}
\end{aligned} \tag{A.11}
$$

where we have used equations (3.26).

If we compare equations (A.1) with equations (4.5), we get

$$i\sum_{n\neq 0}\frac{\tilde{a}^{\mu}_{n}(\tau)\tilde{a}^{\nu}_{-n}(\tau)}{n} = i\sum_{n\neq 0}\frac{A^{\mu}_{n}(\tau)a^{\nu}_{-n}(\tau)}{n} + \qquad (A.12)$$

$$- i\sum_{n\neq 0}\frac{A^{\mu}_{n}(\tau)A^{\nu}_{0} - A^{\nu}_{n}(\tau)A^{\mu}_{0}}{2\pi}\int_{-\pi}^{\pi} d\sigma e^{-i\omega_n \frac{B^{+}_{+}(\sigma,\tau)}{2NP^{+}}} + \tilde{G}^{\mu\nu},$$

where

$$\tilde{G}^{\mu\nu} = G^{\mu\nu} + \left[(\eta^{\mu 0} + \eta^{\mu 3})P^{\nu} - (\eta^{\nu 0} + \eta^{\nu 3})P^{\mu}\right]\cdot$$

$$\cdot\left[\frac{X^{+}(\tau)}{P^{+}}\Xi^{-}_{tot}(\tau) + \frac{i}{P^{+}}\sum_{n\neq 0}\frac{1}{n}\frac{\tilde{n(\tau)}}{2\pi}\int_{-\pi}^{\pi} d\sigma e^{-i\omega_n \frac{B^{+}_{+}(\sigma,\tau)}{2NP^{+}}}\right] \approx 0,$$

and

$$\tilde{a}^{\mu}_{n} \equiv \frac{a^{\mu}_{n}}{\sqrt{a\pi N}}.$$

Equations (A.12) have to be compared with similar results in references [13,23].

## References


[ 1] F.Colomo, G.Longhi and L.Lusanna, "Classical Solutions of the Many-time Functional Equations of Motion of the Nambu String", Florence preprint June 27, 1989.

[ 2] A.J.Hanson, T.Regge and C.Teitelboim, *Constrained Hamiltonian Systems*, Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche, Fisiche e loro Applicazioni, N.22, Accademia Nazionale dei Lincei, Roma, 1975.

[ 3] S.Shanmugadhasan, J.Math.Phys. **14** (1973) 677.

[ 4] E.Del Giudice,P.Di Vecchia and S.Fubini, Ann.Phys.(NY) **70** (1972) 378.

[ 5] H.Leutwyler and J.Stern, Ann.Phys. (NY) **112** (1978) 94.

[ 6] J.Govaerts, "The Nambu-Goto String: its Phase Space Path Integral", Cern preprint TH-4950/88.

[ 7] K.Pohlmeyer and K.H.Rehren, Commun.Math.Phys., **105** (1086); ibid, **114** (1988) 55. K.H.Rehren, "Geometric Charges of the Nambu-Goto String", in Proceeding of the Paris-Meudon Colloquium, World Scientific, 1986.

[ 8] Y.Nambu, Proc.Int.Conf. on *Symmetries and Quark Models*, 1969, Wayne State Univ.;
H.B.Nielsen, Proc. 15-th Conf. in *High Energy Physics*, Kiev Conference, 1970;
L.Susskind, Nuovo Cimento **69** (1970) 457;
T.Goto, Prog.Theor.Phys. **46** (1971) 1560.

[ 9] D.C.Salisbury and K.Sundermeyer, Nucl.Phys. **B191**(1981) 250.

[10] J.Scherk, Rev.Mod.Phys. **47** (1975) 123;
J.H.Schwarz, Phys.Reports **89** (1982) 223;

M.B.Green, J.H.Schwarz and E.Witten, *Superstring Theory*, Cambridge University Press, Cambridge, 1987.

[11] R.C.Brower and C.B.Thorn, Nucl.Phys. **B31** (1971) 163;
R.C.Brower, Phys.Rev. **D6** (1972) 1655;
P.Goddard and C.B.Thorn, Phys.Lett. **40B** (1972) 2, 235;
P.Goddard, C.Rebbi and C.B.Thorn, Nuovo Cimento **12A** (1972) 425;
L.Brink and D.Olive, Nucl.Phys. **B56** (1973) 256;
L.Brink, D.Olive and J.Scherk, Nucl. Phys. **B61** (1973) 173;
P.Goddard, A.J.Hanson and G.Ponzano, Nucl.Phys. **B89** (1975) 76;
C.B.Thorn, Nucl.Phys. **B248** (1984) 551.

[12] R.C.Brower, Phys.Rev. **D6** (1972) 1655.

[13] R.Marnelius, Nucl.Phys. **B211** (1983) 14;
R.Marnelius, Nucl.Phys. **B104** (1976) 477;
R.Marnelius, Phys.Lett. **B172** (1986) 337; in Proc. Workshop *Constraint's Theory and Relativistic Dynamics*, Firenze 1986, pg.277, eds. G.Longhi and L.Lusanna, World Scientific, Singapore, 1986.

[14] A.Barducci and L.Lusanna, Nuovo Cimento **66A** (1970) 253.

[15] M.Henneaux, Phys.Reports **126** (1985) 1.

[16] P.A.M.Dirac, Rev.Mod.Phys. **21** (1949) 392.

[17] H. Leutwyler and J. Stern, Ann. Phys. (NY) **112** (978) 94.

[18] P.Goddard, C.Rebbi and C.B.Thorn, Nuovo Cimento **12A** (1972) 425.

[19] E.Angelopoulos, F.Bayen and M.Flato, Phys.Scr., **9** (1974) 173.

[20] A.Patrascioiu, Lett. Nuovo Cimento **10** (1974) 676 ;
A.Patrascioiu, Nucl.Phys. **B81** (1974) 525 .

[21] V.I.Arnold, *Mathematical Methods in Classical Mechanics*, Mir Publishers, Moscow.

[22] L.Corgnier and A.D'Adda, Nuovo Cimento **66A** (1970) 253;
C.B.Chiu, R.L.Heimann and A.Schwimmer, Phys.Rev **D4** (1971) 3177;
T.L.Curtright and C.B.Thorn, Nucl.Phys. **B274** (1986) 520;
T.L.Curtright, J.Goldstone and C.B.Thorn, Phys.Letters **175B**, (1986) 47,
and Phys.Letters **182B** (1986) 45.

[23] P.Goddard, A.J.Hanson and G.Ponzano, Nucl.Phys. **B89** (1975) 76.